Some Problems in the Theory of Risk

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Warszawa, December 13-15, 2010

Contents

Chapter I

Concepts from insurance

Insurance mathematical theory can be divided into three parts:

1. life insurance,
2. non-life insurance,
3. risk theory.

In these series of lectures we will review some notions, concepts and results from the third group.

1 Basic notions of risk theory

In the chapters dealing with insurance aspects, we will restrict our attention to one specific portfolio. Such a portfolio is characterized by a number of ingredients of both a deterministic and a stochastic nature.

Among the first we mention the *starting position* and a *time period.* Usually, data referring to an insurance portfolio refer to a time span of one year in accordance with the bookkeeping of the company. Far more important is the *initial reserve* or *initial capital.* One interpretation of the latter is the amount of capital set aside to cover costs occurring during the initial period of the portfolio when the company has not yet received the yearly premiums. In the sequel the initial reserve will be denoted by u.

Among the elements that usually have a stochastic nature are the following:

- The *epochs of the claims*; denote them by $\sigma_1, \sigma_2, \ldots$. In some cases we consider an additional claim arrival epoch at time zero denoted by

$\sigma_0 = 0$. Apart from the fact that the epochs form a nondecreasing sequence we do not in general assume anything specific about their interdependence. The random variables defined by $T_n = \sigma_n - \sigma_{n-1}$, $n \geq 1$, are called the *inter-occurrence times* in between successive claims.

- The *number of claims* up to time t is denoted by $N(t)$ where $N(t) = \sup\{n : \sigma_n \leq t\}$. The intrinsic relation between the sequence of claim arrivals $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ and the counting process $\{N(t), t \geq 0\}$ is given by $\{N(t) = n\} = \{\sigma_n \leq t < \sigma_{n+1}\}$. Process $N(t)$ is sometimes called a *counting process*

- The claim occurring at time σ_n has size U_n. The sequence $\{U_n, n = 1, 2, \ldots\}$ of consecutive *claim sizes* is often assumed to consist of independent and identically distributed random variables. However, other possibilities will show up in the text as well.

- The *aggregate claim amount* up to time t is given by $X(t) = \sum_{i=1}^{N(t)} U_i$ while $X(t) = 0$ if $N(t) = 0$. By its very definition, the aggregate claim amount is in general a *random sum* of random variables.

- The *premium income*. In the course of time 0 to t we assume that a total of $\Pi(t)$ has been received through premiums.

- The *risk reserve* at time t is then $R(t) = u + \Pi(t) - X(t)$.

The above setup allows flexibility in that an individual claim may mean a claim from an individual customer (e.g. third-liability insurance) or a claim caused by a single event (e.g. windstorm insurance).

1.1 Examples of counting processes

Renewal process. In this case $T_1, T_2, \ldots$ are nonnegative i.i.d. random variables. If $T_1, T_2 \ldots$ are indepednent and $T_2, \ldots$ then $(N(t))$ is a delayed renewal process.

Poisson process is a special case of renewal process, when $T \sim$Exp(λ). Then $(N(t))_{t\geq 0}$ is a process with stationary and independent increments. Furthermore

$$p_k(t) = \mathbb{P}(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

We have

$$\mathbb{E}\, N(t) = \lambda t, \qquad \operatorname{Var} N(t) = \lambda t$$

and hence the index of dispertion $I(t) = \mathrm{Var}\, N(t)/\mathbb{E}\, N(t) = 1$.

Mixed Poisson process. Two step random mechanizm: first: choose λ according to distribution F_λ, second: generate the counting Poisson process $N(t)$ with rate λ. Then

$$p_k(t) = \mathbb{P}(N(t) = k) = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t}\, F_\lambda(dt).$$

Process $(N(t))$ is with stationary increments, however it is not ergodic. Notice that

$$N(t)/t \to \lambda, \qquad \text{a.s.}$$

2 Ruin Problems

Ruin theory has always been a vital part of actuarial mathematics. At first glance, some of the theoretically derived results seem to have limited scope in practical situations. Nevertheless, calculation of and approximation to ruin probabilities have been a constant source of inspiration and technique development in actuarial mathematics.

Assume an insurance company is willing to risk a certain amount u in a certain branch of insurance, i.e. if the claim surplus exceeds the level u some drastic action will have to be taken for that branch. Because in some sense this part of the business starts with the capital u we can safely call u the *initial capital.* The actuary now has to make some decisions, for instance which premium should be charged and which type of reinsurance to take. Often, the premium is determined by company policies and by tariffs of rivals. A possible criterion for optimizing the reinsurance treaty would be to minimize the probability that the claim surplus ever exceeds the level u. To be more specific, consider the risk reserve $R(t) = u + \Pi(t) - X(t)$ and define the random variable $\tau = \inf\{t \geq 0 : R(t) < 0\}$. The instant τ gives us the *ruin time* of the portfolio, where we interpret ruin in a technical sense. Of course, we should allow the possibility that no ruin ever occurs, which means that $\tau = \infty$. We should realize that τ is dependent on all the stochastic elements in the risk reserve process $\{R(t)\}$ as well as on the deterministic value u. For this reason one often singles out the latter quantity in the notation for the ruin time by writing $\tau(u)$ for τ. More specifically, the *survival* or *nonruin probability in finite time* will be defined and denoted by

$$\overline{\psi}(u; x) = \mathbb{P}\big(\inf_{0\leq t\leq x} R(t) \geq 0\big) = \mathbb{P}(\tau(u) > x)$$

when we consider a *finite horizon* $x > 0$. The *survival probability* over an infinite time horizon is defined by the quantity

$$\overline{\psi}(u) = \mathbb{P}\big(\inf_{t\geq 0} R(t) \geq 0\big) = \mathbb{P}(\tau(u) = \infty)\,.$$

Alternative notations that are in constant use refer to the *ruin probabilities* which are defined by the equalities

$$\psi(u;x) = 1 - \overline{\psi}(u;x)\,, \qquad \psi(u) = 1 - \overline{\psi}(u)\,.$$

The *risk reserve process* $\{R(t),\ t \geq 0\}$ is then given by

$$R(t) = u + \beta t - \sum_{i=1}^{N(t)} U_i\,, \tag{2.1}$$

while the *claim surplus process* $\{S(t),\ t \geq 0\}$ is

$$S(t) = \sum_{i=1}^{N(t)} U_i - \beta t\,. \tag{2.2}$$

The *time of ruin* $\tau(u) = \min\{t : R(t) < 0\} = \min\{t : S(t) > u\}$ is the first epoch when the risk reserve process becomes negative or, equivalently, when the claim surplus process crosses the level u. We will mainly be interested in the *ruin probabilities* $\psi(u;x) = \mathbb{P}(\tau(u) \leq x)$ and $\psi(u) = \lim_{x\to\infty} \psi(u;x) = \mathbb{P}(\tau(u) < \infty)$. Here $\psi(u;x)$ is called the *finite-horizon ruin probability* and $\psi(u)$ the *infinite-horizon ruin probability*. Alternatively, $\psi(u)$ can be called the *probability of ultimate ruin*. We will further need the notion of the *survival probability* $\overline{\psi}(u) = 1 - \psi(u)$.

There is a relationship between infinite-horizon ruin probabilities of risk models in discrete time and in continuous time. To get $\tau(u)$ it is sufficient to check the claim surplus process $\{S(t)\}$ at the embedded epochs σ_k ($k = 1, 2, \ldots$); see Figure 2.1. Indeed, the largest value $M = \max_{t\geq 0} S(t)$ of the claim surplus process can be given by $M = \max_{n\geq 0} \sum_{k=1}^{n} (U_k - \beta T_k)$ and consequently

$$\psi(u) = \mathbb{P}(M > u)\,. \tag{2.3}$$

The representation formula (2.3) gives us the possibility to interpret the ruin function $\psi(u)$ as the tail function of the stationary waiting time in a *single-server system* of queueing theory.

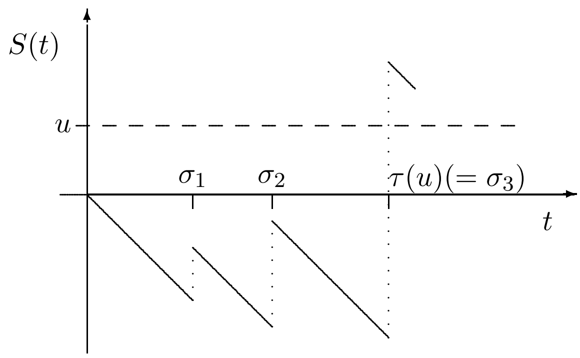

Figure 2.1: Claim surplus process

Note, however, that one has to be careful when comparing finite-horizon ruin probabilities in discrete time with those in continuous time because in general

$$\mathbb{P}\Big(\max_{0\le t\le x} S(t) > u\Big) \ne \mathbb{P}\Big(\max_{0<n\le x} \sum_{k=1}^{n}(U_k - \beta T_k) > u\Big).$$

Anyhow, in order to keep the notation simple we will use the same symbol for the finite-horizon ruin function in the continuous-time risk model as in the discrete-time risk model, i.e.

$$\psi(u;x) = \mathbb{P}(\tau(u) \le x) = \mathbb{P}\Big(\max_{0\le t\le x} S(t) > u\Big).$$

Apart from the time of ruin $\tau(u)$, there are other characteristics related to the concept of technical ruin. The *overshoot* above the level u of the random walk $\{S_n\}$ crossing this level for the first time is defined by

$$Y^+(u) = \begin{cases} S(\tau(u)) - u & \text{if}\tau(u) < \infty, \\ \infty & \text{if}\tau(u) = \infty. \end{cases}$$

Note that it is possible to express $Y^+(u)$ in terms of the risk reserve process:

$$Y^+(u) = \begin{cases} -R(\tau(u)) & \text{if}\tau(u) < \infty, \\ \infty & \text{if}\tau(u) = \infty. \end{cases}$$

In other words, $Y^+(u)$ can be interpreted as the *severity of ruin* at time $\tau(u)$; see Figure 2.2.

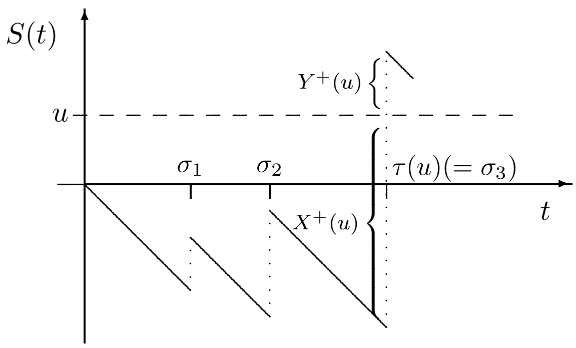

Figure 2.2: Severity of ruin and surplus prior to ruin

Another quantity of interest is the *surplus prior to ruin* given by

$$X^+(u) = \begin{cases} u - S(\tau(u)-) & \text{if} \tau(u) < \infty, \\ \infty & \text{if} \tau(u) = \infty. \end{cases}$$

Clearly, $X^+(u)+Y^+(u)$ is the *size of the claim causing ruin* at time $\tau(u)$. In order to determine the joint distribution of $X^+(u), Y^+(u)$, we will consider the *multivariate ruin function* $\psi(u, x, y)$ given by

$$\psi(u, x, y) = \mathbb{P}(\tau(u) < \infty, X^+(u) \le x, Y^+(u) > y)\,, \tag{2.4}$$

where $u, x, y \ge 0$ or its dual

$$\varphi(u, x, y) = \mathbb{P}(\tau(u) < \infty, X^+(u) > x, Y^+(u) > y)\,, \tag{2.5}$$

when the latter is more convenient. Another characteristic related to the severity of ruin is the time $\tau'(u) = \inf\{t : t > \tau(u), R(t) > 0\}$ at which the risk reserve process $\{R(t)\}$ crosses the level zero from below for the first time after the ruin epoch $\tau(u)$. Then

$$T'(u) = \begin{cases} \tau'(u) - \tau(u) & \text{if} \tau(u) < \infty, \\ 0 & if \tau(u) = \infty, \end{cases}$$

is the *time in the red* (see Figure 2.3), the amount of time the risk reserve process $\{R(t)\}$ stays below zero after the ruin time $\tau(u)$. It is clear that $T'(u)$ does not fully describe the severity of ruin, because it does not carry any information about the behaviour of the risk reserve between $\tau(u)$ and $\tau'(u)$. However, for the insurer it makes a difference whether $\{R(t)\}$ remains slightly below zero for a long time, or whether the *total maximal deficit*

$$Z^+(u) = \max\{-R(t) : \tau(u) \le t\}$$

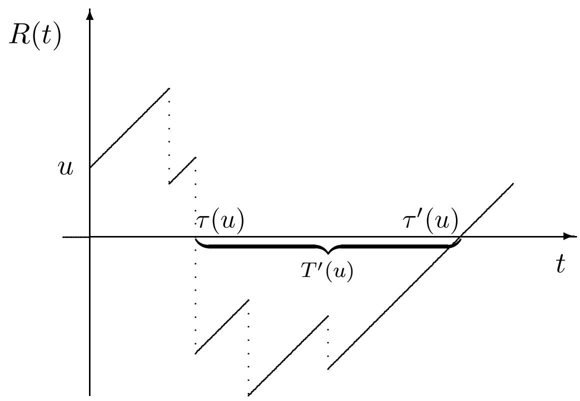

Figure 2.3: Time in the red

after $\tau(u)$ is large. In the latter case, all successive times in the red are taken into account. We can finally consider the *maximal deficit*

$$Z_1^+(u) = \max\{-R(t) : \tau(u) \leq t \leq \tau'(u)\}$$

during the first period in the red, that is between $\tau(u)$ and $\tau'(u)$.

We can modify the notion of ruin, if it happens only when the risk process stays in the red for a prescribed time a. It is said then that Parisian ruin happened.

3 Recent developments

Recently various modifications of classical ruin problem attract attention of researchers. An important contributions were done in the case when dividends are allowed. Suppose that claim surpluss process $S(t) = \sum_{i=1}^{N(t)} U_i - \beta t$ is given and it generates filtration $(\mathcal{F}_t)_{t\geq 0}$. One introduces then a divident process $V(t)$ adapted to $\mathcal{F}_t$ and such that

- ruin does not occur due to divident payments, i.e. $\Delta V(t) \leq R^V(t)$, where R^V denotes the controlled risk process,
- $V(0) = 0$ and the path of V is non-decreasing,
- payments have to stop after the event of ruin,
- decision have to be fixed in a predictable way.

The controlled processes is defined by

$$R^V(t) = x + \beta t - \sum_{i=1}^{N(t)} U_i - V(t).$$

As before we define the ruin by

$$\tau^V = \inf_{t\geq 0}\{R^V(t) < 0\}.$$

In particular two strategies are of interest:

- *Threshold strategies.* The cumulated divident payment is given by

$$V(t) = \int_0^{t\wedge\tau^V} a1(R^V(s-)\, ds.$$

- *Barrier strategies.*

$$V(t) = (x-b)1(x>b) + \int_0^{t\wedge\tau^V} c1(R^V(s-) = b)\, dt.$$

Bibliographical comments. In these notes the basic fact from risk are taken from Rolski *et al* (1999). The literature on dividend problems is already quite big; see for example a recent survay by Albrecher and Thonhauser (2009). Parisian ruin was recently studied by Czarna and Palmowski (2010).

Chapter II

Classes of distributions; light versus heavy tailed.

1 Facts about distributions of random random variables

In this lecture we consider some specific properties of distributions of random variables. Here X, Y are random variables defined on $\Omega, \mathcal{F}, \mathbb{P})$ with distribtutions F_X, F_Y respectively. The *tail distribution* of $F(x)$ we denote by $\overline{F}(x) = 1 - F(x)$. Unless it is said otherwise, we consider nonnegative random variables. Let $I = \{s \in \mathbb{R} : \mathbb{E}\, \mathrm{e}^{sX} < \infty\}$. Note that I is an interval which can be the whole real line $\mathbb{R}$, a halfline or even the singleton $\{0\}$. The *moment generating function* $\hat{m} : I \to \mathbb{R}$ of X is defined by $\hat{m}(s) = \mathbb{E}\, \mathrm{e}^{sX}$.

For purposes, which will be obvious in the sequel of these lecture notes we divide the class of distributions of nonnegative random variables into two groups:

- the class of *light-tailed distributions* if for some $0 < s$ we have $\hat{m}(s) < \infty$,
- the class of *heavy-tailed distributions* if we have $\hat{m}(s) = \infty$, for all $s > 0$. We call them *heavy-tailed distributions.*

Problems

1.1 Let $F(x)$ be the distribution function of a nonnegative random variable X. Show that $\hat{m}(s_0) < \infty$ for some $s_0 > 0$ if and only if for some

$a, b > 0$ the inequality $\overline{F}(x) \le a\,\mathrm{e}^{-bx}$ holds for all $x \ge 0$. Conclude from this that X has all moments finite if $\hat{m}(s_0) < \infty$ for some $s_0 > 0$. Give an example of a distribution F of a nonnegative random variable such that $\hat{m}(s) = \infty$ for all $s > 0$.

1.2 Show by examples that the following cases are possible for the moment generating function $\hat{m}(s)$:

(a) $\hat{m}(s) < \infty$ for all $s \in \mathbb{R}$

(b) there exists $s_0 > 0$ such that $\hat{m}(s) < \infty$ for $s < s_0$ and $\hat{m}(s) = \infty$ for $s \ge s_0$,

(c) there exists $s_0 > 0$ such that $\hat{m}(s) < \infty$ for $s \le s_0$ and $\hat{m}(s) = \infty$ for $s > s_0$.

[Hint. Use the inverse Gaussian distribution as an example for (c).]

1.3 Suppose that F is light tailed. Show that there exists s_0 such that $\hat{m}(s_0) < \infty$.

2 Heavy-Tailed Distributions

Unless it is said, in this section we study classes of distributions of nonnegative random variables such that $\hat{m}(s) = \infty$ for all $s > 0$. We call them *heavy-tailed distributions*. Prominent examples of heavy-tailed distributions are the lognormal, Pareto and Weibull distributions with shape parameter smaller than 1. Another important class of heavy-tailed distributions is *Pareto-type* distribution: Before we recall this notion we need a definition.

Definition 2.1 We say that a positive function $L : \mathbb{R}_+ \to (0, \infty)$ is a *slowly varying function of x at ∞* if for all $y > 0$, $L(xy)/L(x) \to 1$ as $x \to \infty$.

Examples of such functions are $|\log^p x|$, and functions converging to a positive limit as $x \to \infty$. Note that (2.8) gives in particular that, if $F \in \mathcal{S}$, then $\overline{F}(\log x)$ is a slowly varying function of x at ∞.

Definition 2.2 It is said the distribution F is *Pareto-type* with exponent $\alpha > 0$ if $\overline{F}(x) \sim L(x)x^{-\alpha}$ as $x \to \infty$ for a slowly varying function $L(x)$.

In the literature, Pareto-type distributions are also called *distributions with regular varying tails*.

Problems

2.1 Show that the following distribution functions are Pareto-type:

- Pareto Par(α) wtih density function $f(x) = \alpha(1+x)^{-(\alpha+1)}$,
- loggamma with density function

$$f(x) = \frac{\lambda^a}{\Gamma(a)}(\log x)^{a-1} x^{-\lambda-1}, \qquad x > 1,$$

- Burr distribution with

$$\overline{F}(x) = \left(\frac{b}{b+x}\right)^{\alpha}, \qquad x \geq 0.$$

2.1 Definition and Basic Properties

Let $\alpha_F = \limsup_{x\to\infty} M(x)/x$, where $M(x) = -\log \overline{F}(x)$ is the *hazard function* of F. If F has a continuous density, then $M(x)$ is differentiable and $\mathrm{d}M(x)/\mathrm{d}x = m(x)$, where $m(x)$ is the hazard rate function.

Theorem 2.3 *If $\alpha_F = 0$, then F is heavy-tailed.*

Proof Suppose that $\alpha_F = 0$. Then $\lim_{x\to\infty} M(x)/x = 0$. Thus, for each $\varepsilon > 0$ there exists an $x' > 0$ such that $M(x) \leq \varepsilon x$ for all $x \geq x'$. Therefore for some $c > 0$ we have $\overline{F}(x) \geq c\mathrm{e}^{-\varepsilon x}$ for all $x \geq 0$ and hence

$$\int_0^\infty \mathrm{e}^{sx}\overline{F}(x)\,\mathrm{d}x = \infty \tag{2.1}$$

for all $s \geq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, (2.1) holds for all $s > 0$, which means that F is heavy-tailed. $\square$

Remark For a heavy-tailed distribution F we have

$$\limsup_{x\to\infty} \mathrm{e}^{sx}\overline{F}(x) = \infty \tag{2.2}$$

for all $s > 0$. We leave it to the reader to show this as an exercise.

2.2 Long-tailed distribution

Definition 2.4 A distribution F is *long-tailed* if $\bar{F}(x) > 0$ for all x, and for any fixed $y > 0$

$$\overline{F}(x+y) \sim \overline{F}(x), \qquad x \to \infty.$$

We denote the class of long-tailed distributions by $\mathcal{L}$. Note that from the definition of long-tailness we have that $\overline{F}(x+y) \sim \overline{F}(x)$ uniformly with respect y in compact intervals.

Lemma 2.5 *Long-tailed distributions are heavy-tailed ones: for any $a > 0$*

$$\lim_{x\to\infty} \overline{F}(x)e^{ax} = \infty.$$

Lemma 2.6 *Suppose that $\overline{F}$ is long-tailed. The there exists a function $h(x)$ swuch that $h(x) \to \infty$ as $x \to \infty$ and*

$$\sup_{|y|\le h(x)} \overline{F}(x+y) - \overline{F}(x)| = o(\overline{F}(x)),$$

as $x \to \infty$.

In this case it is said that function h in Lemma 2.6 is *h-sensitive.*

For example if $h(x) = \epsilon x$, then

$$\frac{\overline{F}((1+\epsilon)x)}{\overline{F}(x)} \to 1$$

for $x \to \infty$, so $\overline{F}(x)$ is slowly varying (very heavy indeed!).

Suppose now F is Pareto-type; let $\overline{F}(x) = x^{-\alpha}L(x)$ where L is slowly varying. Then immediately

$$\lim_{x\to\infty} \frac{\overline{F}((1+\epsilon)x)}{\overline{F}(x)} = (1+\epsilon)^{-\alpha} \lim_{x\to\infty} \frac{L((1+\epsilon)x)}{L(x)} = 1$$

and hence it fulfills

$$\lim_{\epsilon\downarrow 0} \liminf_{x\to\infty} \frac{\overline{F}((1+\epsilon)x)}{\overline{F}(x)} = 1. \tag{2.3}$$

A distribtuion F is called intermediate regularly varying if (2.3) holds. A recent discovery of Foss *et al* is the following fact.

Theorem 2.7 *A distribution F is intermediate regularly varying if and only if, for any function h such that $h(x) = o(x)$*

$$\overline{F}(x+h(x)) \sim \overline{F}(x).$$

Problems

2.1 Show that the following distributions are heavy-tailed: Pareto-type, Weibull (when?), log-normal, log-gamma, Burr.

2.3 Subexponential Distributions

A distribution F on $\mathbb{R}_+$ is said to be *subexponential* if

$$\lim_{x\to\infty} \frac{1-F^{*2}(x)}{1-F(x)} = 2\,. \tag{2.4}$$

Let $\mathcal{S}$ denote the class of all subexponential distributions. We show later that the following important (parametrized) families of distributions are in $\mathcal{S}$: the lognormal distributions, Pareto distributions and Weibull distributions with shape parameter smaller than 1.

Sometimes we need the concept of subexponentiality for distributions on the real line $\mathbb{R}$. Before we exend this definition we state the following lemma.

Lemma 2.8 *[Foss et al] Let F be a distribution on $\mathbb{R}$ and X be a random variable with distribution F. Then the following are equivalent:*
(i) *F is long-tailed and*

$$\lim_{x\to\infty} \frac{1-F^{*2}(x)}{1-F(x)} = 2\,.$$

(ii) *the distribution F^+ of X_+ is subexponential,*
(iii) *the conditional distribution G of $X|X>0$ is subexponential.*

Thus it is said that a distribtion F on $\mathbb{R}$ with right-unbounded support is subexponential on the whole line if F is long-tailed and

$$\lim_{x\to\infty} \frac{1-F^{*2}(x)}{1-F(x)} = 2\,.$$

A direct consequence of (2.4) is that $\overline{F}(x)>0$ for all $x\ge 0$. However, not all distributions with this property are subexponential. Note, for example, that trivially the exponential distribution is not subexponential because in this case $(1-F^{*2}(x))/(1-F(x)) = e^{-\lambda x}(1+\lambda x)/e^{-\lambda x} \to \infty$ as $x\to\infty$. On the other hand, it is easy to see that if F is subexponential and X_1, X_2 are independent and identically distributed random variables with distribution F, then we have for $x\to\infty$ that

$$\mathbb{P}(X_1+X_2>x) \sim \mathbb{P}(\max\{X_1,X_2\}>x)\,, \tag{2.5}$$

since $\mathbb{P}(\max\{X_1,X_2\}>x) = 1-F^2(x) = (1-F(x))(1+F(x))$ and hence

$$1 = \lim_{x\to\infty} \frac{1-F^{*2}(x)}{2(1-F(x))} = \lim_{x\to\infty} \frac{1-F^{*2}(x)}{(1+F(x))(1-F(x))} = \lim_{x\to\infty} \frac{1-F^{*2}(x)}{1-F^2(x)}\,.$$

Lemma 2.9
$$\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 1 + \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\, \mathrm{d}F(y)\,, \tag{2.6}$$
from which we obtain that always

$$\liminf_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \geq 2\,. \tag{2.7}$$

Note that (2.7) implies that the limit value 2 in (2.4) is minimal. Furthermore, (2.6) yields two useful properties of subexponential distributions.

Lemma 2.10 *If $F \in \mathcal{S}$, then for all $x' > 0$,*

$$\lim_{x\to\infty} \frac{\overline{F}(x-x')}{\overline{F}(x)} = 1 \tag{2.8}$$

and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\, \mathrm{d}F(y) = 1\,. \tag{2.9}$$

From the above lemma we see that a subexponential distribution is from $\mathcal{L}$ and hence also is heavy-tailed.

Lemma 2.11 *Let $F \in \mathcal{S}$ and F' be a distribution with $F'(0) = 0$ such that $\lim_{x\to\infty} \overline{F'}(x)/\overline{F}(x) = c$ for some $c \in [0,\infty)$. Then*

$$\lim_{x\to\infty} \frac{\overline{F * F'}(x)}{\overline{F}(x)} = 1 + c\,. \tag{2.10}$$

Using Lemma 2.11 we get the following characterization of subexponential distributions.

Theorem 2.12 *Let F be a distribution on $\mathbb{R}_+$. Then, $F \in \mathcal{S}$ if and only if for each $n = 2, 3, \ldots$*
$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n\,. \tag{2.11}$$

We recommend the reader to show the following natural extension of (2.5) to an arbitrary (finite) number of random variables with subexponential distribution: if $X_1, \ldots, X_n$ are independent and identically distributed with distribution $F \in \mathcal{S}$, then $\mathbb{P}(\sum_{i=1}^n X_i > x) \sim \mathbb{P}(\max_{1\leq i\leq n} X_i > x)$

as $x \to \infty$. Furthermore, Theorem 2.12 immediately yields that for distributions of the form $F(x) = \sum_{k=0}^{n} p_k G^{*k}(x)$, where $\{p_0, p_1, \ldots, p_n\}$ is a probability function and G a subexponential distribution, we have

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\overline{G}(x)} = \sum_{k=0}^{n} kp_k \,. \tag{2.12}$$

Such *compound distributions* F are important in insurance mathematics and will be studied later. For example, ruin functions of some risk processes can be expressed by compound distributions. To study the asymptotic behaviour of ruin functions in the case of subexponential claim size distributions we need an extended version of (2.12) for compound distributions of type $F(x) = \sum_{k=0}^{\infty} p_k G^{*k}(x)$ where $\{p_0, p_1, \ldots\}$ is a probability function. In connection with this the following lemma is useful.

Lemma 2.13 *[Kesten lemma] If $F \in \mathcal{S}$, then for each $\varepsilon > 0$ there exists a constant $c < \infty$ such that for all $n \ge 2$*

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \le c(1+\varepsilon)^n, \qquad x \ge 0\,. \tag{2.13}$$

Theorem 2.14 *Let $F(x) = \sum_{k=0}^{\infty} p_k G^{*k}(x)$, where $\{p_0, p_1, \ldots\}$ is a probability function and $G \in \mathcal{S}$. If $\sum_{n=1}^{\infty} p_n (1+\varepsilon)^n < \infty$ for some $\varepsilon > 0$, then*

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\overline{G}(x)} = \sum_{k=0}^{\infty} kp_k \,. \tag{2.14}$$

We close this section showing subexponentiality for an important class of distributions, containing Pareto distributions and other parametrized families of distributions like loggamma distributions.

Theorem 2.15 *If F is Pareto-type, then $F \in \mathcal{S}$.*

Proof Let X, X_1 and X_2 be independent and identically distributed risks with Pareto-type distribution F. Note that $\{X_1 + X_2 > x\}$ implies that for $\varepsilon \in (0,1)$

$$\{X_1 > (1-\varepsilon)x\} \quad \text{or} \quad \{X_2 > (1-\varepsilon)x\} \quad \text{or} \quad \{X_1 > \varepsilon x \quad \text{and} \quad X_2 > \varepsilon x\},$$

which yields $\mathbb{P}(X_1 + X_2 > x) \le 2\mathbb{P}(X > (1-\varepsilon)x) + (\mathbb{P}(X > \varepsilon x))^2$. Hence

$$\limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{L(x)x^{-\alpha}} \le 2(1-\varepsilon)^{-\alpha}.$$

Since $\varepsilon > 0$ is arbitrary, $\limsup_{x\to\infty} \overline{F^{*2}}(x)/\overline{F}(x) \le 2$. However, in view of (2.7) this gives $\lim_{x\to\infty} \overline{F^{*2}}(x)/\overline{F}(x) = 2$ and the proof is completed. $\square$

2.4 Criteria for Subexponentiality and the Class $\mathcal{S}^*$

In most cases it is not an easy task to prove directly that a given distribution is subexponential. In Theorem 2.15 we were able to verify subexponentiality for Pareto-type distributions. However, for future applications in risk theory, we need the integrated tail of the distribution F to be subexponential rather than the distribution itself. Recall that for a distribution F of a nonnegative random variable with finite expectation $\mu > 0$, the *integrated tail distribution* F^{s} is given by

$$F^{\mathrm{s}}(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \mu^{-1} \displaystyle\int_0^x \overline{F}(y)\,\mathrm{d}y & \text{if } x > 0. \end{cases} \tag{2.15}$$

It seems to be not yet known whether $F \in \mathcal{S}$ and $0 < \mu < \infty$ imply $F^{\mathrm{s}} \in \mathcal{S}$ in general. Thus, it is useful to have conditions for a distribution with finite expectation to be subexponential jointly with its integrated tail distribution. On the other hand, there exist examples of distributions F on $\mathbb{R}_+$ such that $F^{\mathrm{s}} \in \mathcal{S}$, but $F \notin \mathcal{S}$.

We now show that, for a certain subset $\mathcal{S}^*$ of $\mathcal{S}$ which is defined below, $F \in \mathcal{S}^*$ implies $F^{\mathrm{s}} \in \mathcal{S}$. Throughout this section we only consider distributions F on $\mathbb{R}_+$ such that $F(0) = 0, F(x) < 1$ for all $x \in \mathbb{R}_+$.

Definition 2.16 (a) We say that F belongs to the class $\mathcal{S}^*$ if F has finite expectation μ and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\,\mathrm{d}y = 2\mu\,. \tag{2.16}$$

(b) We say that F belongs to $\mathcal{L}$ if for all $y \in \mathbb{R}$

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1\,. \tag{2.17}$$

Note that Lemma 2.10 implies $\mathcal{S} \subset \mathcal{L}$. Class $\mathcal{L}$ will serve to show that class $\mathcal{S}^*$ of distributions on $\mathbb{R}_+$ has some desired properties. We leave it to the reader to show as an exercise that all distribution functions with hazard rate functions tending to 0 are in $\mathcal{L}$. We also have the identity

$$\int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\,\mathrm{d}y = \int_0^{x/2} \ldots + \int_{x/2}^x \ldots = 2\int_0^{x/2} \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\,\mathrm{d}y\,, \tag{2.18}$$

from which we get that (2.16) is equivalent to

$$\lim_{x\to\infty}\int_0^{x/2}\frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{F}(y)\,\mathrm{d}y=\mu\,. \tag{2.19}$$

We now study the relationship between $\mathcal{S}^*$ and $\{F : F\in\mathcal{S}$ and $F^{\mathrm{s}}\in\mathcal{S}\}$. For this we need three lemmas. In the first we give an equivalence relation for subexponential distributions.

Lemma 2.17 *Let F, G be two distributions on $\mathbb{R}_+$ and assume that there exists a constant $c\in(0,\infty)$ such that*

$$\lim_{x\to\infty}\frac{\overline{G}(x)}{\overline{F}(x)}=c\,. \tag{2.20}$$

Then, $F\in\mathcal{S}$ if and only if $G\in\mathcal{S}$.

The above lemma justifies the following definition. Two distributions F and G on $\mathbb{R}_+$ are said to be *tail-equivalent* if $\lim_{x\to\infty}\overline{G}(x)/\overline{F}(x)=c$ for some $0<c<\infty$. This will be denoted by $G\sim^{\mathrm{t}} F$. It turns out that for distributions from $\mathcal{S}^*$, condition (2.20) can be weakened.

Lemma 2.18 *Let $F, G\in\mathcal{L}$. Suppose there exist $c_-, c_+\in(0,\infty)$ such that*

$$c_-\le\frac{\overline{G}(x)}{\overline{F}(x)}\le c_+ \tag{2.21}$$

for all $x\ge 0$. Then, $F\in\mathcal{S}^*$ if and only if $G\in\mathcal{S}^*$.

It can be proved that for a distribution function F with hazard rate function $m_F(x)$, we have $F\in\mathcal{L}$ if $\lim_{x\to\infty}m_F(x)=0$. A certain conversion of this statement is given in the following lemma.

Lemma 2.19 *For each $F\in\mathcal{L}$ there exists a distribution $G\in\mathcal{L}$ with $F\sim^{\mathrm{t}} G$ such that its hazard function $M_G(x)=-\log\overline{G}(x)$ and its hazard rate function $m_G(x)=\mathrm{d}M_G(x)/\mathrm{d}x$ have the following properties: $M_G(x)$ is continuous and almost everywhere differentiable with the exception of points in $\mathbb{N}$, and $\lim_{x\to\infty}m_G(x)=0$.*

Remark A consequence of Lemmas 2.17, 2.18 and 2.19 is that to check subexponentiality for $F\in\mathcal{L}$ it suffices to verify this for G, which is tail-equivalent to F and for which $\lim_{x\to\infty}m_G(x)=0$. Moreover, if G^s belongs to $\mathcal{S}$, then F^{s} belongs to $\mathcal{S}$, too. The proof is left to the reader.

We use the idea from the above remark in the proof of the following theorem.

Theorem 2.20 *If* $F \in \mathcal{S}^*$*, then* $F \in \mathcal{S}$ *and* $F^{\mathrm{s}} \in \mathcal{S}$*.*

Corollary 2.21 *Assume that the hazard rate function* $m_F(x)$ *of* F *exists and* $\mu < \infty$*. If* $\limsup_{x\to\infty} x m_F(x) < \infty$*, then* $F \in \mathcal{S}$ *and* $F^{\mathrm{s}} \in \mathcal{S}$*.*

In the case that $\limsup_{x\to\infty} x m_F(x) = \infty$, one can use the following criterion for $F \in \mathcal{S}^*$.

Theorem 2.22 *Assume that the hazard rate function* $m_F(x)$ *of* F *exists and is ultimately decreasing to* 0*. If* $\int_0^\infty \exp(x m_F(x))\overline{F}(x)\,\mathrm{d}x < \infty$ *then* $F \in \mathcal{S}^*$*.*

Examples 1. For the Weibull distribution $F = \mathrm{W}(r,c)$ with $0 < r < 1, c > 0$ we have $\overline{F}(x) = \exp(-cx^r)$ and $m_F(x) = crx^{r-1}$. Hence $\lim_{x\to\infty} x m_F(x) = \infty$ and Corollary 2.21 cannot be applied. But, the function $\exp(x m_F(x))\overline{F}(x) = \exp(c(r-1)x^r)$ is integrable and so $F = \mathrm{W}(r,c) \in \mathcal{S}^*$ by Theorem 2.22.

2. Consider the standard lognormal distribution $F = \mathrm{LN}(0,1)$. Let $\Phi(x)$ be the standard normal distribution function with density denoted by $\phi(x)$. Then, F has the tail and hazard rate functions

$$\overline{F}(x) = 1 - \Phi(\log x)\,, \qquad m_F(x) = \frac{\phi(\log x)}{x(1-\Phi(\log x))}\,.$$

Furthermore, $\phi(x) \sim x(1-\Phi(x))$ as $x \to \infty$. This follows from the fact that

$$\begin{aligned} \frac{\mathrm{e}^{-x^2/2}}{(2\pi)^{1/2}}\frac{1}{x} &= \frac{1}{(2\pi)^{1/2}}\int_x^\infty \mathrm{e}^{-y^2/2}\Big(1+\frac{1}{y^2}\Big)\,\mathrm{d}y \\ &> 1-\Phi(x) > \frac{1}{(2\pi)^{1/2}}\int_x^\infty \mathrm{e}^{-y^2/2}\Big(1-\frac{3}{y^4}\Big)\,\mathrm{d}y \\ &= \frac{\mathrm{e}^{-x^2/2}}{(2\pi)^{1/2}}\Big(\frac{1}{x}-\frac{1}{x^3}\Big)\,. \end{aligned}$$

Thus, we have $\mathrm{e}^{x m_F(x)}\overline{F}(x) \sim x(1-\Phi(\log x))$ as $x \to \infty$. For $x \to \infty$, the function

$$x(1-\Phi(\log x)) \sim \frac{x\phi(\log x)}{\log x}$$

is integrable, because $\phi(\log x) = (2\pi)^{-1/2}x^{-(\log x)/2}$ and $\int_1^\infty x^{1-(\log x)/2}\,\mathrm{d}x < \infty$. Hence the standard lognormal distribution $\mathrm{LN}(0,1)$ belongs to $\mathcal{S}^*$ and therefore F and F^{s} are subexponential. The case of a general lognormal distribution can be proved analogously.

To show that the integrated tail distribution of Pareto-type distributions is subexponential, we need the following result, known as *Karamata's theorem.* We state this theorem without proof, for which we refer to Feller (1971).

Theorem 2.23 *If $L_1(x)$ is a slowly varying function and locally bounded in $[x_0, \infty)$ for some $x_0 > 0$, then for $\alpha > 1$*

$$\int_x^\infty y^{-\alpha} L_1(y)\,\mathrm{d}y = x^{-\alpha+1} L_2(x)\,, \tag{2.22}$$

where $L_2(x)$ is also a slowly varying function of x at ∞ and moreover $\lim_{x\to\infty} L_1(x)/L_2(x) = \alpha - 1$. If $L_1(y)/y$ is integrable, then the result also holds for $\alpha = 1$.

As proved in Section 2.3, every Pareto-type distribution F with exponent $\alpha > 1$ is subexponential. We now get that the corresponding integrated tail distribution F^{s} is also subexponential, because Theorem 2.23 implies that $\overline{F}^{\mathrm{s}}(x) = x^{-\alpha+1} L_2(x)$ is Pareto-type too. This yields that many distributions, like Pareto and loggamma distributions as well as Pareto mixtures of exponentials studied in the next section, have the desired property that $F \in \mathcal{S}$ and $F^{\mathrm{s}} \in \mathcal{S}$.

2.5 Maximum of random walk

In this section we study a random walk, whose increments have heavy-tailed distribution with a negative mean. To be more specific, let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables with distribution F on $\mathbb{R}$. The sequence $\{S_n,\ n \in \mathbb{N}\}$ with $S_0 = 0$ and $S_n = Y_1 + \ldots + Y_n$ for $n = 1, 2, \ldots$ is called a *random walk*. We assume that the first moment $\mathbb{E}Y$ exists and that Y is not concentrated at 0, i.e. $\mathbb{P}(Y = 0) < 1$. Define the maximum of the random walk $M = \sum_{n\geq 0} S_n$. Define for a distribution F on $\mathbb{R}$ such that

$$\int_0^\infty \overline{F}(y)\,dy < \infty$$

the integrated tail function

$$\overline{F}_I(x) = \min\left(1, \int_x^\infty \overline{F}(y)\,dy\right).$$

If F is a distribtution F on $\mathbb{R}_+$, then we have the following relationship between the so called integrated tail distribution F^{s} of a distribtution F on $\mathbb{R}_+$:

$$\overline{F}^{\mathrm{s}}(x) = \frac{\overline{F}_I(x)}{\mu}$$

where $\mu = \int_0^\infty \overline{F}(x)\,dx$.

Theorem 2.24 *[Foss et al] Suppose that* $\mathbb{E}\, Y = -a < 0$*. Then for any* $x > 0$

$$\mathbb{P}(M > x) \geq \frac{\int_x^\infty \overline{F}(y)\, dy}{a + \int_x^\infty \overline{F}(y)\, dy}.$$

Hence

$$\liminf_{x \to \infty} \frac{\mathbb{P}(M > x)}{\overline{F}_I(x)} \geq \frac{1}{a}.$$

The following result is the most general form of what is called Embrecht-Verabeveke theorem.

Theorem 2.25 *Suppose that* $\mathbb{E}\, X = -a < 0$*. Then* $\mathbb{P}(M > x) \sim \overline{F}(x)/a$ *if and only if* F_I *is subexponential.*

Bibliographical comments.

Most of the properties about heavy-tailed distributions can be found in the monography of Rolski el al (1999). A small monography devoted especially to heavy-tailed distributions was recently written by Foss et al (2010), where from are given the recent discoveries in the area of heavy-tailed distributions. For regularly varying functions we also refer to Bingam et al (1987).

Chapter III

Classical Risk Model

1 Poisson Arrival Processes

1.1 Homogeneous Poisson Processes

Let $\{T_n\}$ be a sequence of independent random variables with exponential distribution Exp(λ); $\lambda > 0$. Then, the counting process $\{N(t)\}$ is called a *homogeneous Poisson process* with *intensity* λ.

Definition 1.1 A real-valued stochastic process $\{X(t), t \geq 0\}$ is said to have
(a) independent increments if for all $n = 1, 2, \ldots$ and $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables $X(t_0), X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent,
(b) stationary increments if for all $n = 1, 2, \ldots,$ $0 \leq t_0 < t_1 < \ldots < t_n$ and $h \geq 0$, the distribution of $(X(t_1+h)-X(t_0+h), \ldots, X(t_n+h)-X(t_{n-1}+h))$ does not depend on h.

We now give some equivalent definitions of a Poisson process.

Theorem 1.2 *Let $\{N(t), t \geq 0\}$ be a counting process. Then the following statements are equivalent:*
(a) *$\{N(t)\}$ is a Poisson process with intensity λ.*
(b) *For all $t \geq 0$, $n = 1, 2, \ldots$ the random variable $N(t)$ has distribution* Poi(λt) *and, given $\{N(t) = n\}$, the random vector $(\sigma_1, \ldots, \sigma_n)$ has the same distribution as the order statistics of n independent points uniformly distributed on $[0, t]$.*
(c) *$\{N(t)\}$ has independent increments such that $\mathbb{E}\, N(1) = \lambda$ and for all $t \geq 0$, $n = 1, 2, \ldots,$ given $\{N(t) = n\}$, the random vector $(\sigma_1, \ldots, \sigma_n)$ has*

the same distribution as the order statistics of n independent points uniformly distributed on $[0,t]$.
(d) $\{N(t)\}$ *has stationary and independent increments and satisfies as* $h \downarrow 0$,

$$\mathbb{P}(N(h)=0)=1-\lambda h+o(h)\,, \qquad \mathbb{P}(N(h)=1)=\lambda h+o(h)\,. \tag{1.1}$$

(e) $\{N(t)\}$ *has stationary and independent increments and, for each fixed* $t \ge 0$, *the random variable* $N(t)$ *is* $\mathrm{Poi}(\lambda t)$ *distributed.*

1.2 Compound Poisson Processes

We continue to assume that the inter-occurrence times $\{T_n\}$ are exponentially distributed with parameter $\lambda > 0$ or that the counting process $\{N(t)\}$ is a Poisson process with intensity λ. Let the claim sizes $\{U_n\}$ be independent and identically distributed with distribution F_U and let $\{U_n\}$ be independent of

$$\{N(t)\}_{j=1}^{N(t)} U_j.$$

Then the cumulative arrival process $\{X(t),\ t \ge 0\}$, where $X(t)=\sum_{j=1}^{N(t)} U_j$ is called a *compound Poisson process* with characteristics (λ, F_U), i.e. with intensity λ and jump size distribution F_U. This terminology is motivated by the property that $X(t)$ has a compound Poisson distribution with characteristics $(\lambda t, F_U)$. Since $X(t)=\sum_{i=1}^{N(t)} U_i$, it suffices to observe that, by the result of Theorem 1.2, $N(t)$ is Poisson distributed with parameter λt.

The next result follows from Theorem 1.2.

Corollary 1.3 *Let* $\{X(t)\}$ *be a compound Poisson process with characteristics* (λ, F_U). *Then,*
(a) *the process* $\{X(t)\}$ *has stationary and independent increments,*
(b) *the moment generating function of* $X(t)$ *is given by*

$$\hat{m}_{X(t)}(s)=\mathrm{e}^{\lambda t(\hat{m}_U(s)-1)}, \tag{1.2}$$

and the mean and variance by

$$\mathbb{E}\,X(t)=\lambda t\mu_U\,, \qquad \mathrm{Var}\,X(t)=\lambda t\mu_U^{(2)}\,. \tag{1.3}$$

Problems

1.1 Assume that $\{X(t), t \ge 0\}$ is a process with independent increments and for each $t \ge 0$ the distribution of $X(t+h)-X(h)$ does not depend on $h \ge 0$. Show that then the process $\{X(t)\}$ has stationary increments.

1.2 Let $\{N(t), t \geq 0\}$ be a stochastic process which has stationary and independent increments and satisfies as $h \downarrow 0$,

$$\mathbb{P}(N(h) = 0) = 1 - \lambda h + o(h)\,, \qquad \mathbb{P}(N(h) = 1) = \lambda h + o(h)$$

for some $\lambda > 0$. Show that then the probability $p_n(t) = \mathbb{P}(N(t) = n)$ is given by

$$p_n(t) = \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}$$

for all $t \geq 0$ and $n \in \mathbb{N}$.

1.3 Let $\{N(t), t \geq 0\}$ be a Poisson process with intensity λ and let $c > 0$ be some constant. Show that $\{N(ct), t \geq 0\}$ is a Poisson process and determine its intensity.

1.4 Let $\{X(t)\}$ be a compound Poisson process with characteristics (λ, F_U). Show that for all $t \geq 0$

$$\hat{m}_{X(t)}(s) = \mathrm{e}^{\lambda t(\hat{m}_U(s)-1)}$$

and consequently $\mathbb{E}\, X(t) = \lambda t \mu_U$,

$$\operatorname{Var} X(t) = \lambda t \mu_U^{(2)}\,, \qquad \mathbb{E}\left((X(t) - \mathbb{E}\, X(t))^3\right) = -\lambda t \mu_U^{(3)}\,.$$

[Hint. Notice that the random variable $X(t)$ has a compound Poisson distribution.]

1.5 Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be two independent compound Poisson processes with characteristics (λ_1, F_1), (λ_2, F_2). Show that $\{X(t)\} = \{X_1(t) + X_2(t)\}$ is a compound Poisson process and determine its characteristics.

1.6 Consider a compound Poisson process with claim occurrence times σ_n and claim sizes U_n. Define, for some fixed $u \geq 0$,

$$X_1(t) = \sum_{k=1}^{\infty} U_k 1(U_k \leq u) 1(\sigma_k \leq t)$$

and

$$X_2(t) = \sum_{k=1}^{\infty} U_k 1(U_k > u) 1(\sigma_k \leq t)\,.$$

Show that $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ are two independent compound Poisson processes. Determine their characteristics.

1.7 Assume that there is a *delay in claim settlement* modelled by the sequence $D_1, D_2, \ldots$ of nonnegative independent and identically distributed random variables which are independent of $\{(\sigma_n, U_n)\}$. Show that the process $\{X(t), t \geq 0\}$ defined by $X(t) = \sum_{k=1}^{\infty} U_k \mathbb{1}(\sigma_k + D_k < t)$ has independent increments. Determine the distribution of $X(t+h) - X(h)$ for $t, h \geq 0$.

1.8 Let $\{(\sigma_n, U_n)\}$ and $\{D_n\}$ be the same as in Exercise 1.7 and consider the following model of *gradual claim settlement*. Let $g : \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}_+$ be a measurable function. Then $\{X(t)\}$ with $X(t) = \sum_{k=1}^{\infty} g(\sigma_k - t, D_k, U_k)$ is called a *shot-noise process* with *response function* g. Determine the Laplace–Stieltjes transform of $X(t)$.

Bibliographical comments. The material covered in Section 1 can be found in a large number of textbooks, such as Billingsley (1995). For a discussion of Poisson processes in the context of risk theory, see also Kingman (1996).

2 Ruin Probabilities: The Compound Poisson Model

In the sequel of this chapter we consider the compound Poisson model. The risk reserve process $\{R(t), t \geq 0\}$ is defined in (I.2.1) and claims occur according to a compound Poisson process with characteristics (λ, F_U). The most frequently used property of the process $\{R(t)\}$ is the independence and stationarity of its increments. Considering $\{R(t)\}$ from time t onwards is like restarting a risk reserve process with an identically distributed claim arrival process but with initial reserve $R(t)$. In particular, if $R(t) = y$ and ruin has not yet occurred by time t, then the (conditional) ruin probability is $\psi(y)$. Furthermore, considering $\{R(t)\}$ from the first claim occurrence epoch σ_1 on is like starting a risk reserve process with initial reserve $R(\sigma_1) = u + \beta\sigma_1 - U_1$.

Let $\{S_n, n \geq 0\}$ be the random walk given by

$$S_n = \sum_{i=1}^{n} Y_i \,, \qquad Y_i = U_i - \beta T_i \,. \tag{2.4}$$

In Theorem IV.1.1 we will show that $\limsup_{n\to\infty} S_n = \infty$ if $\mathbb{E}\, Y \geq 0$. Thus, (I.2.3) implies that $\psi(u) \equiv 1$ in this case. Let us therefore assume that $\mathbb{E}\, Y < 0$, i.e. $\beta > \lambda\mu$, where $\mu = \mu_U$ denotes the expected claim size. Recall that β is the premium income in the unit time interval and that $\lambda\mu$

is the expected aggregate claim over the unit time interval (see (1.3)). The condition

$$\beta > \lambda\mu \tag{2.5}$$

is therefore called the *net profit condition.* Throughout the rest of this chapter we will assume (2.5). Note that in this case $\lim_{n\to\infty} S_n = -\infty$, since from the strong law of large numbers we have $S_n/n \to \mathbb{E}\, Y < 0$. Thus, the maximum of $\{S_n\}$ is finite. Using (I.2.3) we get $\lim_{u\to\infty}\psi(u) = 0$. Moreover, we will see later in Theorem 2.4 that this implies $\psi(u) < 1$ for all $u \geq 0$.

2.1 An Integro-Differential Equation

In this section we study the survival probability $\overline{\psi}(u) = 1 - \psi(u)$. We show that $\overline{\psi}(u)$ is differentiable everywhere on $\mathbb{R}_+$ with the exception of an at most countably infinite set of points. Furthermore, we prove that $\overline{\psi}(u)$ fulfils an integro-differential equation.

Theorem 2.1 *The survival function $\overline{\psi}(u)$ is continuous on $\mathbb{R}_+$ with right and left derivatives $\overline{\psi}_+^{(1)}(u)$ and $\overline{\psi}_-^{(1)}(u)$, respectively. Moreover*

$$\beta\overline{\psi}_+^{(1)}(u) = \lambda\Big(\overline{\psi}(u) - \int_0^u \overline{\psi}(u-y)\,\mathrm{d}F_U(y)\Big) \tag{2.6}$$

and

$$\beta\overline{\psi}_-^{(1)}(u) = \lambda\Big(\overline{\psi}(u) - \int_0^{u-} \overline{\psi}(u-y)\,\mathrm{d}F_U(y)\Big)\,. \tag{2.7}$$

An immediate consequence of Theorem 2.1 is that the continuous function $\overline{\psi}(u)$ is differentiable everywhere except for the countable set, where $F_U(y)$ is not continuous. The importance of this fact is that it implies

$$\int_u^\infty \overline{\psi}^{(1)}(v)\,\mathrm{d}v = \psi(u)\,, \qquad u \geq 0\,.$$

In the terminology of measure theory, this means that $\overline{\psi}(u)$ is absolutely continuous with respect to the Lebesgue measure.

Note that in general (2.6) cannot be solved analytically. However, one can compute the survival probability $\overline{\psi}(u)$ in (2.6) numerically.

Example Assume that the claim sizes are exponentially distributed with parameter δ. Then the net profit condition (2.5) takes the form $\delta\beta > \lambda$.

Furthermore, (2.6) can be solved analytically. The survival function $\overline{\psi}(u)$ is differentiable everywhere and satisfies the integral equation

$$\beta\overline{\psi}^{(1)}(u) = \lambda\Big(\overline{\psi}(u) - \mathrm{e}^{-\delta u}\int_0^u \overline{\psi}(y)\delta\mathrm{e}^{\delta y}\,\mathrm{d}y\Big)\,. \tag{2.8}$$

This equation implies that $\overline{\psi}^{(1)}(u)$ is differentiable and that

$$\beta\overline{\psi}^{(2)}(u) = \lambda\Big(\overline{\psi}^{(1)}(u) + \delta\mathrm{e}^{-\delta u}\int_0^u \overline{\psi}(y)\delta\mathrm{e}^{\delta y}\,\mathrm{d}y - \delta\overline{\psi}(u)\Big) = (\lambda - \delta\beta)\overline{\psi}^{(1)}(u)\,.$$

The general solution to this differential equation is

$$\overline{\psi}(u) = c_1 - c_2\mathrm{e}^{-(\delta-\lambda/\beta)u}, \tag{2.9}$$

where $c_1, c_2 \in \mathbb{R}$. Since $\lim_{u\to\infty}\overline{\psi}(u) = 1$ it follows that $c_1 = 1$. Plugging (2.9) into (2.8) yields

$$\begin{aligned} &c_2(\delta\beta - \lambda)\mathrm{e}^{-(\delta-\lambda/\beta)u} \\ &= \lambda\Big(1 - c_2\mathrm{e}^{-(\delta-\lambda/\beta)u} - (1-\mathrm{e}^{-\delta u}) + c_2\mathrm{e}^{-\delta u}\frac{\beta\delta}{\lambda}(\mathrm{e}^{\lambda u/\beta} - 1)\Big) \end{aligned}$$

from which $c_2 = \lambda(\beta\delta)^{-1}$ is obtained. Thus,

$$\psi(u) = \frac{\lambda}{\beta\delta}\mathrm{e}^{-(\delta-\lambda/\beta)u}. \tag{2.10}$$

2.2 An Integral Equation

Equation (2.6) is not easily solved because it involves both the derivative and an integral of $\overline{\psi}(u)$. It would be more convenient to get rid of the derivative. Indeed, integrating (2.6) we arrive at the following result.

Theorem 2.2 *The ruin function $\psi(u)$ satisfies the integral equation*

$$\beta\psi(u) = \lambda\Big(\int_u^\infty \overline{F}_U(x)\,\mathrm{d}x + \int_0^u \psi(u-x)\overline{F}_U(x)\,\mathrm{d}x\Big). \tag{2.11}$$

2.3 Laplace Transforms, Pollaczek–Khinchin Formula

In this section we compute the Laplace transforms

$$\hat{L}_\psi(s) = \int_0^\infty \psi(u)\mathrm{e}^{-su}\,\mathrm{d}u\,, \qquad \hat{L}_{\overline{\psi}}(s) = \int_0^\infty \overline{\psi}(u)\mathrm{e}^{-su}\,\mathrm{d}u\,.$$

Note that both integrals make sense for all $s > 0$. Furthermore, we have

$$\hat{L}_\psi(s) = \int_0^\infty (1-\overline{\psi}(u))\,\mathrm{e}^{-su}\,\mathrm{d}u = \frac{1}{s} - \hat{L}_{\overline{\psi}}(s)\,. \tag{2.12}$$

Theorem 2.3 *The Laplace transforms $\hat{L}_{\overline{\psi}}(s)$ and $\hat{L}_{\psi}(s)$ are given by*

$$\hat{L}_{\overline{\psi}}(s) = \frac{\beta - \lambda\mu}{\beta s - \lambda(1 - \hat{l}_U(s))}\,, \qquad s > 0 \tag{2.13}$$

and

$$\hat{L}_{\psi}(s) = \frac{1}{s} - \frac{\beta - \lambda\mu}{\beta s - \lambda(1 - \hat{l}_U(s))}\,, \qquad s > 0\,. \tag{2.14}$$

Example In the case of exponentially distributed claims (with $\mu = \delta^{-1}$), (2.13) gives

$$\hat{L}_{\overline{\psi}}(s) = \frac{\beta - \lambda/\delta}{\beta s - \lambda(1 - \delta/(\delta + s))} = \frac{\beta - \lambda/\delta}{s(\beta - \lambda/(\delta + s))} = \frac{(\beta - \lambda/\delta)(\delta + s)}{s(\beta(\delta + s) - \lambda)}$$

and, by (2.12),

$$\begin{aligned} \hat{L}_{\psi}(s) &= \frac{1}{s} - \frac{(\beta - \lambda/\delta)(\delta + s)}{s(\beta(\delta + s) - \lambda)} = \frac{\beta(\delta + s) - \lambda - (\beta - \lambda/\delta)(\delta + s)}{s(\beta(\delta + s) - \lambda)} \\ &= \frac{\lambda}{\delta}\,\frac{1}{\beta(\delta + s) - \lambda} = \frac{\lambda}{\delta\beta}\,\frac{1}{\delta - \lambda/\beta + s}\,. \end{aligned}$$

Hence, by comparison with the Laplace–Stieltjes transform of the exponential distribution we realize that $\psi(u) = \lambda(\delta\beta)^{-1}\mathrm{e}^{-(\delta - \lambda/\beta)u}$, in accordance with (2.10).

Although equation (2.11) is simpler than (2.6), it is generally difficult to solve it in closed form. However, (2.11) leads to a formula for $\psi(u)$ in the form of an infinite series of convolutions. In this connection, we need the integrated tail distribution F_U^{s} of F_U. Remember that F_U^{s} is given by

$$F_U^{\mathrm{s}}(x) = \frac{1}{\mu}\int_0^x \overline{F}_U(y)\,\mathrm{d}y\,, \qquad x \ge 0\,. \tag{2.15}$$

The representation formula for $\psi(u)$ derived in the next theorem is called the *Pollaczek–Khinchin formula.*

Theorem 2.4 *For each $u \ge 0$,*

$$\psi(u) = \Big(1 - \frac{\lambda\mu}{\beta}\Big)\sum_{n=1}^{\infty}\Big(\frac{\lambda\mu}{\beta}\Big)^n \overline{(F_U^{\mathrm{s}})^{*n}}(u)\,. \tag{2.16}$$

Besides the case of exponentially distributed claim sizes, where (2.16) has been written in closed form (see (2.10)), there are other claim size distributions for which (2.16) simplifies. One important class of such claim size distributions is provided by the phase-type distributions.

Problems

2.1 Let the claim sizes U_n be $\mathrm{Erl}(2,\delta)$-distributed. Show, that

$$\psi(u) = a\,\mathrm{e}^{-r_1 u} - b\,\mathrm{e}^{-r_2 u}\,, \tag{2.17}$$

where $r_1 < r_2$ are the solutions to the equation

$$\beta r^2 - (2\delta\beta - \lambda)r + \delta(\delta\beta - 2\lambda) = 0$$

and

$$a = \frac{\lambda(2\lambda - \beta\delta + 2\beta r_2)}{\beta^2\delta(r_2 - r_1)}\,, \qquad b = \frac{\lambda(\beta\delta - 2\lambda - 2\beta r_1)}{\beta^2\delta(r_2 - r_1)}\,.$$

[Hint. Differentiate (2.6) twice.]

2.2 Let the claim sizes U_n be $\mathrm{Erl}(2,\delta)$-distributed. Determine the Laplace transforms of $\overline{\psi}(u)$ and $\psi(u)$. Invert the Laplace transforms in order to verify (2.17).

2.3 Let $\beta = \lambda = 1$ and $F_U(x) = 1 - \frac{1}{3}(\mathrm{e}^{-x} + \mathrm{e}^{-2x} + \mathrm{e}^{-3x})$. Show that

$$\psi(u) = 0.550790\mathrm{e}^{-0.485131u} + 0.0436979\mathrm{e}^{-1.72235u} + 0.0166231\mathrm{e}^{-2.79252u}\,.$$

[Hint. Calculate the Laplace transform $\hat{L}_\psi(s)$ and invert it.]

2.4 Put $\rho = \lambda\mu\beta^{-1}$. Let $\rho = 0.75$, $\beta = 1$ and let the claim sizes have the distribution $F_U = p\mathrm{Exp}(a_1) + (1-p)\mathrm{Exp}(a_2)$, where $p = 2/3$, $a_1 = 2$ and $a_2 = 1/2$. Show that the ruin function $\psi(u)$ is given by

$$\psi(u) = 0.75\left(0.935194\mathrm{e}^{-0.15693u} + 0.0648059\mathrm{e}^{-1.59307u}\right)\,.$$

2.5 Let the distribution F_U of claims sizes U_n be the exponential distribution $\mathrm{Exp}(\delta)$ with parameter $\delta > 0$. Determine the integrated tail distribution F_U^{s}. Use this result to recover formula (2.10) for $\psi(u)$ using the Pollaczek-Khinchin formula (2.16).

2.6 Let the claims sizes U_n be exponentially distributed with parameter $\delta > 0$. Find the multivariate ruin function $\varphi(u, 0, y)$.

2.4 Severity of Ruin

We now want to analyse further what happens if ruin occurs. Consider the ruin probabilities $\varphi(u,x,y) = \mathbb{P}(\tau(u) < \infty, X_+(u) > x, Y_+(u) > y)$ where $X_+(u) = R(\tau(u)-)$ and $Y_+(u) = -R(\tau(u))$ is the surplus just before and at the ruin time $\tau(u)$ respectively. Remember that the random variable $Y_+(u)$ is also called *severity of ruin*.

Since in general we were not able to find an explicit formula for $\psi(u)$ there is no hope of achieving this goal for $\varphi(u,x,y)$. But it is possible to derive integro-differential and integral equations for $\varphi(u,x,y)$. Moreover, we will be able to find $\varphi(x,y) = \varphi(0,x,y)$ explicitly.

We will proceed as in Section 2.1. Condition on the first claim occurrence epoch and on the size of that claim to find that $\varphi(u,x,y)$ satisfies

$$\begin{aligned}\varphi(u,x,y) &= \mathrm{e}^{-\lambda h}\varphi(u+\beta h,x,y) \\ &\quad + \int_0^h \Bigl(\int_0^{u+\beta t} \varphi(u+\beta t - v,x,y)\,\mathrm{d}F_U(v) \\ &\quad + 1(u+\beta t > x)(1-F_U(u+\beta t+y))\Bigr)\lambda \mathrm{e}^{-\lambda t}\,\mathrm{d}t\,,\end{aligned}$$

for all $h,u,x,y > 0$. Thus $\varphi(u,x,y)$ is right-continuous and differentiable from the right with respect to u. Furthermore,

$$\begin{aligned}&\beta \frac{\partial^+}{\partial u}\varphi(u,x,y) \\ &= \lambda\Bigl(\varphi(u,x,y) - \int_0^u \varphi(u-v,x,y)\,\mathrm{d}F_U(v) - 1(u \ge x)\overline{F}_U(u+y)\Bigr)\,.\end{aligned}$$

Analogously, $\varphi(u,x,y)$ is left-continuous and differentiable from the left with respect to u, and satisfies

$$\begin{aligned}\beta \frac{\partial^-}{\partial u}\varphi(u,x,y) &= \lambda\Bigl(\varphi(u,x,y) - \int_0^{u-} \varphi(u-v,x,y)\,\mathrm{d}F_U(v) \\ &\qquad - 1(u > x)\overline{F}_U((u+y)-)\Bigr)\,.\end{aligned}$$

Thus the set of points u where the partial derivative $(\partial/\partial u)\varphi(u,x,y)$ does not exist is countable and therefore $\varphi(u,x,y)$ is absolutely continuous in u.

Proceeding as in Section 2.2 we obtain the integral equation

$$\begin{aligned}&\beta(\varphi(u,x,y) - \varphi(0,x,y)) \\ &= \lambda\Bigl(\int_0^u \varphi(u-v,x,y)\,\overline{F}_U(v)\,\mathrm{d}v - 1(u \ge x)\int_{x+y}^{u+y} \overline{F}_U(v)\,\mathrm{d}v\Bigr).\end{aligned} \tag{2.18}$$

We now let $u \to \infty$. Note that $\int_0^\infty (1-F_U(v))\,\mathrm{d}v = \mu$ allows us to interchange integration and limit on the right-hand side of (2.18). Since $0 \le \varphi(u,x,y) \le \psi(u)$ we find that $\lim_{u\to\infty} \varphi(u,x,y) = 0$ and therefore

$$\varphi(0,x,y) = \frac{\lambda}{\beta} \int_{x+y}^{\infty} \overline{F}_U(v)\,\mathrm{d}v\,. \tag{2.19}$$

In Section IV.1, we will show how (2.19) provides another interpretation to the integrated tail distribution F_U^{s} as the *ladder height distribution* of the random walk $\{S_n\}$ given in (2.4).

Bibliographical comments. The classical compound Poisson risk model was introduced by Filip Lundberg (1903) and extensively studied by Harald Cramér (1930,1955). It is therefore often called the *Cramér–Lundberg model.* In particular, Theorems 2.1, 2.2 and 2.3 go back to these two authors. From the mathematical point of view, the ruin function $\psi(u)$ of the compound Poisson model is equivalent to the tail function of the stationary distribution of virtual waiting time in an M/GI/1 queue. Thus, formula (2.16) is equivalent to the celebrated *Pollaczek–Khinchin formula* of queueing theory; see, for example, Asmussen (2003) and Prabhu (1965). It also is a special case of a more general result on the distribution of the maximum of a random walk with negative drift, see Theorem IV.1.3. Further details on the equivalence between characteristics of queueing and risk processes can be found, for example, in the books by Asmussen (2003) and Prabhu (1965). In risk theory, (2.16) is often called *Beekman's formula.* The notion of severity of ruin was introduced in Gerber, Goovaerts and Kaas (1987). The compound Poisson risk model has been extended in several directions. Some of them will be discussed later in these notes. . For some other extensions we will refer to the literature. Notice that a compound Poisson process has finitely many jumps in bounded time intervals. Examples of claim arrival processes with stationary and independent increments and with infinitely many jumps in bounded intervals have been studied, for instance, in Dufresne, Gerber and Shiu (1991). These processes are called *gamma processes* and belong to the larger class of Lévy processes. We return to this later on.

3 Bounds, and Asymptotics

We have seen that it is generally difficult to determine the function $\psi(u)$ explicitly from formula (2.16). Therefore, bounds and approximations to the ruin probability $\psi(u)$ are requested. Besides this, knowledge of the asymptotic behaviour of $\psi(u)$ as $u \to \infty$ can also be useful in order to get information about the nature of the underlying risks.

3.1 Lundberg Bounds

Since the claim surplus $S(t)$ at time $t \geq 0$ has a shifted compound Poisson distribution with characteristics $(\lambda t, F_U)$ and the shift is on $-\beta t$, the moment generating function of $S(t)$ is

$$\hat{m}_{S(t)}(s) = \mathbb{E}\, \mathrm{e}^{sS(t)} = \exp\left(t(\lambda(\hat{m}_U(s) - 1) - \beta s)\right).$$

If $\hat{m}_U(s_0) < \infty$ for some $s_0 > 0$, then the function $\theta(s) = \lambda(\hat{m}_U(s) - 1) - \beta s$ is infinitely often differentiable in the interval $(-\infty, s_0)$. In particular

$$\theta^{(2)}(s) = \lambda \hat{m}_U^{(2)}(s) = \lambda \mathbb{E}\left(U^2 \mathrm{e}^{sU}\right) > 0\,, \tag{3.20}$$

which shows that $\theta(s)$ is a convex function. For the first derivative $\theta^{(1)}(s)$ at $s = 0$ we have

$$\theta^{(1)}(0) = \lambda \hat{m}_U^{(1)}(0) - \beta = \lambda\mu - \beta < 0\,. \tag{3.21}$$

It is easily seen that $\theta(0) = 0$. Moreover, there may exist a second root of

$$\theta(s) = 0\,. \tag{3.22}$$

If such a root $s \neq 0$ exists, then it is unique and strictly positive. We call this solution, if it exists, the *adjustment coefficient* or the *Lundberg exponent* and denote it by γ.

Note that the adjustment coefficient exists in the following situation.

Lemma 3.1 *Assume that there exists $s_\infty \in \mathbb{R} \cup \{\infty\}$ such that $\hat{m}_U(s) < \infty$ if $s < s_\infty$ and $\lim_{s \uparrow s_\infty} \hat{m}_U(s) = \infty$. Then there exists a unique positive solution γ to the equation* (3.22).

The existence of the adjustment coefficient is important because it allows uniform upper and lower exponential bounds for the ruin function $\psi(u)$. Let $x_0 = \sup\{x : F_U(x) < 1\}$.

Theorem 3.2 *Assume that the adjustment coefficient $\gamma > 0$ exists. Then,*

$$a_- \mathrm{e}^{-\gamma u} \leq \psi(u) \leq a_+ \mathrm{e}^{-\gamma u} \tag{3.23}$$

for all $u \geq 0$, where

$$a_- = \inf_{x \in [0, x_0)} \frac{\mathrm{e}^{\gamma x} \int_x^\infty \overline{F}_U(y)\,\mathrm{d}y}{\int_x^\infty \mathrm{e}^{\gamma y} \overline{F}_U(y)\,\mathrm{d}y}\,, \qquad a_+ = \sup_{x \in [0, x_0)} \frac{\mathrm{e}^{\gamma x} \int_x^\infty \overline{F}_U(y)\,\mathrm{d}y}{\int_x^\infty \mathrm{e}^{\gamma y} \overline{F}_U(y)\,\mathrm{d}y}\,.$$

Results as in Theorem 3.2 are known in risk theory as *two-sided Lundberg bounds* for the ruin function $\psi(u)$. Alternatively, an easy application of integral equation (2.11) leads to (3.23). Moreover, for all $u \ge 0$,

$$\psi(u) \begin{cases} < & a_+ \mathrm{e}^{-\gamma u} \quad \text{if } a_+ > \psi(0), \\ > & a_- \mathrm{e}^{-\gamma u} \quad \text{if } a_- < \psi(0). \end{cases} \tag{3.24}$$

This can be shown in the following way. Note that

$$a_+ \ge \frac{\int_0^\infty \overline{F}_U(y)\,\mathrm{d}y}{\int_0^\infty \mathrm{e}^{\gamma y}\overline{F}_U(y)\,\mathrm{d}y} = \frac{\mu}{\gamma^{-1}(\hat{m}_U(\gamma)-1)} = \frac{\lambda\mu}{\beta} = \psi(0)\,,$$

and analogously $a_- \le \psi(0)$. Let $b \ge a_+$ such that $b > \psi(0)$.

3.2 The Cramér–Lundberg Approximation

In Section 3.1 we have found exponential upper and lower bounds for the ruin function $\psi(u)$. We are now interested in the asymptotic behaviour of $\psi(u)\mathrm{e}^{\gamma u}$. The question is whether $\psi(u)\mathrm{e}^{\gamma u}$ converges to a limit or fluctuates between two bounds as $u \to \infty$. We will see that the limit $\lim_{u\to\infty}\psi(u)\mathrm{e}^{\gamma u}$ exists. However, to show this we need the following auxiliary result.

Lemma 3.3 *Assume that the function $z_1 : \mathbb{R}_+ \to (0,\infty)$ is increasing and let $z_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be decreasing, such that*

$$\int_0^\infty z_1(x)z_2(x)\,\mathrm{d}x < \infty \tag{3.25}$$

and

$$\lim_{h\to 0} \sup\{z_1(x+y)/z_1(x) : x \ge 0, 0 \le y \le h\} = 1\,. \tag{3.26}$$

Then, for $z(x) = z_1(x)z_2(x)$ and for each nonlattice distribution F on $\mathbb{R}_+$,

$$g(u) = z(u) + \int_0^u g(u-v)\,\mathrm{d}F(v)\,, \qquad u \ge 0\,, \tag{3.27}$$

admits a unique locally bounded solution $g(u)$ such that

$$\lim_{u\to\infty} g(u) = \begin{cases} \mu_F^{-1}\displaystyle\int_0^\infty z(u)\,\mathrm{d}u & \text{if } \mu_F < \infty, \\ 0 & \text{if } \mu_F = \infty. \end{cases} \tag{3.28}$$

Note that Lemma 3.3 is a version of the so-called *key renewal theorem.* Furthermore, (3.27) is called a *renewal equation.*

Theorem 3.4 *Assume that the adjustment coefficient $\gamma > 0$ exists. Then*

$$\lim_{u\to\infty} \psi(u)\mathrm{e}^{\gamma u} = \begin{cases} \dfrac{\beta-\lambda\mu}{\lambda\hat{m}_U^{(1)}(\gamma)-\beta} & \text{if } \hat{m}_U^{(1)}(\gamma) < \infty, \\ 0 & \text{if } \hat{m}_U^{(1)}(\gamma) = \infty. \end{cases} \tag{3.29}$$

The asymptotic result obtained in Theorem 3.4 for the ruin probability $\psi(u)$ gives rise to the so-called *Cramér–Lundberg approximation*

$$\psi_{\mathrm{app}}(u) = \frac{\beta-\lambda\mu}{\lambda\hat{m}_U^{(1)}(\gamma)-\beta}\mathrm{e}^{-\gamma u}. \tag{3.30}$$

The following numerical investigation shows that the above approximation works quite well even for small values of u.

Example Let $\beta = \lambda = 1$ and $F_U(x) = 1 - \frac{1}{3}(\mathrm{e}^{-x} + \mathrm{e}^{-2x} + \mathrm{e}^{-3x})$. In this example we use the expected inter-occurrence time as the time unit and the premium per unit time as the monetary unit. The mean value of claim sizes is $\mu = 0.611111$, i.e. the net profit condition (2.5) is fulfilled. Furthermore, computing the Laplace transform $\hat{L}_\psi(s)$ and inverting it, we get

$$\psi(u) = 0.550790\mathrm{e}^{-0.485131u} + 0.0436979\mathrm{e}^{-1.72235u} + 0.0166231\mathrm{e}^{-2.79252u}. \tag{3.31}$$

On the other hand, (3.30) implies that in this case the Cramér–Lundberg approximation to $\psi(u)$ is $\psi_{\mathrm{app}}(u) = 0.550790\mathrm{e}^{-0.485131u}$. By comparison to the exact formula given in (3.31), the accuracy of this approximation can be analysed. Table 3.1 shows the ruin function $\psi(u)$, its Cramér–Lundberg approximation $\psi_{\mathrm{app}}(u)$ and the relative error $(\psi_{\mathrm{app}}(u)-\psi(u))/\psi(u)$ multiplied by 100. Note that the relative error is below 1% for $u \geq 1.71358 = 2.8\mu$.

u	0	0.25	0.5	0.75	1
$\psi(u)$	0.6111	0.5246	0.4547	0.3969	0.3479
$\psi_{\mathrm{app}}(u)$	0.5508	0.4879	0.4322	0.3828	0.3391
Er	-9.87	-6.99	-4.97	-3.54	-2.54
u	1.25	1.5	1.75	2	2.25
$\psi(u)$	0.3059	0.2696	0.2379	0.2102	0.1858
$\psi_{\mathrm{app}}(u)$	0.3003	0.2660	0.2357	0.2087	0.1849
Er	-1.82	-1.32	-0.95	-0.69	-0.50

Table 3.1: Cramér–Lundberg approximation to ruin probabilities

Remark In the case of exponentially distributed claim sizes, the constant on the right-hand side of (3.29) is $(\beta\delta)^{-1}\lambda$. Thus the Cramér–Lundberg approximation (3.29) becomes exact in this case. Vice versa, assume that the Cramér–Lundberg approximation is exact, i.e. there exists a constant $c \ge 0$ such that $\psi(u) = c\mathrm{e}^{-\gamma u}$ for all $u \ge 0$. Then from (2.13) we have

$$\frac{\beta - \lambda\mu}{\beta s - \lambda(1 - \hat{l}_U(s))} = \frac{1}{s} - \frac{c}{\gamma + s}\,.$$

A rearrangement of the terms in this equation yields

$$\begin{aligned} \hat{l}_U(s) &= 1 - \frac{\beta s(\gamma + s - cs) - s(\gamma + s)(\beta - \lambda\mu)}{\lambda(\gamma + s - cs)} \\ &= 1 + \frac{(\beta c - \lambda\mu)s^2 - \lambda\mu\gamma s}{\lambda\gamma + (\lambda - \lambda c)s}\,. \end{aligned}$$

Since $\lim_{s\to\infty} \hat{l}_U(s) = 0$, we find that $c = \lambda\mu(\beta)^{-1}$ and $\gamma = \mu^{-1} - \lambda\beta^{-1}$. Thus the claim sizes must be exponentially distributed.

3.3 Subexponential Claim Sizes

In Section 3.2 we found the asymptotic behaviour of the ruin function $\psi(u)$ when the initial risk reserve u tends to infinity. However our result was limited to claim sizes for which the tail of the distribution function decreases exponentially fast. For many applications such an assumption is unrealistic. For instance, data from motor third liability insurance, fire insurance or catastrophe insurance (earthquakes, flooding etc.) clearly show heavy tail behaviour. In particular, Pareto, lognormal and loggamma distributions are popular in actuarial mathematics.

In Section II.2, we have shown that several families of heavy-tailed claim size distributions belong to the class of subexponential distributions. It turns out (see Section II.2.4) that also their integrated tail distributions are subexponential. Note that in such a case the Pollaczek–Khinchin formula (2.16) implies that the ruin function $\psi(u)$ decreases more slowly than any exponential function. Indeed, by (II.2.2) and (2.16), we have for $u \to \infty$

$$\psi(u)\mathrm{e}^{su} \ge (1 - \lambda\mu\beta^{-1})\lambda\mu\beta^{-1}\overline{F^{\mathrm{s}}_U}(u)\mathrm{e}^{su} \longrightarrow \infty$$

for all $s > 0$. This simple result indicates that, in the case of heavy-tailed claim sizes, the asymptotic behaviour of $\psi(u)$ is very different from that in Theorem 3.4. If the integrated tail distribution F^{s}_U is subexponential, then we have the following result.

Theorem 3.5 *Let $\rho = \lambda\mu\beta^{-1}$ and assume that $F_U^{\mathrm{s}} \in \mathcal{S}$. Then*

$$\lim_{u\to\infty} \frac{\psi(u)}{1 - F_U^s(u)} = \frac{\rho}{1-\rho}\,. \tag{3.32}$$

The above theorem suggests the approximation

$$\psi_{\mathrm{app}}(u) = \frac{\rho}{1-\rho}(1 - F_U^s(u))\,. \tag{3.33}$$

Note that the quantity ρ captures all the information on the claim number process one needs to know.

Examples 1. Assume that the claim sizes are $\mathrm{Par}(\alpha, c)$ distributed. In order to have a finite mean (which is necessary by the net profit condition (2.5)) we must have $\alpha > 1$. The integrated tail distribution F_U^{s} is readily obtained as

$$F_U^{\mathrm{s}}(x) = \begin{cases} (\alpha-1)x/\alpha c & \text{if } x \le c, \\ 1 - \alpha^{-1}\,(x/c)^{-(\alpha-1)} & \text{if } x > c. \end{cases}$$

By Theorem II.2.15, F_U^{s} is subexponential. Thus, Theorem 3.5 leads to the following approximation to the ruin probability $\psi(u)$:

$$\psi_{\mathrm{app}}(u) = \frac{\rho}{\alpha(1-\rho)}\left(\frac{u}{c}\right)^{-(\alpha-1)}$$

for $u > c$. Details are left to the reader.

2. Let $\beta = 1$, $\lambda = 9$ and $F_U(x) = 1 - (1+x)^{-11}$, where we use the premium as the monetary unit. The integrated tail distribution F_U^{s} is readily obtained: $F_U^{\mathrm{s}}(x) = 1 - (1+x)^{-10}$. From Theorem II.2.15 we conclude that both $F_U \in \mathcal{S}$ and $F_U^{\mathrm{s}} \in \mathcal{S}$. Approximation (3.33) then reads $\psi_{\mathrm{app}}(u) = 9(1+u)^{-10}$. Table 3.2 gives some values of $\psi(u)$ and of the approximation $\psi_{\mathrm{app}}(u) = 9(1+u)^{-10}$ as well as 100 times the relative error. The "exact values" of $\psi(u)$ were calculated using a Panjer's algorithm. In order to get a discrete approximation to the claim size distribution, this distribution was discretized with bandwidth $h = 10^{-3}$, i.e. $q_k = \mathbb{P}(k/1000 \le U < (k+1)/1000)$. Consider for instance the initial risk reserve $u = 20$. Then the ruin probability $\psi(u)$ is 1.75×10^{-8}, which is so small that it is not interesting for practical purposes. However, the approximation error is still almost 100%. Thus, in the case of heavy-tailed claim sizes, the approximation (3.33) can be poor, even for large values of u.

Note that (2.19) and (3.33) imply that for u (very) large the ruin probability $\psi(u)$ is $(\beta-\lambda\mu)^{-1}\lambda\mu$ times the probability that the first ladder height

u	$\psi(u)$	$\psi_{\text{app}}(u)$	Er
1	0.364	8.79×10^{-3}	-97.588
2	0.150	1.52×10^{-4}	-99.898
3	6.18×10^{-2}	8.58×10^{-6}	-99.986
4	2.55×10^{-2}	9.22×10^{-7}	-99.996
5	1.05×10^{-2}	1.49×10^{-7}	-99.999
10	1.24×10^{-4}	3.47×10^{-10}	-100
20	1.75×10^{-8}	5.40×10^{-13}	-99.997
30	2.50×10^{-12}	1.10×10^{-14}	-99.56
40	1.60×10^{-15}	6.71×10^{-16}	-58.17
50	1.21×10^{-16}	7.56×10^{-17}	-37.69

Table 3.2: Approximation to ruin probabilities for subexponential claims

of the random walk $\{S_n\}$ considered in (2.4) exceeds u. But $(\beta-\lambda\mu)^{-1}\lambda\mu$ is the expected number of ladder epochs of $\{S_n\}$. Intuitively this means that, for u large, the ruin will occur if one of the ladder heights is larger than u.

3.4 Ordering of Ruin Functions

We compare the ruin functions $\psi(u)$ and $\psi'(u)$ of two compound Poisson models with arrival rates λ and λ', premium rates β and β', and claim size distributions F_U and $F_{U'}$, respectively. If we suppose that

$$\lambda \le \lambda', \quad \mu_U \le \mu_{U'}, \quad \beta \ge \beta' \tag{3.34}$$

and

$$F_U^{\mathrm{s}} \le_{\mathrm{st}} F_{U'}^{\mathrm{s}}, \tag{3.35}$$

then we immediately get $\psi(u) \le \psi'(u)$ for all $u \ge 0$. It turns out that (3.35) can be replaced by a slightly weaker condition.

Theorem 3.6 *If $\lambda \le \lambda'$ and $\beta \ge \beta'$ and if $U \le_{\mathrm{sl}} U'$, then $\psi(u) \le \psi'(u)$ for all $u \ge 0$.*

Corollary 3.7 *If F_U is* NBUE*, then for all $u \ge 0$*

$$\psi(u) \le \frac{\lambda\mu_U}{\beta}\,\mathrm{e}^{-(\mu_U^{-1}-\lambda\beta^{-1})u}\,. \tag{3.36}$$

Similarly, the reversed inequality in (3.36) *is true if F_U is* NWUE.

Bibliographical comments. One-sided bounds of the type $\psi(u) \leq \mathrm{e}^{-\gamma u}$ as well as asymptotic relations $\psi(u) \sim c\mathrm{e}^{-\gamma u}$ for large u-values have been studied by Filip Lundberg (1926,1932,1934). The modern approach to these estimations is due to Cramér (1955). By means of martingale techniques one-sided inequalities were also derived in Gerber (1973) and Kingman (1964) in the settings of risk and queueing theories, respectively. In Taylor (1976), two-sided bounds of the form (3.23) were obtained for the ruin function $\psi(u)$. The renewal approach to Theorem 3.4 is due to Feller (1971).

Chapter IV

Sparre Andersen Model

1 Random Walks

We turn to the discussion of some basic properties of random walks on the real line $\mathbb{R}$. These processes are useful when computing ruin probabilities in the case where premiums are random or when extending bounds and asymptotic results as in Section III.3 to the case of general inter-occurrence times.

Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed (not necessarily integer-valued) random variables with distribution F which can take both positive and negative values. The sequence $\{S_n,\ n \in \mathbb{N}\}$ with $S_0 = 0$ and $S_n = Y_1 + \ldots + Y_n$ for $n = 1, 2, \ldots$ is called a *random walk*. We assume that the first moment $\mathbb{E}\,Y$ exists and that Y is not concentrated at 0, i.e. $\mathbb{P}(Y = 0) < 1$.

1.1 Ladder Epochs

Look at the first entrance time of the random walk $\{S_n\}$ into the positive half-line $(0, \infty)$

$$\nu^+ = \min\{n > 0 :\ S_n > 0\}\,, \tag{1.1}$$

setting $\nu^+ = \infty$ if $S_n \le 0$ for all $n \in \mathbb{N}$, and call ν^+ the (first strong) *ascending ladder epoch* of $\{S_n\}$. Similarly we introduce the first entrance time to the nonpositive half-line $(-\infty, 0]$ by

$$\nu^- = \min\{n > 0 :\ S_n \le 0\}\,, \tag{1.2}$$

setting $\nu^- = \infty$ if $S_n > 0$ for all $n = 1, 2, \ldots$, and call ν^- the (first) *descending ladder epoch* of $\{S_n\}$. As we will see later, we need to know whether $\mathbb{E}\,Y$ is strictly positive, zero or strictly negative, as otherwise we cannot say whether ν^+ or ν^- are proper. In Figures 1.1 and 1.2 we depict the first ladder epochs ν^+ and ν^-. For each $k = 1, 2, \ldots$, the events

$$\{\nu^+ = k\} = \{S_1 \le 0, S_2 \le 0, \ldots, S_{k-1} \le 0, S_k > 0\} \tag{1.3}$$

and

$$\{\nu^- = k\} = \{S_1 > 0, S_2 > 0, \ldots, S_{k-1} > 0, S_k \le 0\} \tag{1.4}$$

are determined by the first k values of $\{S_n\}$. Note that this is a special case of the following, somewhat more general, property. Consider the σ-algebras $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \{\{\omega : (S_1(\omega), \ldots, S_k(\omega)) \in B\}, B \in \mathcal{B}(\mathbb{R}^k)\}$. Then, in view of (1.3) and (1.4), we have $\{\nu^+ = k\} \in \mathcal{F}_k$ and $\{\nu^- = k\} \in \mathcal{F}_k$ for $k \in \mathbb{N}$. This means that the ladder epochs ν^+ and ν^- are so-called *stopping times* with respect to the *filtration* $\{\mathcal{F}_n\}$ generated by $\{S_n\}$. From Wald identity for we have that, for each stopping time τ with respect to $\{\mathcal{F}_n\}$,

$$\mathbb{E}\,S_\tau = \mathbb{E}\,\tau \mathbb{E}\,Y \tag{1.5}$$

provided that $\mathbb{E}\,\tau < \infty$ and $\mathbb{E}\,|Y| < \infty$.

Actually, we can recursively define further ladder epochs. Define the sequence $\{\nu_n^+, n \in \mathbb{N}\}$ by

$$\nu_{n+1}^+ = \min\{j > \nu_n^+ : S_j > S_{\nu_n^+}\}, \tag{1.6}$$

where $\nu_0^+ = 0$ and $\nu_1^+ = \nu^+$and call ν_n^+ the n-th (strong ascending) *ladder epoch.* A priori, we cannot exclude the case that, from some random index on, all the ladder epochs are equal to ∞.

In a similar way, we recursively define the sequence $\{\nu_n^-, n \in \mathbb{N}\}$ of consecutive *descending ladder epochs* by $\nu_0^- = 0$, $\nu_1^- = \nu^-$ and

$$\nu_{n+1}^- = \min\{j > \nu_n^-, S_j \le S_{\nu_n^-}\}, \qquad n = 1, 2, \ldots. \tag{1.7}$$

Another interesting characteristic is the step ν at which the random walk $\{S_n\}$ has a local minimum for the last time before ν^-, i.e.

$$\nu = \max\Big\{n : 0 < n < \nu^-, S_n = \min_{0<j<\nu^-} S_j\Big\},$$

as depicted on Figure 1.1.

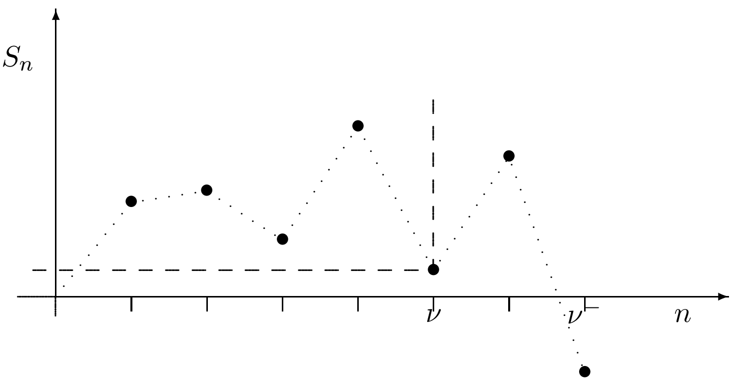

Figure 1.1: Last minimum before "ruin"

1.2 Random Walks with and without Drift

Depending on whether $\mathbb{E}\,Y$ is positive, zero or negative, we have three different kinds of evolution for the random walk $\{S_n\}$.

Theorem 1.1 (a) *If* $\mathbb{E}\,Y > 0$*, then* $\lim_{n\to\infty} S_n = \infty$.
(b) *If* $\mathbb{E}\,Y < 0$*, then* $\lim_{n\to\infty} S_n = -\infty$.
(c) *If* $\mathbb{E}\,Y = 0$*, then* $\limsup_{n\to\infty} S_n = \infty$ *and* $\liminf_{n\to\infty} S_n = -\infty$.

Theorem 1.1 motivates the use of the following terminology. We say that the random walk $\{S_n\}$

- has a *positive drift* provided that $\mathbb{E}\,Y > 0$,
- has a *negative drift* provided that $\mathbb{E}\,Y < 0$,
- is *without drift* or *oscillating* provided that $\mathbb{E}\,Y = 0$.

1.3 Ladder Heights; Negative Drift

In this subsection we assume that the random walk $\{S_n\}$ has a negative drift, i.e. $\mathbb{E}\,Y < 0$. A basic characteristic of $\{S_n\}$ is then the first ascending ladder epoch ν^+. As one can expect, and we confirm this in Theorem 1.2, the distribution of the random variable ν^+ is defective under the assumption of a negative drift. The *overshoot* Y^+ above the zero level is defined by

$$Y^+ = \begin{cases} S_{\nu^+} & \text{if } \nu^+ < \infty, \\ \infty & \text{otherwise} \end{cases}$$

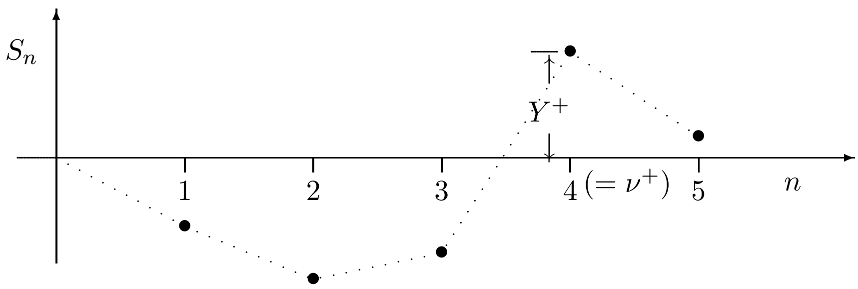

Figure 1.2: Ascending ladder height

and is called the (first strong) *ascending ladder height*. A typical trajectory of the random walk $\{S_n\}$ which reflects this situation is presented in Figure 1.2.

More precisely, we have a result for $G^+(x) = \mathbb{P}(Y^+ \le x)$, the distribution function of Y^+ and $G^+(\infty) = \lim_{x\to\infty} G^+(x)$.

Theorem 1.2 *The following statements are equivalent:*
(a) $\mathbb{E}\,Y < 0$,
(b) M *is finite with probability* 1,
(c) $G^+(\infty) < 1$.

The *proof* of this theorem is easy and is left to the reader.

Suppose that $\nu^+ < \infty$. We can then repeat the same argument as above, but now from the point (ν^+, Y^+), because of our assumption that the increments $Y_1, Y_2, \ldots$ of the random walk $\{S_n\}$ are independent and identically distributed. This means in particular, as illustrated in Figure 1.2, that we can define a new random walk $S_{\nu^++1} - S_{\nu^+}, S_{\nu^++2} - S_{\nu^+}, \ldots$ which can be proved to be an identically distributed copy of the original random walk $\{S_n\}$ and independent of $S_1, S_2, \ldots, S_{\nu^+}$. We leave it to the reader to show this. Iterating this procedure, we can recursively define the sequence $\{\nu_n^+\}$ of consecutive ladder epochs in the same way as this was done in (1.6). The random variable

$$Y_n^+ = \begin{cases} S_{\nu_n^+} - S_{\nu_{n-1}^+} & \text{if } \nu_n^+ < \infty, \\ \infty & \text{otherwise} \end{cases}$$

is called the n-th *ascending ladder height* of $\{S_n\}$. It is not difficult to show that the sequence $\{Y_1^+ + \ldots + Y_n^+, n = 0, 1, \ldots\}$ forms a terminating renewal process. Moreover, for the maximum $M = \max\{0, S_1, S_2, \ldots\}$ of $\{S_n\}$ we have (see also Figure 1.2)

$$M = \sum_{i=1}^{N} Y_i^+, \tag{1.8}$$

where $N = \max\{n : \nu_n^+ < \infty\}$ is the number of finite ladder epochs. Thus, with the notation $G_0(x) = G^+(x)/G^+(\infty)$, where $G_0(x)$ is a proper (i.e. non-defective) distribution function, we arrive at the following result, saying that M has a compound geometric distribution.

Theorem 1.3 *If* $\mathbb{E}\, Y < 0$*, then for all* $x \ge 0$ *and for* $p = G^+(\infty)$

$$\mathbb{P}(M \le x) = (1-p)\sum_{k=0}^{\infty}(G^+)^{*k}(x) = \sum_{k=0}^{\infty}(1-p)p^k G_0^{*k}(x)\,. \tag{1.9}$$

Theorem 1.3 implies the following result for the ruin function $\psi(u) = \mathbb{P}(M > u)$ considered in Section [**??sim.tim.dep??**].

Corollary 1.4 *For any* $u \ge 0$*,* $\psi(u) = \sum_{k=1}^{\infty}(1-p)p^k\overline{G_0^{*k}}(u)$.

We now introduce the dual notions of descending ladder heights. Consider the descending ladder epoch ν^-. The *undershoot* Y^- below the zero level is defined by $Y^- = S_{\nu^-}$ and called the (first) *descending ladder height.* The n-th descending ladder height is defined by $Y_n^- = S_{\nu_n^-} - S_{\nu_{n-1}^-}$. Since $Y_1^-, \ldots, Y_n^-$ are independent and identically distributed copies of Y^-, it is clear that the sequence $\{-\sum_{i=1}^n Y_i^-,\ n \in \mathbb{N}\}$ is a nonterminating renewal process (in the case of the negative drift). Indeed, under our assumption on the negative drift it follows from Theorem 1.1 that all descending ladder epochs and heights are proper random variables.

Bibliographical comments. The basic references for Section 1 are Feller (1971) and Chung (1974).

2 The Wiener–Hopf Factorization

2.1 General Representation Formulae

Define the ladder height distribution G^-, concentrated on $\mathbb{R}_-$, by

$$G^-(x) = \mathbb{P}(Y^- \le x)\,, \qquad x \in \mathbb{R}\,. \tag{2.10}$$

Thus G^- dualizes the ladder height distribution G^+ which is concentrated on $(0, \infty)$ and is given by

$$G^+(x) = \mathbb{P}(Y^+ \le x)\,, \qquad x \in \mathbb{R}\,. \tag{2.11}$$

Let H_0^- be the measure on $\mathbb{R}_-$ given by

$$H_0^-(B) = \sum_{k=0}^{\infty} (G^-)^{*k}(B)\,, \qquad B \in \mathcal{B}(\mathbb{R}_-)\,. \tag{2.12}$$

We also introduce as a dual measure H_0^+ on $\mathbb{R}_+$

$$H_0^+(B) = \sum_{k=0}^{\infty} (G^+)^{*k}(B)\,, \qquad B \in \mathcal{B}(\mathbb{R}_+)\,. \tag{2.13}$$

From (2.12) it follows that

$$H_0^- * G^- = H_0^- - \delta_0\,. \tag{2.14}$$

It turns out that H_0^- is equal to the so-called *pre-occupation measure* γ^- given by $\gamma^-(B) = \mathbb{E}\left(\sum_{i=0}^{\nu^+-1} \mathbf{1}(S_i \in B)\right)$ for $B \in \mathcal{B}(\mathbb{R})$, where obviously $\gamma^-(B) = 0$ for $B \subset (0, \infty)$.

Lemma 2.1 *For each $B \in \mathcal{B}(\mathbb{R})$ and $H_0^-(B) = H_0^-(B \cap \mathbb{R}_-)$ we have $H_0^-(B) = \gamma^-(B)$.*

Next we show that the distribution F of the increments $Y_1, Y_2, \ldots$ of the random walk $\{S_n\}$ can be expressed in terms of the ladder height distributions G^+ and G^-. This is the so-called *Wiener–Hopf factorization* of F, which is sometimes useful when computing the distribution of the maximum M of the random walk $\{S_n\}$.

Theorem 2.2 *The following relationship holds:*

$$F = G^+ + G^- - G^- * G^+. \tag{2.15}$$

If we want to compute ruin probabilities, we need to determine the ladder height distribution G^+ that appears in Theorem 1.3. The Wiener–Hopf factorization (2.15) yields the following representation formula for G^+.

Corollary 2.3 *For $B \in \mathcal{B}((0, \infty))$,*

$$G^+(B) = F * H_0^-(B) = \int_{-\infty}^{0} F(B - y)\, \mathrm{d}H_0^-(y)\,, \tag{2.16}$$

while for $B \in \mathcal{B}(\mathbb{R}_-)$

$$G^-(B) = F * H_0^+(B) = \int_{0}^{\infty} F(B - y)\, \mathrm{d}H_0^+(y)\,. \tag{2.17}$$

2.2 Ladder Height Distributions

In Theorem 1.3 we showed that the probability of ruin $\psi(u) = \mathbb{P}(\tau(u) < \infty)$ is closely related to the ladder height distribution G^+ of the random walk $\{S_n\}$ with $S_n = \sum_{i=1}^n (U_i - \beta T_i)$. We now compute G^+, $p = G^+(\infty)$ and $G_0(x) = G^+(x)/G^+(\infty)$ for two further cases, i.e. the compound Poisson model with general claim size distribution, and the Sparre Andersen model with exponentially distributed claim sizes. We again assume that the drift of the random walk $\{S_n\}$ is negative, or equivalently that $\mathbb{E}\,U - \beta\mathbb{E}\,T < 0$.

We start with the compound Poisson model. We first prove a lemma of independent interest, which gives a simple expression for the pre-occupation measure $\gamma^- = H_0^-$ introduced in Section 2.1.

Lemma 2.4 *For the compound Poisson model,*

$$H_0^-((-x,0]) = 1 + \lambda\beta^{-1}x\,, \qquad x > 0\,, \tag{2.18}$$

or, alternatively, $\mathrm{d}H_0^-(x) = \mathrm{d}\delta_0(x) + \lambda\beta^{-1}\,\mathrm{d}x$.

Next, we derive an expression for the tail function $\overline{G^+}(x) = G^+(\infty) - G^+(x)$. It turns out that, in the compound Poisson model, the conditional ladder height distribution G_0 coincides with the integrated tail distribution F_U^{s} of claim sizes.

Theorem 2.5 *For the compound Poisson model,*

$$\overline{G^+}(x) = \lambda\beta^{-1}\int_x^\infty \overline{F}_U(v)\,\mathrm{d}v\,, \qquad x \ge 0\,. \tag{2.19}$$

Hence

$$p = \lambda\mu_U\beta^{-1} \tag{2.20}$$

and

$$\overline{G}_0(x) = \mu_U^{-1}\int_x^\infty \overline{F}_U(v)\,\mathrm{d}v\,, \qquad x \ge 0\,. \tag{2.21}$$

We turn to the Sparre Andersen model with general inter-occurrence time distribution but with exponentially distributed claim sizes. In particular, we derive the ladder height distribution G^+ and the probability p that the first ascending ladder epoch ν^+ is finite.

Theorem 2.6 *If the claim size distribution F_U is exponential with parameter $\delta > 0$, then G_0 is exponential with the same parameter δ and $\delta(1-p)$ is the unique positive root of*

$$\hat{m}_Y(s) = \frac{\delta}{\delta - s}\hat{l}_T(\beta s) = 1\,. \tag{2.22}$$

The following result is an obvious consequence of Theorems 2.5 and 2.6.

Corollary 2.7 *Consider the compound Poisson model with intensity* λ *and exponential claim size distribution* $F_U = \mathrm{Exp}(\delta)$. *Then* $G_0 = \mathrm{Exp}(\delta)$ *and* $p = \lambda(\delta\beta)^{-1}$.

Bibliographical comments. Factorization theorems for random walks appear in many books and articles and in different forms. We refer, for example, to Chung (1974), Feller (1971), Prabhu (1980).

3 Ruin Probabilities: Sparre Andersen Model

3.1 Formulae of Pollaczek–Khinchin Type

Sometimes it is more convenient to consider the claim surplus process $\{S(t)\}$ with $S(t) = \sum_{i=1}^{N(t)} U_i - \beta t$ for $t \geq 0$ instead of the risk reserve process $\{R(t)\}$. The ruin function $\psi(u)$ is then given by $\psi(u) = \mathbb{P}(\tau(u) < \infty)$, where $\tau(u) = \min\{t : S(t) > u\}$ is the time of ruin for the initial risk reserve u. As already stated, a fundamental question of risk theory is how to derive pleasing formulae for $\psi(u)$. However, most often this is impossible, as formulae turn out to be too complicated. As a result, various approximations are considered. From random walk theory, applied to the independent increments $Y_n = U_n - \beta T_n$, we already know that there is only one case that is interesting, namely when the coefficient $\rho = (\lambda \mathbb{E}\, U)/\beta$ is less than 1, as otherwise $\psi(u) \equiv 1$ (see Theorem 1.1). If $\rho < 1$, then the drift $\mathbb{E}\, U - \beta \mathbb{E}\, T$ of the random walk $\{S_n\}$ with $S_n = Y_1 + \ldots + Y_n$ is negative. In risk theory it is customary to express this condition in terms of the *relative safety loading* η, which is defined as

$$\eta = \frac{\beta \mathbb{E}\, T - \mathbb{E}\, U}{\mathbb{E}\, U} = \frac{1}{\rho} - 1 \; . \tag{3.23}$$

Obviously, $\eta > 0$ if and only if $\rho < 1$. The concept of relative safety loading comes from the following considerations. Consider a risk reserve process in the compound Poisson model,

$$R(t) = u + \lambda \mathbb{E}\, U\, t - \sum_{n=1}^{N(t)} U_n \,, \qquad t \geq 0 \,,$$

where the premium rate $\beta = \lambda \mathbb{E}\, U$ is computed by the net premium principle. From random walk theory, we already know that the risk reserve process without drift will have unbounded large fluctuations as time goes on, and so ruin happens with probability 1. If we add a safety loading $\varepsilon\lambda\mathbb{E}\,U$ for some $\varepsilon > 0$, then ruin in the risk reserve process $\{R(t)\}$ with

$$R(t) = u + (1+\varepsilon)\lambda\mathbb{E}\,U\,t - \sum_{n=1}^{N(t)} U_n\,, \qquad t \geq 0\,,$$

will no longer occur with probability 1. Solving equation (3.23) for $\beta = (1+\varepsilon)\lambda\mathbb{E}\,U$, we have the relative safety loading $\eta = \varepsilon$.

In the sequel to this chapter, we always assume that $0 < \mathbb{E}\,T < \infty$, $0 < \mathbb{E}\,U < \infty$ and that the relative safety loading η is positive so that $\mathbb{E}\,U - \beta\mathbb{E}\,T < 0$. We know from Section 1.3 that the survival probability $1-\psi(u)$ is given by the following formula of Pollaczek–Khinchin type.

Theorem 3.1 *For all $u \geq 0$,*

$$1-\psi(u) = (1-p)\sum_{k=0}^{\infty}(G^+)^{*k}(u) = \sum_{k=0}^{\infty}(1-p)p^k G_0^{*k}(u)\,, \tag{3.24}$$

where G^+ is the (defective) distribution of the ladder height of the random walk $\{S_n\}$; $S_n = \sum_{i=1}^{n}(U_i - \beta T_i)$, $p = G^+(\infty)$ and $G_0(u) = G^+(u)/G^+(\infty)$.

Note that (3.24) implies

$$\psi(u) = (1-p)\sum_{k=1}^{\infty} p^k\,\overline{G_0^{*k}}(u)\,, \qquad u \geq 0\,. \tag{3.25}$$

After some simple algebraic manipulations, this reads

$$\psi(u) = \sum_{k=0}^{\infty} p^{k+1}\int_0^u \overline{G}_0(u-v)\,\mathrm{d}G_0^{*k}(v)\,, \qquad u \geq 0\,. \tag{3.26}$$

In the case of a compound Poisson model we know from Theorem 2.5 that G_0 is equal to the integrated tail distribution F_U^{s} of claim sizes. We rediscover the classical Pollaczek–Khinchin formula for the ruin probability $\psi(u)$ from Theorem III.2.4.

Corollary 3.2 *The ruin function in the compound Poisson model is*

$$\psi(u) = \sum_{k=1}^{\infty}(1-\rho)\rho^k\,\overline{(F_U^{\mathrm{s}})^{*k}}(u), \tag{3.27}$$

which is the same as

$$\psi(u) = \sum_{k=0}^{\infty} \rho^{k+1} \int_0^u \overline{F_U^{\mathrm{s}}}(u-v)\,\mathrm{d}(F_U^{\mathrm{s}})^{*k}(v)\,. \tag{3.28}$$

The *proof* is immediate as it suffices to insert (2.20) and (2.21) into (3.25). In the same way (3.28) follows from (3.26).

Corollary 3.3 *In the Sparre Andersen model with exponential claim size distribution* Exp(δ)*,*

$$\psi(u) = (1-\gamma/\delta)\,\mathrm{e}^{-\gamma u} \tag{3.29}$$

for all $u \geq 0$, where γ is the unique positive root of (2.22)*.*

Corollaries 3.2 and 3.3 yield the following result, which coincides with (III.2.10).

Corollary 3.4 *In the compound Poisson model with exponential claim size distribution* Exp(δ)*,*

$$\psi(u) = \frac{\lambda}{\beta\delta}\,\mathrm{e}^{-(\delta-\lambda/\beta)u}\,, \qquad u \geq 0\,. \tag{3.30}$$

We now determine the joint distribution of $(X^+(u), Y^+(u))$, where $X^+(u)$ is the surplus just before ruin time $\tau(u)$ and $Y^+(u)$ is the severity of ruin. More generally, we consider the multivariate ruin function

$$\psi(u,x,y) = \mathbb{P}(\tau(u) < \infty, X^+(u) \leq x, Y^+(u) > y)\,, \tag{3.31}$$

where $u, x, y \geq 0$. We derive a representation formula for $\psi(u,x,y)$, which generalizes the representation formula (3.24) for the (univariate) ruin function $\psi(u)$ and expresses $\psi(u,x,y)$ in terms of p, G_0 and $\psi(0,x,y)$. Here, $\psi(0,x,y)$ is obtained from the distribution of $(X^+(0), Y^+(0))$. Recall the pre-occupation measure $\gamma^- = H_0^+ = \sum_{k=0}^{\infty}(G^+)^{*k} = \sum_{k=0}^{\infty} p^k {G_0}^{*k}$ introduced in Section 2.1.

Theorem 3.5 *The multivariate ruin function $\psi(u,x,y)$ satisfies the integral equation*

$$\psi(u,x,y) = \psi(0,x-u,y+u) + p\int_0^u \psi(u-v,x,y)\,\mathrm{d}G_0(v) \tag{3.32}$$

for all $u,x,y \geq 0$; its solution is

$$\psi(u,x,y) = \int_0^u \psi(0,x-u+v,y+u-v)\,\mathrm{d}H_0^+(v)\,. \tag{3.33}$$

Corollary 3.6 *For all $u, y \geq 0$,*

$$\psi(u,\infty,y) = p\int_0^u \overline{G}_0(y+u-v)\,\mathrm{d}H_0^+(v)\,. \tag{3.34}$$

Note that formulae (3.33) and (3.34) are extensions of (3.26), since $\psi(u) = \psi(u,\infty,0)$. Furthermore, recall that in the case of the compound Poisson model, the characteristics p, G_0 and

$$\varphi(0,x,y) = \mathbb{P}(\tau(0)<\infty, X^+(0)>x, Y^+(0)>y) \tag{3.35}$$

can be easily expressed in terms of λ, β and F_U as shown in Sections III.2.4 and 2.2. In particular, we have the representation formula (III.2.19):

$$\varphi(0,x,y) = \lambda\beta^{-1}\int_{x+y}^\infty (1-F_U(v))\,\mathrm{d}v\,, \qquad x,y\geq 0\,. \tag{3.36}$$

Clearly, then $\psi(0,x,y) = \varphi(0,0,y) - \varphi(0,x,y)$ can also be expressed by λ, β and F_U. Using (3.36), from (3.33) we immediately obtain the following formula for $\psi(u,x,y)$.

Theorem 3.7 *In the compound Poisson model,*

$$\begin{aligned}&\psi(u,x,y)\\ &\quad= \rho\int_0^u \big(F_U^{\mathrm{s}}((x-u+v)_+ + (y+u-v)) - F_U^{\mathrm{s}}(y+u-v)\big)\,\mathrm{d}H_0^+(v)\,,\end{aligned} \tag{3.37}$$

for all $u,x,y\geq 0$, where $\rho = \lambda\beta^{-1}\mu_U$ and $H_0^+(v) = \sum_{k=0}^\infty \rho^k F_U^{\mathrm{s}\,*k}(v)$.

Note that the marginal ruin function $\psi(u,\infty,y)$ can be obtained directly from Corollary 3.6 and Theorem 2.5.

Corollary 3.8 *The probability $\psi(u,\infty,y)$ that, in the compound Poisson model, the overshoot $Y^+(u)$ at ruin time $\tau(u)$ exceeds y is given by*

$$\psi(u,\infty,y) = \sum_{k=0}^\infty \rho^{k+1}\int_0^u \overline{F_U^{\mathrm{s}}}(y+u-v)\,\mathrm{d}(F_U^{\mathrm{s}})^{*k}(v)\,. \tag{3.38}$$

The marginal ruin function $\psi(u,\infty,y)$ can also be obtained in the Sparre Andersen model with exponentially distributed claim sizes if one uses Corollary 3.6 and Theorem 2.6.

Corollary 3.9 *In the Sparre Andersen model with exponential claim size distribution* Exp(δ)

$$\psi(u,\infty,y)=\psi(u)\,\mathrm{e}^{-\delta y}=(1-\gamma/\delta)\,\mathrm{e}^{-(\gamma u+\delta y)} \tag{3.39}$$

for all $u,y\ge 0$*, where* γ *is the unique positive root of* (2.22)*.*

In order to determine the probability that, besides the overshoot $Y^+(u)$, the total maximal deficit $Z^+(u)$ after time $\tau(u)$ exceeds level z we define for $u,x,y,z\ge 0$:

$$\psi(u,x,y,z)=\mathbb{P}(\tau(u)<\infty, X^+(u)\le x, Y^+(u)>y, Z^+(u)>z)\,.$$

Clearly, for $y\ge z$ we have $\psi(u,x,y,z)=\psi(u,x,y,y)=\psi(u,x,y)$. Using the same argument as in the proof of Theorem 3.5 we get the following defective renewal equation for $\psi(u,x,y,z)$. For all $u,x,y,z\ge 0$, we have

$$\psi(u,x,y,z)=\psi(0,x-u,y+u,z+u)+p\int_0^u \psi(u-v,x,y,z)\,\mathrm{d}G_0(v)\,. \tag{3.40}$$

Hence, using a result from the renwal theory,

$$\psi(u,x,y,z)=\int_0^u \psi(0,x-u+v,y+u-v,z+u-v)\,\mathrm{d}H_0^+(v) \tag{3.41}$$

for all $u,x,y,z\ge 0$, where here and below $H_0^+(v)=\sum_{k=0}^\infty p^k G_0^{*k}(v)$. Moreover, since

$$\psi(0,\infty,y,z)=p\Big(\int_y^{\max\{y,z\}} \psi(z-v')\,\mathrm{d}G_0(v')+\overline{G}_0(\max\{y,z\})\Big)\,,$$

(3.41) yields the following extension to (3.34). For all $u,y,z\ge 0$

$$\begin{aligned}\psi(u,\infty,y,z) \;&=\; p\int_0^u\int_{y+u-v}^{\max\{y,z\}+u-v} \psi(z+u-v-v')\,\mathrm{d}G_0(v')\\ &\qquad\times\overline{G}_0(\max\{y,z\}+u-v)\,\mathrm{d}H_0^+(v)\,.\end{aligned}$$

3.2 Lundberg Bounds

In Theorem 3.10 below, we extend the result of Theorem III.3.2 and derive a two-sided Lundberg bound for the ruin function $\psi(u)$ in the Sparre Andersen model with general distributions of inter-occurrence times and claim sizes. Consider the equation

$$\hat{m}_Y(s)=\hat{m}_U(s)\hat{l}_T(\beta s)=1\,. \tag{3.42}$$

Clearly, $\hat{m}_Y(0) = 1$. This equation may have a second root. If such a root $s \neq 0$ exists, then it is unique and strictly positive. The solution to (3.42), if it exists, is called the *adjustment coefficient* and is denoted by γ. For the compound Poisson model, the solutions to (III.3.22) and (3.42) coincide. As we will see in the next Section 3.3, the adjustment coefficient γ in Theorem 3.10 satisfies $\int_0^\infty \mathrm{e}^{\gamma x}\,\mathrm{d}G_0(x) = p^{-1}$. Let $x_0 = \sup\{x : F_Y(x) < 1\}$.

Theorem 3.10 *Suppose that there exists a positive solution γ to* (3.42). *Then*

$$b_- \,\mathrm{e}^{-\gamma u} \le \psi(u) \le b_+ \,\mathrm{e}^{-\gamma u} \tag{3.43}$$

for all $u \ge 0$, where

$$b_- = \inf_{x\in[0,x_0)} \frac{\mathrm{e}^{\gamma x}\overline{F}_Y(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\,\mathrm{d}F_Y(y)}\,, \qquad b_+ = \sup_{x\in[0,x_0)} \frac{\mathrm{e}^{\gamma x}\overline{F}_Y(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\,\mathrm{d}F_Y(y)}\,. \tag{3.44}$$

A somewhat weaker though probably more useful bound is obtained if we express the prefactors in the two-sided Lundberg inequality (3.43) via the claim size distribution F_U. Thus we define further constants b_-^*, b_+^* by

$$b_-^* = \inf_{x\in[0,x_0')} \frac{\mathrm{e}^{\gamma x}\overline{F}_U(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\,\mathrm{d}F_U(y)}\,, \qquad b_+^* = \sup_{x\in[0,x_0')} \frac{\mathrm{e}^{\gamma x}\overline{F}_U(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\,\mathrm{d}F_U(y)}\,, \tag{3.45}$$

where γ is the solution to (3.42) and $x_0' = \sup\{x : F_U(x) < 1\}$. Note that

$$\frac{1}{b_-} = \sup_x \mathbb{E}\,(\mathrm{e}^{\gamma(Y-x)} \mid Y > x), \quad \frac{1}{b_-^*} = \sup_x \mathbb{E}\,(\mathrm{e}^{\gamma(U-x)} \mid U > x) \tag{3.46}$$

and that $(b_+)^{-1}$ and $(b_+^*)^{-1}$ can be expressed in a similar way.

Theorem 3.11 *The constants b_-^*, b_-, b_+, b_+^* defined in* (3.44) *and* (3.45), *respectively, satisfy $0 \le b_-^* \le b_- \le b_+ \le b_+^* \le 1$.*

Corollary 3.12 *Suppose that* (3.42) *has a positive solution and F_U is* IHR. *Then $0 < b_-^* \le b_-$.*

3.3 The Cramér–Lundberg Approximation

In this section we assume that the distribution F of Y is nonlattice and $\mathbb{E}\,Y < 0$. The reader should prove that then the ladder height distribution G^+ corresponding to F is nonlattice too. Furthermore, we assume that (3.42) has a positive solution γ. The following theorem deals with the asymptotic behaviour of $\psi(u)$ as u becomes unbounded large. It extends Theorem III.3.4 of Section III.3.2 from the compound Poisson model to the Sparre Andersen model.

Theorem 3.13 *For the Sparre Andersen model,*

$$\lim_{u\to\infty} \mathrm{e}^{\gamma u}\psi(u) = c \tag{3.47}$$

where the constant $c \geq 0$ is finite and given by

$$c = \frac{1 - G^+(\infty)}{\gamma \int_0^\infty v\mathrm{e}^{\gamma v}\,\mathrm{d}G^+(v)}\ . \tag{3.48}$$

Note that if $c > 0$ the asymptotic result obtained in Theorem 3.13 gives rise to the *Cramér–Lundberg approximation* $\psi_{\mathrm{app}}(u) = c\mathrm{e}^{-\gamma u}$ to the ruin function $\psi(u)$ when u is large.

Remark The constant c in Theorem 3.13 is positive if $\int_0^\infty v\mathrm{e}^{\gamma v}\,\mathrm{d}G^+(v) < \infty$. This condition holds if, for example, $\hat{m}_F(s) < \infty$ for $s < \gamma + \varepsilon$ for some $\varepsilon > 0$. Then $\hat{m}_F(s)$ is continuously differentiable in the interval $0 < s < \gamma + \varepsilon$ and hence from the Wiener–Hopf identity the same property holds for $\hat{m}_{G^+}(s)$. Consequently, $\hat{m}^{(1)}_{G^+}(\gamma) = \int_0^\infty v\mathrm{e}^{\gamma v}\,\mathrm{d}G^+(v) < \infty$.

3.4 Compound Poisson Model with Aggregate Claims

In the compound Poisson modelruin could occur anytime whenever the risk reserve became negative. What happens if we are only able to inspect the value of the risk reserve at countably many, equally spaced time epochs $t = h, 2h, \ldots$ for some $h > 0$? To specify the problem, we consider the risk reserve process $\{R(t)\}$ given by $R(t) = u + \beta t - \sum_{i=1}^{N(t)} U_i = u + \beta t - X(t)$, where $\{X(t)\}$ is the compound Poisson process with the increments $X(t + h) - X(h) = \sum_{i=N(h)+1}^{N(t+h)} U_i$. We now say that ruin occurs if $R(kh) < 0$ for some $k = 1, 2, \ldots$. In terms of the claim surplus process $\{S(t)\}$ with $S(t) = X(t) - \beta t$, this can be written as $S(kh) > u$ for some $k = 1, 2, \ldots$. Since the compound Poisson process $\{X(t)\}$ has independent and stationary increments, the random variables $Y_k(h) = X(kh) - X((k-1)h) - \beta h$, $k = 1, 2, \ldots$, are independent and identically distributed. Hence, ruin occurs if the random walk $\{S(nh),\ n = 0, 1 \ldots\}$ with $S(nh) = \sum_{k=1}^n Y_k(h)$ crosses the level u. We call this model the *compound Poisson model with aggregate claims* as it is closely related to the risk model with discrete time. However, now the aggregate claims do not necessarily take values in $\mathbb{N}$.

Another interpretation of a compound Poisson model with aggregate claims is that of a Sparre Andersen model with constant inter-occurrence times $T_n = h$, premium rate $\beta > 0$ and (individual) claim sizes $U_n(h) = X(nh) - X((n-1)h)$ having a compound Poisson distribution with characteristics $(\lambda h, F_U)$. For the initial reserve u, the ruin probability is then given by $\psi_h(u) = \mathbb{P}(\max_{n\geq 0} S(nh) > u)$, and $\psi_h(u)$, as a function of u, is called the *ruin function* of the compound Poisson model with aggregate claims. Below we derive a Lundberg bound and a Cramér–Lundberg approximation for this model. Note that in these results the adjustment coefficient γ is the same as for the ordinary compound Poisson model.

Theorem 3.14 *In the compound Poisson model with aggregate claims there exist constants* $0 \leq b_-(h) \leq b_+(h) \leq 1$ *such that*

$$b_-(h)\mathrm{e}^{-\gamma u} \leq \psi_h(u) \leq b_+(h)\mathrm{e}^{-\gamma u}\,, \tag{3.49}$$

for all $u \geq 0$, *where the adjustment coefficient* γ *is the positive solution to* (III.3.22) *which is assumed to exist.*

Assume now that the distribution of U is nonlattice and (III.3.22) has a positive solution γ. We next derive a version of the Cramér–Lundberg approximation (3.47) for the compound Poisson model with aggregate claims.

Theorem 3.15 *There exists a positive and finite constant* $c(h)$ *such that*

$$\lim_{u\to\infty} \mathrm{e}^{\gamma u}\psi_h(u) = c(h)\,. \tag{3.50}$$

In general it is difficult to compare the constant $c(h)$ with the constant c that appears in the original Cramér–Lundberg approximation (III.3.29) for the compound Poisson model with permanent (time-continuous) inspection. Nevertheless, the following asymptotic result holds.

Theorem 3.16 *If* $\hat{m}_U(\gamma+\varepsilon) < \infty$ *for some* $\varepsilon > 0$, *then*

$$\lim_{h\to\infty} hc(h) = \frac{c}{\gamma\beta(1-\rho)} = \frac{c}{\eta\gamma\lambda\mu_U}\,, \tag{3.51}$$

where $\eta = \beta(\lambda\mu_U)^{-1} - 1$ *is the relative safety loading of the compound Poisson model with permanent (time-continuous) inspection of risk reserve and* $c = ((1-\rho)\beta)/(\lambda\hat{m}^{(1)}(\gamma) - \beta)$ *is the constant appearing in the original Cramér–Lundberg approximation* (III.3.29).

3.5 Subexponential Claim Sizes

The Cramér–Lundberg approximation studied in Section 3.3 to the probability of ruin is valid for claim sizes having exponentially bounded or light-tailed distribution. To be more precise, the assumption that (3.42) has a positive solution γ means that the moment generating function $\hat{m}_{F_U}(s)$ is finite in a right neighbourhood of $s = 0$. Furthermore, for all $s > 0$ with $\hat{m}_{F_U}(s) < \infty$, the moment generating function $\hat{m}_{F_U^{\mathrm{s}}}(s)$ of the integrated tail distribution F_U^{s} is

$$\hat{m}_{F_U^{\mathrm{s}}}(s) = \frac{\hat{m}_{F_U}(s) - 1}{s\mu_{F_U}} \, .$$

Consequently, $\hat{m}_{F_U^{\mathrm{s}}}(s)$ is finite in the same right neighbourhood of $s = 0$. When modelling large claims, one often uses claim size distributions F_U like the Pareto or the lognormal distribution and that do not have this property. In the present section, we consider the Sparre Andersen model where the integrated tail distribution F_U^{s} of claim sizes belongs to the class $\mathcal{S}$ of subexponential distributions introduced in Section II.2. We will show that the ruin function $\psi(u)$ has then the same asymptotic behaviour as the tail function $\overline{F_U^{\mathrm{s}}}(x)$. See Section II.2.4 for sufficient conditions to have $F_U^{\mathrm{s}} \in \mathcal{S}$, in terms of the hazard rate function of F_U.

For heavy-tailed claim size distributions, the following result is an analogue to the Cramér–Lundberg approximation from Theorem 3.13. It extends Theorem III.3.5 and shows that, for $\mathbb{E}\,U$ fixed, the asymptotics of the ruin function $\psi(u)$ depends on the claim size distribution $F_U(x)$ only through its behaviour for large values of x. Another interesting fact is that, in the case of a heavy-tailed claim size distribution, the asymptotic behaviour of $\psi(u)$ does not depend on the form of the inter-occurrence time distribution but only on its mean $\mathbb{E}\,T$.

Theorem 3.17 *If $F_U^{\mathrm{s}} \in \mathcal{S}$, then*

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F_U^{\mathrm{s}}}(u)} = \frac{\mathbb{E}\,U}{\beta\mathbb{E}\,T - \mathbb{E}\,U} \, . \tag{3.52}$$

The *proof* of Theorem 3.17 will be partitioned into several steps. First we show the following auxiliary result for the integrated tail distribution $F_{Y_+}^{\mathrm{s}}$ of the generic increment $Y_+ = (U - \beta T)_+$. Recall that $Y_+ = \max\{0, U - \beta T\}$ and note that Y_+ is not the generic ladder height of a random walk, which we denote by Y^+.

Lemma 3.18 *If $F_U^{\mathrm{s}} \in \mathcal{S}$, then $F_{Y_+}^{\mathrm{s}} \in \mathcal{S}$ and*

$$\lim_{x\to\infty} \frac{\overline{F_{Y_+}^{\mathrm{s}}}(x)}{\overline{F_U^{\mathrm{s}}}(x)} = \frac{\mathbb{E}\,U}{\mathbb{E}\,Y_+}\,. \tag{3.53}$$

We are now in a position to prove that subexponentiality of the integrated tail distribution F_U^{s} of claim sizes implies subexponentiality of the conditional ladder height distribution G_0, where $G_0(x) = p^{-1}G^+(x)$ and $p = G^+(\infty)$.

Lemma 3.19 *If $F_{Y_+}^{\mathrm{s}} \in \mathcal{S}$, then $G_0 \in \mathcal{S}$ and*

$$\lim_{x\to\infty} \frac{\overline{F_{Y_+}^{\mathrm{s}}}(x)}{\overline{G}_0(x)} = \frac{p}{\mathbb{E}\,Y_+}\int_{-\infty}^{0} |t|\,\mathrm{d}G^-(t)\,. \tag{3.54}$$

Examples 1. We showed in Section II.2.4 that the Weibull distribution $F = \mathrm{W}(r,c)$ with $0 < r < 1, c > 0$ belongs to $\mathcal{S}^*$. Furthermore, using Theorems 3.17 and II.2.20 we have (for $F_U = \mathrm{W}(r,1)$)

$$\psi(u) \sim \frac{1}{r\beta\mathbb{E}\,T - \Gamma(1/r)}\int_{u^r}^{\infty} \mathrm{e}^{-y}y^{1/r-1}\,\mathrm{d}y\,, \qquad u\to\infty\,. \tag{3.55}$$

Note that the integral in (3.55) is the tail of an incomplete gamma function.
2. Let $F_U \in \mathcal{S}$ be the Pareto distribution with density

$$f_U(x) = \begin{matrix} \alpha c^{\alpha} x^{-(\alpha+1)} & \mathrm{if} x \geq c, \\ 0 & if x < c, \end{matrix}$$

with $\alpha > 1, c > 0$. We leave it to the reader to show that then $\mu_U = \alpha c/(\alpha-1)$, $F_U^{\mathrm{s}} \in \mathcal{S}$ and $\psi(u) \sim c(\beta\mathbb{E}\,T(\alpha-1) - \alpha c)^{-1}(c/u)^{\alpha-1}$ as $u\to\infty$, where it suffices to prove that the condition of Corollary II.2.21 is fulfilled and to use Theorem 3.17.
3. Let $F_U \in \mathcal{S}$ be the lognormal distribution $\mathrm{LN}(a,b)$ with $-\infty < a < \infty$, $b > 0$. If we show first that

$$\overline{F_U^{\mathrm{s}}}(x) \sim \frac{b^3\exp(-b^2/2)}{\mathrm{e}^a\sqrt{2\pi}}\,\frac{x}{(\log x - a)^2}\exp\Big(-\frac{(\log x-a)^2}{2b^2}\Big)\,,$$

and then that the right-hand side belongs to $\mathcal{S}$, then we can conclude that $F_U^{\mathrm{s}} \in \mathcal{S}$. Now it is not difficult to show that

$$\psi(u) \sim c\frac{u}{(\log u - a)^2}\exp\Big(-\frac{(\log u-a)^2}{2b^2}\Big)\,, \qquad u\to\infty\,,$$

where $c = b^3(\sqrt{2\pi}(\beta\mathbb{E}\,T - \exp(a + b^2/2)))^{-1}$.

Bibliographical comments. The surplus just before ruin and the severity of ruin were studied by many authors, mostly for the compound Poisson model; see the bibliographical notes to Section III.2. Note however that results like formula (3.36) remain true even for much more general arrival processes with stationary increments. The original proof of Theorem 3.13 given by H. Cramér is analytical, using Wiener–Hopf techniques and expansions of the resulting solutions. The approach via ladder heights, as presented in Section 3.3, is due to W. Feller. Theorem 3.16 is from Cramér (1955), p. 75. The exposition of Section 3.5 follows Embrechts and Veraverbeke (1982). Properties of subexponential distributions like those used in the proof of Theorem 3.17 can be found, for example, in Athreya and Ney (1972), Pakes (1975), Teugels (1975) and Veraverbeke (1977); see also Section II.2.

Chapter V

Levy insurance risk model

Recently the classical risk model has been generalized by the use of Levy processes. Let us start with a simple observation.

In the classical model the risk reserve process is defined as

$$R(t) = x + \beta t - \sum_{j=1}^{N(t)} U_j,$$

where (U_j) are i.i.d. r.v.s independent of Poisson process $N(t)$ with intensity λ. This is of course a process with independent and stationary increments, and since $R(t)$ has a.s. right continous and with limits from the left realizations, $R(t)$ is a spectrally negative Levy process.

1 Basic facts from Lévy processes

Recall now the basic facts from Levy processes that is processes with independent and stationary increments with a.s. right continous and with limits from the left realizations. Here we follow Kyprianou (2006) for definitions, notations and basic facts on Lévy processes. Let in the sequel $X \equiv (X(t))_t$ be a Lévy process which is defined on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the natural filtration that satisfies the usual assumptions of right continuity and completion. Later if we write $\mathbb{P}_x$, it means that $\mathbb{P}_x(X(0) = x) = 1$ and $\mathbb{P}_0 = \mathbb{P}$; similarly, $\mathbb{E}_x$ is expectation with respect to $\mathbb{P}_x$. We denote by $\Pi(\cdot)$ the jump measure of X. We define the characteristic exponent $\Psi(\vartheta)$ by

$$\mathbb{E}\, e^{i\vartheta X(t)} = e^{-t\Psi(\vartheta)}.$$

From Lévy-Khintchine formula is is known that

$$\Psi(\vartheta) = ia\vartheta + \frac{1}{2}\sigma^2\vartheta^2 + \int_{\mathbb{R}-\{0\}} (1 - e^{i\vartheta x} + i\vartheta 1(|x| < 1)\,\Pi(dx)$$

for some spectral measure Π on $\mathbb{R} - \{0\}$ fulfilling

$$\int_{\mathbb{R}-\{0\}} (1 \wedge x^2)\,\Pi(dx)$$

. If $\Pi(0,\infty) = 0$, then the process is said that to be *spectrally negative.*

Theorem 1.1 *Suppose* $\mathbb{E}\,|X(1)| < \infty$*. The process* X *drifts to infinity if and only if* $\mathbb{E}\,X(1) > 0$*, oscilates if and only if* $\mathbb{E}\,X(1) = 0$*, and drifts to minus infinity if and only if* $\mathbb{E}\,X(1) < 0$*.*

We also define the first passage time

$$\tau_y^- := \inf\{t > 0 : X(t) < y\}.$$

Notice that for $y = 0$, when starting at zero from x this passage time corresponds to ruin time τ from Chapter I. Thus $\mathbb{P}_x(\tau_0^- < \infty)$, as a function of x, is the ruin function. In this section we do not use notation $\psi(x)$ for the ruin function since this notation is reserved for the Laplace exponent.

We now survay basic notions from the theory.

Dual process. The so-called *dual process* is $\hat{X}_t = -X_t$ with jump measure $\hat{\Pi}\,(0, y) = \Pi\,(-y, 0)$. Characteristics of $\hat{X}$ will be indicated by using the same symbols as for X, but with a '^' added.

Ladder heights. For the process X we define the associated $(L^{-1}(t), H(t))_{t\geq 0}$:

$$L^{-1}(t) := \begin{cases} \inf\{s > 0 : L(s) > t\} & \text{if } t < L(\infty), \\ \infty & \text{otherwise,} \end{cases}$$

and

$$H(t) := \begin{cases} X_{L^{-1}(t)} & \text{if } t < L(\infty), \\ \infty & \text{otherwise,} \end{cases}$$

where $L \equiv (L(t))_t$ is the local time at the maximum [10, p. 140]. Recall that (L^{-1}, H) is a bivariate subordinator with the Laplace exponent

$$\kappa(\vartheta, \beta) := -\frac{1}{t}\log \mathbb{E}\,\left(e^{-\vartheta L^{-1}(t) - \beta H(t)}\mathbf{1}_{\{t \leq L(\infty)\}}\right)$$

and with the jump measure Π_H. In addition we define the *descending ladder height process* $(\hat{L}^{-1}(t), \hat{H}(t))_{t\geq 0}$ with the Laplace exponent $\hat{\kappa}(\vartheta, \beta)$ constructed from the dual process $\hat{X}$. Recall that under the stability assumption $\mathbb{E}\, X(1) < 0$, the random variable $L(\infty)$ has an exponential distribution with parameter $\kappa(0,0)$. Moreover, for a spectrally negative Lévy process the Wiener-Hopf factorization states that

$$\kappa(\vartheta, \beta) = \Phi(\vartheta) + \beta, \qquad \hat{\kappa}(\vartheta, \beta) = \frac{\vartheta - \psi(\beta)}{\Phi(\vartheta) - \beta}; \tag{1.1}$$

see Kyprianou (2006) p. 169-170]. It follows that $\kappa(0,0) = \psi'(0+)$.

2 Asymptotic estimates for ruin function

In this section we extend the classical estimation for ruin function. We begin with the Cramer's estimate of ruin.

Theorem 2.1 *Assume that $X(t)$ is a Lévy process which does not have monotone path, for which*

- $\lim_{x\to\infty} X(t) = -\infty$,
- *there exists $\gamma > 0$ such that $\psi(\gamma) = 0$,*
- *the support of Π is not lattice.*

Then

$$\lim_{x\to\infty} e^{\gamma x}\mathbb{P}_x(\tau_0^- < \infty) = \hat{\kappa}(0,0)\left(\gamma \frac{\partial \hat{\kappa}(0,\beta)}{\partial \beta}\Big|_{\beta=-\gamma}\right),$$

where the linit is interpreted to be zero if the derivative on the right-hand side is infinite.

The corresponding to Embrecht-Verabeveke theorem is as follows.

Theorem 2.2 *Suppose that X is a spectrally positive Lévy process with mean $\mathbb{E}\, X(1) < 0$ and $\Pi(-\infty, x)$ regularly varying at $-\infty$. Then*

$$\mathbb{P}_x(\tau_0^- < \infty) \sim \frac{1}{\mathbb{E}\,(X(1))} \int_{-\infty}^{-x} \Pi(-\infty, y)\, dy$$

as $x \to \infty$.

Bibliographical comments. The basic facts on Levy insurance risk processes can be found in Kyprianou (2006).

3 Exit problems for spectrally negative processes

In this section we will study spectrally negative Lévy processes. For a spectrally negative Lévy process it is convenient to work with the *Laplace exponent* $\psi(\vartheta)$ by

$$\mathbb{E}\, e^{\vartheta X(t)} = e^{t\psi(\vartheta)}, \tag{3.2}$$

for $\vartheta \in \Theta$ such that the left hand side of (3.2) is well-defined (from now on we will assume that that this set Θ is not empty). Recall that in the case of spectrally negative for Lévy processes we have the following celebrated formula

$$\psi(\vartheta) = -a\vartheta + \frac{1}{2}\sigma^2\vartheta^2 + \int_{(-\infty,0)} (e^{\vartheta x} - 1 - \vartheta x 1(x > -1))\, \Pi(dx)$$

where the spectral measure Π fulfills

$$\int_{-\infty}^{0} (1 \wedge x^2)\, \Pi(dx) < \infty.$$

When X has bounded variation we may always write

$$\psi(\vartheta) = d\vartheta - \int_{-1<x<0} (e^{\vartheta x} - 1)\, \Pi(dx)$$

where $d \geq 0$. Let $\Phi(q) = \sup\{\vartheta : \psi(\vartheta) = q\}$ be the right inverse of ψ.

3.1 Scale functions

We now define a family of scale functions $W^{(q)}(x)$. It is the function which is 0 for $x < 0$ and strictly increasing and continuous with Laplace transform

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)\, dx = \frac{1}{\psi(\beta) - q}$$

for $\beta > \Phi(q)$ and the second scale function

$$Z^{(q)}(x) = 1 + q \int_0^\infty W^{(q)}\, dy.$$

For short we write $W^{(0)}(x) = W(x)$.

Theorem 3.1 *For any $x \in \mathbb{R}$ and $q \geq 0$*

$$\mathbb{E}_x \left(e^{-q\tau_0^-} 1(\tau_0^- < \infty) \right) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x),$$

where we understand $q/\Phi(q)$ in the limiting senses for $q = 0$. In particular

$$\mathbb{P}_x(\tau_0^-) = \begin{cases} 1 - \psi'(0+)W(x) & \textit{if} \quad \psi'(0+) > 0, \\ 1 & \textit{if} \quad \psi'(0+) \le 0. \end{cases}$$

Notice that, up to the multiplicative constant

$$W(x) = \mathbb{P}_x(\underline{X}_\infty \ge 0).$$

For some cases we have more explicit formulas for q-scale functions.

- *Linear Brownian Motion.* In this case $\Psi(\vartheta) = \sigma^2\vartheta^2/2 + \delta\vartheta$, where $\sigma > 0$ and $\delta \in \mathbb{R}$. Then

$$W^{(q)}(x) = \frac{2}{\sqrt{2q\sigma^2 + \delta}} e^{-\delta x/\sigma^2} \sinh\left(\frac{x}{\sigma^2}\sqrt{2q\sigma^2 + \delta}\right)$$

  for $q \ge 0$.

- *Spectrally negative stable process.* Let $\alpha \in (1, 2)$ be the stability parameter; and take $\psi(\vartheta) = \vartheta^\alpha$. Then

$$W^{(q)}(x) = \alpha x^{\alpha-1} E'_{\alpha,1}(qx^\alpha),$$

  for $q \ge 0$, where $E_{\beta,1}(z) = \sum_{k\ge 0} z^k/\Gamma(1 + \alpha k)$ is the Mittag-Lefler function. If the stable process is with drift d, then we know only the scale function for $q = 0$:

$$W(x) = \frac{1}{d}(1 - E_{\alpha-1,1}(-dx^{\alpha-1})),$$

  for $x \ge 0$.

- Spectrally negative Levy process of bounded variation. Suppose that $X(t) = dt - S(t)$, where $S(t)$ is a subordinator with jump meather Π such that $X(t) \to \infty$ a.s.. Then we necessarily have $d\int_0^\infty \Pi(x, \infty)\, dx < 1$. Furthermore

$$\int_{[0,\infty)} e^{-\beta x}\, W(dx) = \frac{1}{d - \int_0^\infty e^{-\beta y}\Pi(y, \infty)\, dy},$$

  and hence

$$W(dx) = \frac{1}{d}\sum_{n\ge 0} \nu^*(dx),$$

where $\nu(dx) = d^{-1}\Pi(x,\infty)$. In particular if $S(t) = \sum_{j=1}^{N(t)} U_j$, where $N(t)$ is Poisson process with intensity $\lambda > 0$ and U is exponentially distributed with parameter $\delta > 0$

$$W(x) = \frac{1}{d}\left(1 + \frac{\lambda}{d\delta - \lambda}(1 - e^{-(\delta - d^{-1}\lambda)x})\right).$$

Bibliographical comments. More scacle functions can be found in the paper Hubalek and Kyprianou (2010).

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