

# Financial modeling with Lévy processes

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Lecture notes

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## 1 Introduction

Exponential Lévy models generalize the classical Black and Scholes setup by allowing the stock prices to jump while preserving the independence and stationarity of returns. There are ample reasons for introducing jumps in financial modeling. First of all, asset prices do jump, and some risks simply cannot be handled within continuous-path models. Second, the well-documented phenomenon of *implied volatility smile* in option markets shows that the risk-neutral returns are non-gaussian and leptokurtic. While the smile itself can be explained within a model with continuous paths, the fact that it becomes much more pronounced for short maturities is a clear indication of the presence of jumps. In continuous-path models, the law of returns for shorter maturities becomes closer to the Gaussian law, whereas in reality and in models with jumps returns actually become less Gaussian as the

horizon becomes shorter. Finally, jump processes correspond to genuinely incomplete markets, whereas all continuous-path models are either complete or 'completable' with a small number of additional assets. This fundamental incompleteness makes it possible to carry out a rigorous analysis of the hedging error and find ways to improve the hedging performance using additional instruments such as liquid European options.

A great advantage of exponential Lévy models is their mathematical tractability, which makes it possible to perform many computations explicitly and to present deep results of modern mathematical finance in a simple manner. This has led to an explosion of the literature on option pricing and hedging in exponential Lévy models in the late 90s and early 2000s, the literature which now contains hundreds of research papers and several monographs. However, some fundamental aspects such as asymptotic behavior of implied volatility or the computation of hedge ratios have only recently been given a rigorous treatment.

For background on exponential Lévy models, the reader may refer to textbooks such as [20, 60] for a more financial perspective or [3, 44] for a more mathematical perspective.

## 2 Lévy processes: basic facts

Lévy processes are a class of stochastic processes with discontinuous paths, which is at the same time simple enough to study and rich enough for applications, or at least to be used as building blocks of more realistic models.

**Definition 1.** A stochastic process  $X$  is a Lévy process if it is càdlàg, satisfies  $X_0 = 0$  and possesses the following properties:

- Independent increments;
- Stationary increments;

From these properties it follows that

- $X$  is continuous in probability:  $\forall \varepsilon, \lim_{s \rightarrow 0} P[|X_{s+t} - X_t| > \varepsilon] = 0$ .
- At any fixed time, the probability of having a jump is zero:  $\forall t, P[X_{t-} = X_t] = 1$ .

Lévy processes are essentially processes with jumps, because it can be shown that any Lévy process which has a.s.continuous trajectories is a Brownian motion with drift.

**Proposition 1.** *Let  $X$  be a continuous Lévy process. Then there exist  $\gamma \in \mathbb{R}^d$  and a symmetric positive definite matrix  $A$  such that*

$$X_t = \gamma t + W_t,$$

where  $W$  is a Brownian motion with covariance matrix  $A$ .

*Proof.* This result is a consequence of the Feller-Lévy central limit theorem, but since it is important for the understanding of Lévy processes, we give here a short proof (for the one-dimensional case).

It is enough to show that  $X_1$  has Gaussian law, the rest will follow from the stationarity and independence of increments.

**Step 1** Let  $\xi_n^k := X_{\frac{k}{n}} - X_{\frac{k-1}{n}}$ . The continuity of  $X$  implies that

$$\lim_n P[\sup_k |\xi_n^k| > \varepsilon] = 0,$$

for all  $\varepsilon$ . Let  $a_n = P[|\xi_n^1| > \varepsilon]$ . Since

$$P[\sup_k |\xi_n^k| > \varepsilon] = 1 - (1 - P[|\xi_n^1| > \varepsilon])^n,$$

we get that  $\lim_n (1 - a_n)^n = 1$ , from which it follows that  $\lim_n n \log(1 - a_n) = 0$ . But  $n \log(1 - a_n) \leq -na_n \leq 0$ . Therefore,

$$\lim_n nP[|X_{\frac{1}{n}}| > \varepsilon] = 0 \tag{1}$$

**Step 2** Using the independence and stationarity of increments, we can show that

$$\lim_n nE[\cos X_{\frac{1}{n}} - 1] = \frac{1}{2} \{ \log Ee^{iX_1} + \log Ee^{-iX_1} \} := -A; \tag{2}$$

$$\lim_n nE[\sin X_{\frac{1}{n}}] = \frac{1}{2i} \{ \log Ee^{iX_1} - \log Ee^{-iX_1} \} := \gamma. \tag{3}$$

**Step 3** The equations (1) and (2) allow to prove that for every function  $f$  such that  $f(x) = o(|x|^2)$  in a neighborhood of 0,  $\lim_n nE[f(X_{\frac{1}{n}})] = 0$  which implies that  $\varepsilon > 0$

$$\lim_n nE[X_{\frac{1}{n}} 1_{|X_{\frac{1}{n}}| \leq \varepsilon}] = \gamma, \quad (4)$$

$$\lim_n nE[X_{\frac{1}{n}}^2 1_{|X_{\frac{1}{n}}| \leq \varepsilon}] = A, \quad (5)$$

$$\lim_n nE[|X_{\frac{1}{n}}|^3 1_{|X_{\frac{1}{n}}| \leq \varepsilon}] = 0. \quad (6)$$

$$(7)$$

**Step 4** Assembling together the different equations, we finally get

$$\begin{aligned} \log E[e^{iuX_1}] &= n \log E[e^{iuX_{\frac{1}{n}}} 1_{|X_{\frac{1}{n}}| \leq \varepsilon}] + o(1) \\ &= n \log \left\{ 1 + iuE[X_{\frac{1}{n}} 1_{|X_{\frac{1}{n}}| \leq \varepsilon}] - \frac{u^2}{2} E[X_{\frac{1}{n}}^2 1_{|X_{\frac{1}{n}}| \leq \varepsilon}] + o(1/n) \right\} + o(1) \\ &= iu\gamma - \frac{Au^2}{2} + o(1) \xrightarrow{n \rightarrow \infty} iu\gamma - \frac{Au^2}{2} \end{aligned}$$

where  $o(1)$  denotes a quantity which tends to 0 as  $n \rightarrow \infty$ . □

The second fundamental example of Lévy process is the Poisson process.

## 2.1 The Poisson process

**Definition 2.** Let  $(\tau_i)_{i \geq 1}$  be a sequence of exponential random variables with parameter  $\lambda$  and let  $T_n = \sum_{i=1}^n \tau_i$ . Then the process

$$N_t = \sum_{n \geq 1} 1_{t \geq T_n} \quad (8)$$

is called the Poisson process with parameter (or intensity)  $\lambda$

**Proposition 2** (Properties of the Poisson process).

1. For all  $t \geq 0$ , the sum in (8) is finite a.s.
2. The trajectories of  $N$  are piecewise constant with jumps of size 1 only.
3. The trajectories are càdlàg.

4.  $\forall t > 0$ ,  $N_{t-} = N_t$  with probability 1.

5.  $\forall t > 0$ ,  $N_t$  follows the Poisson law with parameter  $\lambda t$ :

$$P[N_t = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

6. The characteristic function of the Poisson process is

$$E[e^{iuN_t}] = \exp\{\lambda t(e^{iu} - 1)\}.$$

7. The Poisson process is a Lévy process

The Poisson process counts the events with exponential interarrival times. In a more general setting, one speaks of a counting process.

**Definition 3.** Let  $(T_n)$  be a sequence of times with  $T_n \rightarrow \infty$  a.s. Then the process

$$N_t = \sum_{n \geq 1} 1_{t \geq T_n}$$

is called a counting process.

In other words, a counting process is an increasing piecewise constant process with jumps of size 1 only and almost surely finite.

The first step towards the characterization of Lévy processes is to characterize Lévy processes which are counting processes.

**Proposition 3.** Let  $(N_t)$  be a Lévy process and a counting process. Then  $(N_t)$  is a Poisson process.

*Proof.* The proof uses the characterization of the exponential distribution by its memoryless property: if a random variable  $T$  satisfies

$$P[T > t + s | T > t] = P[T > s]$$

for all  $t, s > 0$  then  $T$  has exponential distribution.

Let  $T_1$  be the first jump time of the process  $N$ . The independence and stationarity of increments give us:

$$\begin{aligned} P[T_1 > t + s | T_1 > t] &= P[N_{t+s} = 0 | N_t = 0] \\ &= P[N_{t+s} - N_t = 0 | N_t = 0] = P[N_s = 0] = P[T_1 > s], \end{aligned}$$

which means that the first jump time  $T_1$  has exponential distribution.

Now, it suffices to show that the process  $(X_{T_1+t} - X_{T_1})_{t \geq 0}$  is independent from  $T_1$  and has the same law as  $(X_t)_{t \geq 0}$ . Let  $f(t) := E[e^{iuX_t}]$ . Then using once again the independence and stationarity of increments we get that  $f(t+s) = f(t)f(s)$  and  $M_t := \frac{e^{iuX_t}}{f(t)}$  is a martingale. Let  $T_1^n := n \wedge T_1$ . Then by Doob's optional sampling theorem,

$$E[e^{iu(X_{T_1^n+t} - X_{T_1^n}) + ivT_1^n}] = E\left[\frac{f(T_1^n + t)}{f(T_1^n)} e^{ivT_1^n}\right] = E[e^{iuX_t}]E[e^{ivT_1^n}].$$

The proof is finished with an application of the dominated convergence theorem.  $\square$

**Compound Poisson process** The Poisson process itself cannot be used to model asset prices because the condition that the jump size is always equal to 1 is too restrictive, but it can be used as building block to construct richer models.

**Definition 4** (Compound Poisson process). The compound Poisson process with jump intensity  $\lambda$  and jump size distribution  $\mu$  is a stochastic process  $(X_t)_{t \geq 0}$  defined by

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where  $\{Y_i\}_{i \geq 1}$  is a sequence of independent random variables with law  $\mu$  and  $N$  is a Poisson process with intensity  $\lambda$  independent from  $\{Y_i\}_{i \geq 1}$ .

In other words, a compound Poisson process is a piecewise constant process which jumps at jump times of a standard Poisson process and whose jump sizes are i.i.d. random variables with a given law.

**Proposition 4** (Properties of the compound Poisson process). *Let  $(X_t)_{t \geq 0}$  be a compound Poisson process with jump intensity  $\lambda$  and jump size distribution  $\mu$ . Then  $X$  is a piecewise constant Lévy process and its characteristic function is given by*

$$E[e^{iuX_t}] = \exp\left\{t\lambda \int_{\mathbb{R}} (e^{iux} - 1)\mu(dx)\right\}.$$

*Example 1* (Merton's model). The Merton (1976) model is one of the first applications of jump processes in financial modeling. In this model, to take into account price discontinuities, one adds Gaussian jumps to the log-price.

$$S_t = S_0 e^{r t + X_t}, \quad X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad Y_i \sim N(\mu, \delta^2) \text{ independents.}$$

The advantage of this choice of jump size distribution is to have a series representation for the density of the log-price (as well as for the prices of European options).

$$p_t(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k \exp\left\{-\frac{(x-\gamma t-k\mu)^2}{2(\sigma^2 t+k\delta^2)}\right\}}{k! \sqrt{2\pi(\sigma^2 t+k\delta^2)}}.$$

## 2.2 Poisson random measures

The notion of the Poisson random measure is central for the theory of Lévy processes: we shall use it in the next section to give a full characterization of their path structure.

**Definition 5** (Random measure). Let  $(\Omega, P, \mathcal{F})$  be a probability space and  $(E, \mathcal{E})$  a measurable space. Then  $M : \Omega \times \mathcal{E} \rightarrow \mathbb{R}$  is a random measure if

- For every  $\omega \in \Omega$ ,  $M(\omega, \cdot)$  is a measure on  $\mathcal{E}$ .
- For every  $A \in \mathcal{E}$ ,  $M(\cdot, A)$  is measurable.

**Definition 6** (Poisson random measure). Let  $(\Omega, P, \mathcal{F})$  be a probability space,  $(E, \mathcal{E})$  a measurable space and  $\mu$  a measure on  $(E, \mathcal{E})$ . Then  $M : \Omega \times \mathcal{E} \rightarrow \mathbb{R}$  is a Poisson random measure with intensity  $\mu$  if

- For all  $A \in \mathcal{E}$  with  $\mu(A) < \infty$ ,  $M(A)$  follows the Poisson law with parameter  $E[M(A)] = \mu(A)$ .
- For any disjoint sets  $A_1, \dots, A_n$ ,  $M(A_1), \dots, M(A_n)$  are independent.

In particular, the Poisson random measure is a positive integer-valued random measure. It can be constructed as the counting measure of randomly scattered points, as shown by the following proposition.



**Proposition 5.** *Let  $\mu$  be a  $\sigma$ -finite measure on a measurable subset  $E$  of  $\mathbb{R}^d$ . Then there exists a Poisson random measure on  $E$  with intensity  $\mu$ .*

*Proof.* Suppose first that  $\mu(E) < \infty$ . Let  $\{X_i\}_{i \geq 1}$  be a sequence of independent random variables such that  $P[X_i \in A] = \frac{\mu(A)}{\mu(E)}$ ,  $\forall i$  and  $\forall A \in \mathcal{B}(E)$ , and let  $M(E)$  be a Poisson random variable with intensity  $\mu(E)$  independent from  $\{X_i\}_{i \geq 1}$ . It is then easy to show that the random measure  $M$  defined by

$$M(A) := \sum_{i=1}^{M(E)} 1_A(X_i), \quad \forall A \in \mathcal{B}(E),$$

is a Poisson random measure on  $E$  with intensity  $\mu$ .

Assume now that  $\mu(E) = \infty$ , and choose a sequence of disjoint measurable sets  $\{E_i\}_{i \geq 1}$  such that  $\mu(E_i) < \infty$ ,  $\forall i$  and  $\bigcup_i E_i = E$ . We construct a Poisson random measure  $M_i$  on each  $E_i$  as described above and define

$$M(A) := \sum_{i=1}^{\infty} M_i(A), \quad \forall A \in \mathcal{B}(E).$$

□

**Corollary 1** (Exponential formula). *Let  $M$  be a Poisson random measure on  $(E, \mathcal{E})$  with intensity  $\mu$ ,  $B \in \mathcal{E}$  and let  $f$  be a measurable function with  $\int_B |e^{f(x)} - 1| \mu(dx) < \infty$ . Then*

$$E \left[ e^{\int_B f(x) M(dx)} \right] = \exp \left[ \int_B (e^{f(x)} - 1) \mu(dx) \right].$$

**Definition 7** (Jump measure). Let  $X$  be a  $\mathbb{R}^d$ -valued càdlàg process. The jump measure of  $X$  is a random measure on  $\mathcal{B}([0, \infty) \times \mathbb{R}^d)$  defined by

$$J_X(A) = \#\{t : \Delta X_t \neq 0 \text{ and } (t, \Delta X_t) \in A\}.$$

The jump measure of a set of the form  $[s, t] \times A$  counts the number of jumps of  $X$  between  $s$  and  $t$  such that their sizes fall into  $A$ . For a counting process, since the jump size is always equal to 1, the jump measure can be seen as a random measure on  $[0, \infty)$ .

**Proposition 6.** *Let  $X$  be a Poisson process with intensity  $\lambda$ . Then  $J_X$  is a Poisson random measure on  $[0, \infty)$  with intensity  $\lambda \times dt$ .*

Maybe the most important result of the theory of Lévy processes is that the jump measure of a general Lévy process is also a Poisson random measure.

**Exercise 1.** Let  $X$  and  $Y$  be two independent Lévy processes. Use the definition to show that  $X + Y$  is also a Lévy process.

**Exercise 2.** Show that the memoryless property characterizes the exponential distribution: if a random variable  $T$  satisfies

$$\forall t, s > 0, \quad P[T > t + s | T > t] = P[T > s]$$

then either  $T \equiv 0$  or  $T$  has exponential law.

**Exercise 3.** Prove that if  $N$  is a Poisson process then it is a Lévy process.

**Exercise 4.** Prove that if  $N$  and  $N'$  are independent Poisson processes with parameters  $\lambda$  and  $\lambda'$  then  $N + N'$  is a Poisson process with parameter  $\lambda + \lambda'$ .

**Exercise 5.** Let  $X$  be a compound Poisson process with jump size distribution  $\mu$ . Establish that

- $E[|X_t|] < \infty$  if and only if  $\int_{\mathbb{R}} |x|f(dx)$  and in this case

$$E[X_t] = \lambda t \int_{\mathbb{R}} xf(dx).$$

- $E[|X_t|^2] < \infty$  if and only if  $\int_{\mathbb{R}} x^2f(dx)$  and in this case

$$\text{Var}[X_t] = \lambda t \int_{\mathbb{R}} x^2f(dx).$$

- $E[e^{X_t}] < \infty$  if and only if  $\int_{\mathbb{R}} e^x f(dx)$  and in this case

$$E[e^{X_t}] = \exp\left(\lambda t \int_{\mathbb{R}} (e^x - 1)f(dx)\right).$$

**Exercise 6.** The goal is to show that to construct a Poisson random measure on  $\mathbb{R}$ , one needs to take two Poisson processes and make the first one run towards  $+\infty$  and the second one towards  $-\infty$ .

Let  $N$  and  $N'$  be two Poisson processes with intensity  $\lambda$ , and let  $M$  be a random measure defined by

$$M(A) = \#\{t > 0 : t \in A, \Delta N_t = 1\} + \#\{t > 0 : -t \in A, \Delta N'_t = 1\}.$$

Show that  $M$  is a Poisson random measure with intensity  $\lambda$ .

### 3 Path structure of a Lévy process

**Definition 8** (Lévy measure). Let  $X$  be an  $\mathbb{R}^d$ -valued Lévy process. The measure  $\nu$  defined by

$$\nu(A) = E[\#\{t \in [0, 1] : \Delta X_t \neq 0 \text{ et } \Delta X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^d)$$

is called the Lévy measure of  $X$ .

**Theorem 1** (Lévy-Itô-decomposition). *Let  $X$  be an  $\mathbb{R}^d$ -valued Lévy process with Lévy measure  $\nu$ . Then*

1. *The jump measure  $J_X$  of  $X$  is a Poisson random measure on  $[0, \infty) \times \mathbb{R}^d$  with intensity  $dt \times \nu$ .*
2. *The Lévy measure  $\nu$  satisfies  $\int_{\mathbb{R}^d} (\|x\|^2 \wedge 1) \nu(dx) < \infty$ .*
3. *There exist  $\gamma \in \mathbb{R}^d$  and a  $d$ -dimensional Brownian motion  $B$  with covariance matrix  $A$  such that*

$$\begin{aligned} X_t &= \gamma t + B_t + N_t + M_t, \quad \text{where} & (9) \\ N_t &= \int_{|x|>1, s \in [0, t]} x J_X(ds \times dx) \quad \text{and} \\ M_t &= \int_{0 < |x| \leq 1, s \in [0, t]} x \{J_X(ds \times dx) - \nu(dx) ds\} \\ &\equiv \int_{0 < |x| \leq 1, s \in [0, t]} x \tilde{J}_X(ds \times dx). \end{aligned}$$

*The three terms are independent and the convergence in the last term is almost sure and uniform in  $t$  on compacts.*

The triple  $(A, \nu, \gamma)$  is called *the characteristic triple* of  $X$ .

The proof is based on the following lemma.

**Lemma 1.** *Let  $(X, Y)$  be a 2-dimensional Lévy process such that  $Y$  is piecewise constant and  $\Delta X_t \Delta Y_t = 0$  for all  $t$  a.s. Then  $X$  and  $Y$  are independent.*

*Proof.* In view of the independence and stationarity of increments, it is enough to show that  $X_1$  and  $Y_1$  are independent. Let  $M_t = \frac{e^{iuX_t}}{E[e^{iuX_t}]}$  and  $N_t = \frac{e^{ivY_t}}{E[e^{ivY_t}]}$ . Then  $M$  and  $N$  are martingales on  $[0, 1]$ . From the independence and stationarity of increments we deduce that for every Lévy process  $Z$ ,

$$E[e^{iuZ_t}] = E[e^{iuZ_1}]^t \quad \text{and} \quad E[e^{iuZ_1}] \neq 0, \forall u.$$

This means that  $M$  is bounded. By Proposition 3, the number of jumps of  $Y$  on  $[0, 1]$  is a Poisson random variable. Therefore,  $N$  has integrable variation on this interval. By the martingale property and dominated convergence we finally get

$$\begin{aligned} E[M_1 N_1] - 1 &= E \left[ \sum_{i=1}^n (M_{i/n} - M_{(i-1)/n})(N_{i/n} - N_{(i-1)/n}) \right] \\ &\rightarrow E \left[ \sum_{0 \leq t \leq 1} \Delta M_t \Delta N_t \right] = 0, \end{aligned}$$

which implies  $E[e^{iuX_1 + ivY_1}] = E[e^{iuX_1}]E[e^{ivY_1}]$ .  $\square$

*Proof of Theorem 1.*

**Part 1** Let  $A \in \mathcal{B}(\mathbb{R}^d)$  with  $0 \notin \bar{A}$ . Then  $N_t^A = \#\{s \leq t : \Delta X_s \in A\}$  is a counting process and a Lévy process, hence, by Proposition 3, a Poisson process, which means that  $J_X([t_1, t_2] \times A)$  follows the Poisson law with parameter  $(t_2 - t_1)\nu(A)$  and that  $J_X([t_1, t_2] \times A)$  is independent from  $J_X([s_1, s_2] \times A)$  if  $t_2 \leq s_1$ . Let us now take two disjoint sets  $A$  and  $B$ . By Lemma 1,  $N^A$  and  $N^B$  are independent, which proves that  $J_X([s_1, s_2] \times A)$  and  $J_X([t_1, t_2] \times B)$  are also independent, for all  $s_1, s_2, t_1, t_2$ .

**Part 2** From the previous part, we deduce that  $\nu(A) < \infty$  whenever  $0 \notin \bar{A}$ . It remains to show that

$$\int_{\|x\| \leq \delta} \|x\|^2 \nu(dx) < \infty$$

for some  $\delta > 0$ . Let

$$X_t^\varepsilon = \sum_{\substack{1 \geq |\Delta X_s| > \varepsilon \\ 0 \leq s \leq t}} \Delta X_s = \int_{\varepsilon < |x| \leq 1, s \in [0, t]} x J_X(ds \times dx)$$

and  $R_t^\varepsilon = X_t - X_t^\varepsilon$ . Since  $(X_t^\varepsilon, R_t^\varepsilon)$  is a Lévy process, Lemma 1 implies that  $X_t^\varepsilon$  and  $R_t^\varepsilon$  are independent. In addition,  $|E[e^{iuX_t}]| > 0$  for all  $t, u$ . This means that

$$E[e^{iuX_t}] = E[e^{iuR_t^\varepsilon}]E[e^{iuX_t^\varepsilon}].$$

Therefore,  $|E[e^{iuX_t^\varepsilon}]|$  is bounded from below by a positive number which does not depend on  $\varepsilon$ . By the exponential formula, this is equivalent to

$$\left| \exp \left\{ t \int_{|x| \geq \varepsilon} (e^{iux} - 1) \nu(dx) \right\} \right| \geq C > 0,$$

which gives  $\int_{|x| \geq \varepsilon} (1 - \cos(ux)) \nu(dx) \leq \tilde{C} < \infty$ . Since this result is true for all  $u$ , the proof of part 2 is completed.

**Part 3** Observe first that the process  $M$  is well defined due to the compensation of small jumps and the fact that the Lévy measure integrates  $\|x\|^2$  near zero: introducing the process

$$M_t^\varepsilon = \int_{\varepsilon \leq \|x\| < 1, s \in [0, t]} x \tilde{J}_X(ds \times dx),$$

we get that for  $\varepsilon_1 < \varepsilon_2$ ,

$$E[(M_t^{\varepsilon_1} - M_t^{\varepsilon_2})^2] = t \int_{\varepsilon_2 \leq \|x\| < \varepsilon_1} x^2 \nu(dx)$$

and so, since the space  $L^2$  is complete, for every  $t$ ,  $M_t^\varepsilon$  converges in  $L^2$  as  $\varepsilon \rightarrow 0$ . Using Doob's inequality we show that the convergence is uniform in  $t$  on compact intervals. The process  $X - N - M$  is then a continuous Lévy process independent from  $N$  and  $M$  in view of Lemma 1. We conclude with Proposition 1.  $\square$

**Corollary 2** (Lévy-Khintchine representation). *Let  $X$  be a Lévy process with characteristic triple  $(A, \nu, \gamma)$ . Its characteristic function is given by*

$$E[e^{iuX_t}] = \exp \left\{ t \left( i\gamma u - \frac{Au^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x| \leq 1}) \nu(dx) \right) \right\}, \quad (10)$$

*Proof.* Using the previous theorem and the exponential formula, we get

$$E[e^{iu(\gamma t + B_t + N_t + M_t^\varepsilon)}] = \exp \left\{ t \left( i\gamma u - \frac{Au^2}{2} + \int_{|x| \geq \varepsilon} (e^{iux} - 1 - iux1_{|x| \leq 1}) \nu(dx) \right) \right\},$$

and we conclude using the dominated convergence theorem.  $\square$

*Example 2* (The variance gamma process). One of the simplest examples of Lévy processes with infinite intensity of jumps is the gamma process, a process with stationary independent increments and such that for all  $t$ , the law  $p_t$  of  $X_t$  is the gamma law with parameters  $\lambda$  and  $ct$ :

$$p_t(x) = \frac{\lambda^{ct}}{\Gamma(ct)} x^{ct-1} e^{-\lambda x}.$$

The gamma process is an increasing Lévy process whose characteristic function has a very simple form:

$$E[e^{iuX_t}] = (1 - iu/\lambda)^{-ct}.$$

One can easily show that the Lévy measure of the gamma process has a density given by

$$\nu(x) = \frac{ce^{-\lambda x}}{x} 1_{x>0}. \quad (11)$$

Starting from the gamma process, we can construct a very popular jump model: the variance gamma process [50, 48] which is obtained by changing the time scale of a Brownian motion with drift with a gamma process:

$$Y_t = \mu X_t + \sigma B_{X_t}.$$

Using  $Y_t$  to model the logarithm of the stock price is usually justified by saying that the price follows a geometric Brownian motion on a stochastic time scale given by the gamma process [34]. The variance gamma process provides another example of a Lévy processes with infinite intensity of jumps, and its characteristic function is given by

$$E[e^{iuY_t}] = \left(1 + \frac{\kappa\sigma^2 u^2}{2} - i\mu\kappa u\right)^{-\kappa t}.$$

The parameters have the following intuitive interpretation:  $\sigma$  is the scale parameter,  $\mu$  is the parameter of asymmetry (skewness) and  $\kappa$  is responsible for the kurtosis of the process (thickness of its tails).

**Exercise 7.** Let  $X$  be a Lévy process with characteristic triple  $(A, \nu, \gamma)$ . Compute the probability that  $X$  will have at least one negative jump of size bigger than  $\varepsilon > 0$  on the interval  $[0, T]$ .

**Exercise 8.** Let  $X$  be a Lévy process with Lévy measure  $\nu(dx) = \lambda\nu_0(dx)$ , where  $\nu_0$  has no atom and satisfies  $\nu_0(\mathbb{R}) = \infty$ . For all  $n \in \mathbb{N}$ , let  $k_n > 0$  be the solution of  $\int_{k_n}^{\infty} \nu_0(dx) = n$ . For a fixed  $T$ , give the law of the random variable

$$A_n = \#\{t \leq T : \Delta X_t \in [k_{n+1}, k_n]\}.$$

Use this result to suggest a method for estimating  $\lambda$  from an observation of the trajectory of  $X$  on  $[0, T]$ , supposing that  $\nu_0$  is known.

**Exercise 9.** Let  $X$  be a Lévy process with no diffusion component and a Lévy measure  $\nu$  which satisfies  $\int_{\mathbb{R}} |x|\nu(dx) < \infty$ . Using the Levy-Ito decomposition, show that the trajectories of  $X$  have a.s. finite variation (a function has finite variation if it can be represented as the difference of two increasing functions).

**Exercise 10.** Prove that the Lévy measure of the gamma process is given by equation (11). Show that the variance gamma process can be represented as a difference of two independent gamma process, and use this result to deduce the form of the Lévy measure of the variance gamma process.

## 4 Basic stochastic calculus for jump processes

### 4.1 Integrands et integrators

The main application of the stochastic integral in finance is the representation of self-financing portfolios: in the absence of interest rates, when the price of the risky asset is a continuous process  $S$ , and the quantity of the asset is denoted by  $\phi$ , the portfolio value is

$$V_T = \int_0^T \phi_t dS_t$$

We would like this relationship to hold in the presence of jumps as well, but what are the natural properties to impose on  $S$  and  $\phi$ ? The process  $S$  must be right-continuous, since the price jumps arrive unexpectedly. On the other hand, the hedging strategy  $\phi_t$  is based on the observations of the portfolio manager up to date  $t$ ; it must therefore be left-continuous. The following example illustrates this: suppose that the asset price is given by  $S_t = \lambda t - N_t$ , where  $N_t$  is a Poisson process with intensity  $\lambda$ , and let  $T$

be the time of the first jump of  $N$ . If one could use the (càdlàg) strategy  $\phi_t = 1_{[0,T)}(t)$  amounting to sell the asset just before the jump, there would clearly be an arbitrage opportunity, since

$$V_t = \int_0^t \phi_t dS_t = \lambda t \wedge T.$$

On the other hand, with the càglàd strategy  $\phi_t = 1_{[0,T]}(t)$ ,

$$V_t = \int_0^t \phi_t dS_t = \lambda t \wedge T - N_{t \wedge T},$$

which has zero expectation. It is therefore natural to consider adapted and left-continuous integrands.

The simplest (and the only one which can be realized in practice) form of a portfolio strategy is such where the portfolio is only rebalanced a finite number of times. We define a simple predictable process by

$$\phi_t = \phi_0 1_{t=0} + \sum_{i=0}^n \phi_i 1_{(T_i, T_{i+1}]}(t), \quad (12)$$

where  $T_0 = 0$ ,  $(T_i)_{i \geq 0}$  is a sequence of stopping times, and for each  $i$ ,  $\phi_i$  is  $\mathcal{F}_{T_i}$ -measurable and bounded. The space of simple predictable processes will be denoted by  $\mathcal{S}$ .

For simple predictable processes, the stochastic integral is defined by

$$\int_0^t \phi_s dS_s := \sum_{i=0}^n \phi_i (S_{T_{i+1} \wedge t} - S_{T_i \wedge t}) \quad (13)$$

For a general adapted left-continuous process, the stochastic integral can then be defined as the continuous extension of the integral for simple predictable processes, using the topology of uniform convergence on compacts in probability (ucp).

The sequence of processes  $(X^n)$  is said to converge ucp to the process  $X$  if for every  $t$ ,  $(X^n - X)_t^*$  converges to 0 in probability, where  $Z_t^* := \sup_{0 \leq s \leq t} |Z_s|$ . We denote by  $\mathcal{S}_{ucp}$  the space  $\mathcal{S}$  endowed with the topology of ucp convergence and by  $\mathbb{L}_{ucp}$  and  $\mathbb{D}_{ucp}$  the space of adapted and, respectively, left or right continuous processes, with the same topology. It is then possible to show that the space  $\mathcal{S}_{ucp}$  is dense in  $\mathbb{L}_{ucp}$ , and to associate the ucp topology with a metric



on  $\mathbb{D}_{ucp}$ , for which this space will be complete. To extend the stochastic integration operator defined by (13) from  $\mathcal{S}_{ucp}$  to  $\mathbb{L}_{ucp}$ , this operator must be continuous as a mapping from  $\mathcal{S}_{ucp}$  to  $\mathbb{D}_{ucp}$ . Whether or not this is true, depends on the integrator  $S$ , and we shall limit ourselves to the integrators for which this property holds.

**Definition 9.** The process  $S \in \mathbb{D}$  is a *semimartingale* if the stochastic integration operator defined by (13) is a continuous operator from  $\mathcal{S}_{ucp}$  to  $\mathbb{D}_{ucp}$ .

Every adapted càdlàg process of finite variation on compacts is a semimartingale. This follows from

$$\sup_{0 \leq t \leq T} \left| \int_0^t \phi_s dS_s \right| \leq \text{Var}_0^T(S) \sup_{0 \leq t \leq T} \phi_t,$$

where  $\text{Var}_0^T(S)$  denotes the total variation of  $S$  on  $[0, T]$ . A square integrable càdlàg martingale is a semimartingale. For a simple predictable process  $\phi$  of the form (12), and a square integrable martingale  $M$ ,  $\int_0^\cdot \phi_t dM_t$  is also a martingale and

$$E \left( \int_0^T \phi_t dM_t \right)^2 \leq \sup_{0 \leq t \leq T, \omega \in \Omega} \phi_t^2 E[M_T^2].$$

Suppose now that  $(\phi^n)$  is a sequence of simple predictable processes such that  $\phi^n \rightarrow 0$  ucp. Then, using the Chebyshev's inequality and Doob's inequality, we get that

$$\begin{aligned} P \left[ \left( \int_0^\cdot \phi_t^n dM_t \right)^* > \varepsilon \right] &\leq P \left[ \left( \int_0^\cdot \phi_t^n 1_{|\phi_t^n| \leq C} dM_t \right)^* > \varepsilon \right] + P[(\phi^n)^* > C] \\ &\leq \frac{4C^2 E[M_T^2]}{\varepsilon^2} + P[(\phi^n)^* > C] \rightarrow 0, \end{aligned}$$

because the first term can be made arbitrarily small by choosing  $C$  sufficiently small, and the second term can be made small by choosing  $n$  sufficiently large.

Since the terms  $\gamma t$  and  $N_t$  in the Lévy-Itô decomposition have finite variation and the terms  $B_t$  and  $M_t$  are square integrable martingales, every Lévy process is a semimartingale.

A deep result of the general theory of processes [53] is that every semimartingale is the sum of a finite variation process and a *local* martingale. The notion of local martingale extends that of the martingale: the process  $(X_t)$  is a local martingale if there exists a sequence of stopping times  $\{T_i\}_{i \geq 1}$  such that  $T_i \rightarrow \infty$  when  $i \rightarrow \infty$  and for each  $i$ ,  $(X_{T_i \wedge t})$  is a martingale.

## 4.2 Stochastic integral with respect to a Poisson random measure

In the Lévy-Itô decomposition, we have already encountered integrals of deterministic functions with respect to Poisson random measures and compensated Poisson random measures. In this section, our goal is to extend this notion of integral to stochastic integrands.

Let  $M$  be a Poisson random measure of  $[0, T] \times \mathbb{R}$  with intensity  $\mu$ .  $\mu$  is supposed to be  $\sigma$ -finite: there exists a sequence  $U_n \uparrow \mathbb{R}$  with  $\mu([0, t] \times U_n) < \infty$  for all  $t$ . Typically,  $M$  will be the jump measure of a Lévy process. We would like to define the integral of  $M$  or its compensated version with respect to a predictable function  $\phi : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , that is, a function which satisfies

- (i) For all  $t$ ,  $(\omega, x) \mapsto \phi(\omega, t, x)$  is  $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$ -measurable.
- (ii) For all  $(\omega, x)$   $t \mapsto \phi(\omega, t, x)$  is left-continuous.

The stochastic integral of  $\phi$  with respect to  $M$  will be defined in two different settings:

Case 1:  $\phi$  satisfies

$$\int_0^T \int_{\mathbb{R}} |\phi(t, y)| M(dt \times dy) < \infty \quad \text{p.s.}$$

In this case, the stochastic integral of  $\phi$  with respect to  $M = \sum \delta_{(T_i, y_i)}$  is defined as the absolutely convergent sum

$$\int_0^t \int_{\mathbb{R}} \phi(t, y) M(dt \times dy) := \sum_{i: T_i \leq t} \phi(T_i, y_i).$$

Case 2:  $\phi$  is square integrable, that is, it satisfies

$$E \int_0^T \int_{\mathbb{R}} \phi^2(t, y) \mu(dt \times dy) < \infty$$

In this case, the construction is more involved, since we need to use the  $L^2$  theory and continuous extension once again. We define simple predictable functions  $\phi : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  via

$$\phi(t, y) = \sum_{j=1}^m \phi_{0j} 1_{t=0} 1_{A_j}(y) + \sum_{i=1}^n \sum_{j=1}^m \phi_{ij} 1_{(T_i, T_{i+1}]}(t) 1_{A_j}(y),$$

where  $T_0 = 0$ ,  $(T_i)_{i \geq 1}$  is a sequence of stopping times; for all  $j$ ,  $A_j \in \mathcal{B}(\mathbb{R})$  is such that  $\mu([0, t] \times A_j) < \infty$  for all  $t$ ; and for all  $i$  and  $j$ ,  $\phi_{ij}$  is bounded and  $\mathcal{F}_{T_i}$ -measurable. The stochastic integral of a simple predictable function with respect to  $M$  is defined by

$$\int_0^t \int_{\mathbb{R}} \phi(t, y) M(dt \times dy) := \sum_{i: T_i \leq t} \phi(T_i, y_i) \equiv \sum_{i, j=1}^{n, m} \phi_{ij} M((T_i \wedge t, T_{i+1} \wedge t] \times A_j)$$

In a similar fashion, we can define the integral with respect to the compensated measure  $\tilde{M} = M - \mu$ :

$$\begin{aligned} X_t &= \int_0^t \int_{\mathbb{R}} \phi(t, y) \tilde{M}(dt \times dy) \\ &:= \sum_{i, j=1}^{n, m} \phi_{ij} \{M((T_i \wedge t, T_{i+1} \wedge t] \times A_j) - \mu((T_i \wedge t, T_{i+1} \wedge t] \times A_j)\} \end{aligned}$$

This process is a martingale and satisfies the “isometry relation”:

$$E[X_T^2] = E \int_0^T \int_{\mathbb{R}} \phi^2(t, y) \mu(dt \times dy).$$

This isometry allows to extend the notion of stochastic integral with respect to a compensated Poisson random measure to square integrable predictable functions. Next, the localization procedure can be used to extend the definition to all functions  $\phi$  adapted and left-continuous in  $t$  and measurable in  $y$ , such that the process

$$A_t := \int_0^t \int_{\mathbb{R}} \phi^2(s, y) \mu(ds \times dy)$$

is locally integrable.

The stochastic integral with respect to a Poisson random measure is more general than that with respect to a Poisson process: if  $S$  is a piecewise constant Lévy process,

$$\int_0^T \phi_t dS_t = \sum \phi_t \Delta S_t = \int_0^T \int_{\mathbb{R}} \phi_t y J_S(dt \times dy),$$

that is, the integral with respect to a process can be written as the integral of a specific function with respect to the jump measure of the process.

The stochastic integral with respect to a Poisson random measure allows us to define a new class of processes, which extends the notion of the Lévy process, while still preserving an easy-to-understand mathematical structure: many authors call this class Lévy-Itô processes. Recall that a Lévy process satisfies (with a little change of notation)

$$X_t = \mu t + \sigma W_t + \int_0^t \int_{|x|>1} x M(ds \times dx) + \int_0^t \int_{|x|\leq 1} x \tilde{M}(ds \times dx),$$

where  $M$  is a Poisson random measure with intensity  $dt \times \nu$ . For a Lévy-Itô process, the coefficients can be non-constant and even random:

$$\begin{aligned} X_t &= \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \\ &+ \int_0^t \int_{|x|>1} \gamma_s(x) M(ds \times dx) + \int_0^t \int_{|x|\leq 1} \gamma_s(x) \tilde{M}(ds \times dx), \end{aligned} \quad (14)$$

where  $\mu$  and  $\sigma$  are adapted locally bounded processes and  $\gamma_t(x)$  is an adapted random function, left-continuous in  $t$ , measurable in  $x$ , such that the process:

$$\int_{|x|\leq 1} \gamma_t^2(x) \nu(dx)$$

is locally bounded.

The class of Lévy-Itô processes enjoys better stability properties than that of Lévy processes: if  $(X_t)$  is a Lévy-Itô process then for every function  $f \in C^2$ ,  $(f(X_t))$  is also a Lévy-Itô process.

When  $\int_{|x|>1} |\gamma_t(x)| \nu(dx)$  is also locally bounded, the process  $X$  can be decomposed onto a 'martingale part' and a 'drift part':

$$X_t = \int_0^t (\mu_s + \int_{|x|>1} \gamma_s(x) \nu(dx)) ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} \gamma_s(x) \tilde{M}(ds \times dx),$$

and in the purely martingale case, often found in applications,

$$X_t = \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} \gamma_s(x) \tilde{M}(ds \times dx), \quad (15)$$

and the isometry relation holds:

$$E[X_T^2] = E \left[ \int_0^T \sigma_t^2 dt \right] + E \left[ \int_0^T \int_{\mathbb{R}} \gamma_t^2(x) \nu(dx) dt \right]. \quad (16)$$

### 4.3 Change of variable formula for Lévy-Itô processes

In the absence of jumps, the change of variable formula (Itô formula) for a function  $f \in C^2$  takes the form

$$f(X_T) = f(X_0) + \int_0^T f'(X_t) dX_t + \frac{1}{2} \int_0^T f''(X_t) \sigma_t^2 dt.$$

When the process has a finite number of jumps on  $[0, T]$ , one can write  $X_t := X_t^c + \sum_{s \leq t} \Delta X_s$  and apply the same formula between the jump times:

$$f(X_T) = f(X_0) + \int_0^T f'(X_t) dX_t^c + \frac{1}{2} \int_0^T f''(X_t) \sigma_t^2 dt + \sum_{t \leq T: \Delta X_t \neq 0} \{f(X_t) - f(X_{t-})\}.$$

When the number of jumps is infinite, the latter sum may diverge, but we still have

$$\begin{aligned} f(X_T) = f(X_0) + \int_0^T f'(X_{t-}) dX_t + \frac{1}{2} \int_0^T f''(X_t) \sigma_t^2 dt \\ + \sum_{t \leq T: \Delta X_t \neq 0} \{f(X_t) - f(X_{t-}) - f'(X_{t-}) \Delta X_t\}. \end{aligned} \quad (17)$$

To make the decomposition (14) appear and show that the class of Lévy-Itô processes is stable with respect to transformations with  $C^2$  functions, we rewrite the above expression as follows:

$$\begin{aligned} f(X_T) = f(X_0) + \int_0^T \{ \mu_t f'(X_t) + \frac{1}{2} \sigma_t^2 f''(X_t) \\ + \int_{|x| \leq 1} (f(X_t + \gamma_t(x)) - f(X_t) - \gamma_t(x) f'(X_t)) \nu(dx) \} dt \\ + \int_0^T f'(X_t) \sigma_t dW_t + \int_0^T \int_{|x| \leq 1} (f(X_{t-} + \gamma_t(x)) - f(X_{t-})) \tilde{M}(dt \times dx) \\ + \int_0^T \int_{|x| > 1} (f(X_{t-} + \gamma_t(x)) - f(X_{t-})) M(dt \times dx) \end{aligned}$$

**Exercise 11.** Show that for a simple predictable process  $\phi$  of the form (12), and a square integrable martingale  $M$ ,  $\int_0^\cdot \phi_t dM_t$  is also a martingale and

$$E \left( \int_0^T \phi_t dM_t \right)^2 \leq \sup_{0 \leq t \leq T, \omega \in \Omega} \phi_t^2 E[M_T^2].$$

**Hint:** Use Doob's optional sampling theorem. Let  $(X_t)$  be an  $(\mathcal{F}_t)$ -martingale and let  $S$  and  $T$  be bounded stopping times with  $S \leq T$  a.s. Then,

$$E[X_T | \mathcal{F}_S] = X_S, \quad p.s.$$

**Exercise 12.** The quadratic variation or 'square bracket' of a semimartingale can be defined by

$$[X]_t := X_t^2 - X_0^2 - 2 \int_0^t X_{s-} dX_s.$$

- Compute the quadratic variation for a general Lévy-Itô process, a Lévy process and a Poisson process.
- Show that if  $X$  is a Lévy-Itô process such that  $[X]_t \equiv t$  and  $X$  is a martingale then  $X$  is the standard Brownian motion.

**Exercise 13.** Let  $X$  be a Lévy-Itô process of the form (14), whose coefficients  $\mu$ ,  $\sigma$  and  $\gamma$  are deterministic and bounded. Applying the change of variable formula (17) to the function  $f(x) = e^{iux}$ , show that the characteristic function of  $X_T$  is given by a generalized version of the Lévy-Khintchine formula.

**Exercise 14.** Let  $X$  be a Lévy-Itô process of the form (14) such that

$$\mu_t + \frac{\sigma_t^2}{2} + \int_{\mathbb{R}} (e^{\gamma_t(x)} - 1 - \gamma_t(x)1_{|x| \leq 1}) \nu(dx) = 0$$

a.s. for all  $t$ . Using the change of variable formula (17), show that  $e^{X_t}$  can be written in the form (15) with coefficients to be defined.

Assuming that  $\sigma_t$  and  $\int_{\mathbb{R}} (e^{\gamma_t(x)} - 1)^2 \nu(dx)$  are a.s bounded by a constant  $C$ , use the isometry relation (16) and Gronwall's lemma to show that  $(e^{X_t})$  is a square integrable martingale.

**Gronwall's lemma:** Let  $\phi$  be a positive locally bounded function on  $\mathbb{R}^+$  such that

$$\phi(t) \leq a + b \int_0^t \phi(s) ds$$

for all  $t$  and two constants  $a$  and  $b \geq 0$ . Then  $\phi(t) \leq ae^{bt}$ .

## 5 Stochastic exponential of a jump process

**Proposition 7** (Stochastic exponential). *Let  $(X_t)_{t \geq 0}$  be a Lévy-Itô process with volatility coefficient  $\sigma$ . There exists a unique cadlag process  $(Z)_{t \geq 0}$  such that*

$$dZ_t = Z_{t-} dX_t \quad Z_0 = 1. \quad (18)$$

$Z$  is given by:

$$Z_t = e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} \prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}. \quad (19)$$

*Proof.* Let

$$V_t = \prod_{0 \leq s \leq t; \Delta X_s \neq 0} (1 + \Delta X_s) e^{-\Delta X_s}.$$

The first step is to show that this process exists and is of finite variation. We decompose  $V$  into a product of two terms:  $V_t = V'_t V''_t$ , where

$$V'_t = \prod_{\substack{0 \leq s \leq t \\ |\Delta X_s| \leq 1/2}} (1 + \Delta X_s) e^{-\Delta X_s} \quad \text{and} \quad V''_t = \prod_{\substack{0 \leq s \leq t \\ |\Delta X_s| > 1/2}} (1 + \Delta X_s) e^{-\Delta X_s}.$$

$V''$  for every  $t$  is a product of finite number of factors, so it is clearly of finite variation and there are no existence problems.  $V'$  is positive and we can consider its logarithm.

$$\ln V'_t = \sum_{0 \leq s \leq t; |\Delta X_s| \leq 1/2} (\ln(1 + \Delta X_s) - \Delta X_s).$$

Note that each term of this sum satisfies

$$0 > \ln(1 + \Delta X_s) - \Delta X_s > -\Delta X_s^2.$$

Therefore, the series is decreasing and bounded from below by  $-\sum_{0 \leq s \leq t} \Delta X_s^2$ , which is finite for every Lévy-Itô process. Hence,  $(\ln V'_t)$  exists and is a decreasing process. This entails that  $(V_t)$  exists and has trajectories of finite variation.

The second step is to apply the Itô formula to the function  $Z_t \equiv f(t, X_t, V_t) \equiv e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t$ . This yields (in differential form)

$$\begin{aligned} dZ_t &= -\frac{\sigma_t^2}{2} e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t dt + e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_{t-} dX_t + e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} dV_t \\ &+ \frac{\sigma_t^2}{2} e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t dt + e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_t - e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_{t-} \\ &- e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} V_{t-} \Delta X_t - e^{X_t - \frac{1}{2} \int_0^t \sigma_s^2 ds} \Delta V_t. \end{aligned}$$

Now observe that since  $V_t$  is a pure jump process,

$$dV_t \equiv \Delta V_t = V_{t-} (e^{\Delta X_t} (1 + \Delta X_t) - 1).$$

Substituting this into the above equality and making all the cancellations yields the Equation (18).

To understand why the solution is unique, observe that if  $(Z_t^{(1)})$  and  $(Z_t^{(2)})$  satisfy the Equation (18), then their difference  $\tilde{Z}_t = Z_t^{(1)} - Z_t^{(2)}$  satisfies the same equation with initial condition  $\tilde{Z}_0 = 0$ . From the form of this equation, it is clear that if the solution is equal to zero at some point, it will remain zero.  $\square$

$Z$  is called the stochastic exponential or the Doléans-Dade exponential of  $X$  and is denoted by  $Z = \mathcal{E}(X)$ .

**Relation between ordinary and stochastic exponential** It is clear from the above results that the ordinary exponential and the stochastic exponential of a Lévy process are two different notions: they do not correspond to the same stochastic process. In fact, contrary to the ordinary exponential  $\exp(X_t)$ , which is obviously a positive process, the stochastic exponential  $Z = \mathcal{E}(X)$  is not necessarily positive. It is easy to see that the stochastic exponential is always nonnegative if all jumps of  $X$  are greater than  $-1$ , or, equivalently,  $\nu((-\infty, -1]) = 0$ .

It is therefore natural to ask, which of the two processes is more suitable for modeling price dynamics. The following result, due to Goll and Kallsen [35], shows that the two approaches are equivalent: if  $Z > 0$  is the stochastic exponential of a Lévy process it is also the ordinary exponential of another Lévy process and vice versa. Therefore, the two operations, although they produce different objects when applied to the same Lévy process, end up by giving us the *same class of positive processes*.



**Proposition 8** (Relation between ordinary and stochastic exponentials).

1. Let  $(X_t)_{t \geq 0}$  be a real valued Lévy process with Lévy triplet  $(\sigma^2, \nu, \gamma)$  and  $Z = \mathcal{E}(X)$  its stochastic exponential. If  $Z > 0$  a.s. then there exists another Lévy process  $(L_t)_{t \geq 0}$  with triplet  $(\sigma_L^2, \nu_L, \gamma_L)$  such that  $Z_t = e^{L_t}$  where

$$L_t = \ln Z_t = X_t - \frac{\sigma^2 t}{2} + \sum_{0 \leq s \leq t} \{\ln(1 + \Delta X_s) - \Delta X_s\}. \quad (20)$$

$$\sigma_L = \sigma,$$

$$\nu_L(A) = \nu(\{x : \ln(1+x) \in A\}) = \int 1_A(\ln(1+x)) \nu(dx), \quad (21)$$

$$\gamma_L = \gamma - \frac{\sigma^2}{2} + \int \nu(dx) \{\ln(1+x) 1_{[-1,1]}(\ln(1+x)) - x 1_{[-1,1]}(x)\}.$$

2. Let  $(L_t)_{t \geq 0}$  be a real valued Lévy process with Lévy triplet  $(\sigma_L^2, \nu_L, \gamma_L)$  and  $S_t = \exp L_t$  its exponential. Then there exists a Lévy process  $(X_t)_{t \geq 0}$  such that  $S_t$  is the stochastic exponential of  $X$ :  $S = \mathcal{E}(X)$  where

$$X_t = L_t + \frac{\sigma^2 t}{2} + \sum_{0 \leq s \leq t} \{e^{\Delta L_s} - 1 - \Delta L_s\}. \quad (22)$$

The Lévy triplet  $(\sigma^2, \nu, \gamma)$  of  $X$  is given by:

$$\sigma = \sigma_L,$$

$$\nu(A) = \nu_L(\{x : e^x - 1 \in A\}) = \int 1_A(e^x - 1) \nu_L(dx), \quad (23)$$

$$\gamma = \gamma_L + \frac{\sigma_L^2}{2} + \int \nu_L(dx) \{(e^x - 1) 1_{[-1,1]}(e^x - 1) - x 1_{[-1,1]}(x)\}.$$

*Proof.* 1. The condition  $Z > 0$  a.s. is equivalent to  $\Delta X_s > -1$  for all  $s$  a.s., so taking the logarithm is justified here. In the proof of Proposition 7 we have seen that the sum  $\sum_{0 \leq s \leq t} \{\ln(1 + \Delta X_s) - \Delta X_s\}$  converges and is a finite variation process. Then it is clear that  $L$  is a Lévy process and that  $\sigma_L = \sigma$ . Moreover,  $\Delta L_s = \ln(1 + \Delta X_s)$  for all  $s$ . This entails that

$$J_L([0, t] \times A) = \int_{[0, t] \times \mathbb{R}} 1_A(\ln(1+x)) J_X(ds dx)$$

and also  $\nu_L(A) = \int 1_A(\ln(1+x))\nu(dx)$ . It remains to compute  $\gamma_L$ . Substituting the Lévy-Itô decomposition for  $(L_t)$  and  $(X_t)$  into (20), we obtain

$$\begin{aligned} \gamma_L t - \gamma t + \frac{\sigma^2 t}{2} &+ \int_{s \in [0,t], |x| \leq 1} x \tilde{J}_L(ds dx) + \int_{s \in [0,t], |x| > 1} x J_L(ds dx) \\ &- \int_{s \in [0,t], |x| \leq 1} x \tilde{J}_X(ds dx) - \int_{s \in [0,t], |x| > 1} x J_X(ds dx) \\ &- \sum_{0 \leq s \leq t} \{\ln(1 + \Delta X_s) - \Delta X_s\} = 0. \end{aligned}$$

Observing that

$$\begin{aligned} &\int_{s \in [0,t], |x| \leq 1} x (J_L(ds dx) - J_X(ds dx)) \\ &= \sum_{0 \leq s \leq t} (\Delta X_s 1_{[-1,1]}(\Delta X_s) - \ln(1 + \Delta X_s) 1_{[-1,1]}(\ln(1 + \Delta X_s))) \end{aligned}$$

converges, we can split the above expression into jump part and drift part, both of which must be equal to zero. For the drift part we obtain:

$$\gamma_L - \gamma + \frac{\sigma^2}{2} - \int_{-1}^1 \{x\nu_L(dx) - x\nu(dx)\} = 0,$$

which yields the correct formula for  $\gamma_L$  after a change of variable.

2. The jumps of  $S_t$  are given by  $\Delta S_t = S_{t-}(\exp(\Delta L_t) - 1)$ . If  $X$  is a Lévy process such that  $S = \mathcal{E}(X)$  then since  $dS_t = S_{t-}dX_t$  then  $\Delta S_t = S_{t-}\Delta X_t$  so  $\Delta X_t = \exp(\Delta L_t) - 1$  so  $\nu$  is given by (23). In particular  $\Delta X_t > -1$  a.s. and it is easily verified that  $\ln \mathcal{E}(X)$  is a Lévy process with characteristics matching those of  $L$  only if  $X$  has characteristics given by (23). Conversely if  $X$  is a Lévy process with characteristics given by (23), using (19) we can verify as above that  $\mathcal{E}(X) = \exp L_t$ .  $\square$

In view of this result and given that the formulas involving the ordinary exponential are usually more tractable, the latter is more commonly used for asset price modeling. However, in some situations, the stochastic exponential may be a better choice, as shown by the following example.

**CPPI strategy in the presence of jumps** The CPPI (constant proportion portfolio insurance) is a portfolio insurance strategy which allows (in theory) to keep the portfolio value above a fixed level, while still preserving some upside potential in case of a favorable market evolution.

To fix the ideas, suppose that the portfolio manager has promised to the investor a guaranteed capital of  $N$  at maturity  $T$  ( $N$  can be greater or smaller than the initial investment). To achieve this, the portfolio value  $V_t$  must remain at each date  $t$  above the floor  $B_t$ , which is equal to the price of the zero-coupon bond with notional  $N$  and maturity  $T$ . The difference  $C_t = V_t - B_t$  is called *cushion* and the CPPI strategy uses the following algorithm:

- At each date  $t$ , if  $V_t > B_t$ , invest  $mC_t$  in the risky asset, where  $m > 1$  is called the *multiplier*, and the rest into zero-coupon bonds with maturity  $T$ .
- If  $V_t \leq B_t$ , invest all wealth in zero-coupon bonds with maturity  $T$ .

If the price of the risky asset is a continuous process, the portfolio value remains above the floor, and the dynamics of the cushion is given by

$$\frac{dC_t}{C_t} = m \frac{dS_t}{S_t} + (1 - m)rdt,$$

where  $r$  is the interest rate. In the Black-Scholes model,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

this equation can be solved explicitly, and we obtain the final portfolio value

$$V_T = N + (V_0 - Ne^{-rT}) \exp \left( rT + m(\mu - r)T + m\sigma W_T - \frac{m^2\sigma^2T}{2} \right),$$

whose expectation is

$$E[V_T] = N + (V_0 - Ne^{-rT}) \exp(rT + m(\mu - r)T).$$

If  $\mu > r$ , there seems to be a paradox: there is no risk and the expected return can be made arbitrarily large by choosing  $m$  big enough. This paradox is easily solved if the price trajectories are allowed to jump, since in this case, if the multiplier increases, the loss probability increases as well.

Let

$$\frac{dS_t}{S_{t-}} = rdt + dZ_t,$$

where  $Z$  is a Lévy process and let  $\tau = \inf\{t : V_t \leq B_t\}$  be the first date when the portfolio passes below the floor (it is possible that  $\tau = \infty$ ). Before  $\tau$ , the cushion satisfies

$$\frac{dC_t}{C_{t-}} = mdZ_t + rdt,$$

and the discounted cushion  $C_t^* := \frac{C_t}{e^{rt}}$  is therefore given by

$$C_t^* = \mathcal{E}(mZ)_t, \quad t < \tau.$$

After  $\tau$ , the entire portfolio is invested into the risk-free asset, which means that the discounted cushion remains constant. Therefore,

$$C_t^* = \mathcal{E}(mZ)_{t \wedge \tau}.$$

The loss occurs if at some date  $t \leq T$ ,  $C_t^* \leq 0$ , which can happen if and only if  $Z$  has a jump in the interval  $[0, T]$  whose size is less than  $-1/m$ . We then get (see exercise 7)

$$P[\exists t \in [0, T] : V_t \leq B_t] = 1 - \exp\left(-T \int_{-\infty}^{-1/m} \nu(dx)\right).$$

See [23] for details on portfolio insurance in the presence of jumps.

**Exercise 15.** Show that if  $X$  and  $Y$  are two Lévy processes and  $Y$  has no diffusion component then

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + \sum \Delta X \Delta Y)$$

**Exercise 16** (Martingale property of the stochastic exponential). Let  $X$  be a Lévy process and a martingale. Show that  $\mathcal{E}(X)$  is a martingale as well.

**Hint:** Represent  $X$  as the sum of a compound Poisson process and a Lévy process  $X'$  such that  $|\Delta X'| < 1$ . Use Proposition 8 and the previous exercise.

**Exercise 17.** Use the previous exercise to show that for every Lévy process  $X$  with  $E[|X_t|] < \infty$ ,

$$E[\mathcal{E}(X)_t] = e^{E[X_t]}, \quad t > 0.$$

## 6 Exponential Lévy models

The Black-Scholes model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

can be equivalently rewritten in the exponential form  $S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$ . This gives us two possibilities to construct an exponential Lévy model starting from a (one-dimensional) Lévy process  $X$ : using the stochastic differential equation:

$$\frac{dS_t^{sde}}{S_{t-}^{sde}} = r dt + dX_t, \quad (24)$$

or using the ordinary exponential

$$S_t^{exp} = S_0^{exp} e^{rt + X_t}, \quad (25)$$

where we explicitly included the interest rate  $r$  (assumed constant) in the formulas, to simplify notation later on. The subscripts *sde* for stochastic differential equation and *exp* for exponential, used here to emphasize the fact that  $S^{sde}$  and  $S^{exp}$  are different processes, will be omitted later on when there is no ambiguity. Sometimes it will be convenient to discount the price processes with the numéraire  $B(t, T) = e^{-r(T-t)}$  for some fixed maturity  $T$ . In this case  $\hat{S}_t := \frac{S_t}{B(t, T)} = e^{r(T-t)} S_t$  and the equations become

$$\frac{d\hat{S}_t}{\hat{S}_{t-}} = dX_t \quad (26)$$

$$\text{or } \hat{S}_t = \hat{S}_0 e^{X_t}, \quad (27)$$

**Examples of exponential Lévy models** Parametric exponential Lévy models fall into two categories. In the first category, called *jump-diffusion* models, the “normal” evolution of prices is given by a diffusion process, punctuated by jumps at random intervals. Here the jumps represent rare events — crashes and large drawdowns. Such an evolution can be represented by a Lévy process with a nonzero Gaussian component and a jump part with finitely many jumps:

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad (28)$$

where  $(Y_i)$  are i.i.d. and  $N$  is a Poisson process.

In the *Merton model* (see Example 1), which is the first model of this type, suggested in the literature, jumps in the log-price  $X$  are assumed to have a Gaussian distribution:  $Y_i \sim N(\mu, \delta^2)$ .

In the *Kou model* [42], jump sizes are distributed according to an asymmetric Laplace law with a density of the form

$$\nu_0(dx) = [p\lambda_+ e^{-\lambda_+ x} \mathbf{1}_{x>0} + (1-p)\lambda_- e^{-\lambda_- |x|} \mathbf{1}_{x<0}] dx \quad (29)$$

with  $\lambda_+ > 0$ ,  $\lambda_- > 0$  governing the decay of the tails for the distribution of positive and negative jump sizes and  $p \in [0, 1]$  representing the probability of an upward jump. The probability distribution of returns in this model has semi-heavy (exponential) tails.

The second category consists of models with an infinite number of jumps in every interval, called *infinite activity* or *infinite intensity* models. In these models, one does not need to introduce a Brownian component since the dynamics of jumps is already rich enough to generate nontrivial small time behavior [14].

The variance gamma process [15, 48] (see Example 2) is obtained by time-changing a Brownian motion with a gamma subordinator and has the characteristic exponent of the form:

$$\psi(u) = -\frac{1}{\kappa} \log\left(1 + \frac{u^2 \sigma^2 \kappa}{2} - i\theta \kappa u\right). \quad (30)$$

The density of the Lévy measure of the variance gamma process is given by

$$\nu(x) = \frac{c}{|x|} e^{-\lambda_- |x|} \mathbf{1}_{x<0} + \frac{c}{x} e^{-\lambda_+ x} \mathbf{1}_{x>0}, \quad (31)$$

where  $c = 1/\kappa$ ,  $\lambda_+ = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2} - \frac{\theta}{\sigma^2}$  and  $\lambda_- = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2} + \frac{\theta}{\sigma^2}$ .

To define the *tempered stable process*, introduced by Koponen [41] and also known under the name of CGMY model [14], one specifies directly the Lévy density:

$$\nu(x) = \frac{c_-}{|x|^{1+\alpha_-}} e^{-\lambda_- |x|} \mathbf{1}_{x<0} + \frac{c_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} \mathbf{1}_{x>0} \quad (32)$$

with  $\alpha_+ < 2$  and  $\alpha_- < 2$ .

## 7 The Esscher transform and absence of arbitrage in exponential Lévy models

### 7.1 Measure changes for Lévy processes

To find out whether a given exponential Lévy model is suitable for financial modeling, one needs to ensure that it does not contain arbitrage opportunities, a property which, by the fundamental theorem of asset pricing, is guaranteed by the existence of an equivalent martingale measure. The no arbitrage equivalences for exponential Lévy models were studied in [37, 19, 66] in the one-dimensional unconstrained case and more recently in [40] in the multidimensional case with convex constraints on trading strategies. In this section, we start by reviewing the one-dimensional result, and then provide a multidimensional result (Theorem 3) which is valid in the unconstrained case only but is more explicit than the one in [40] and clarifies the link between the geometric properties of the Lévy measure and the arbitrage opportunities in the model.

In the Black-Scholes model, the unique equivalent martingale measure could be obtained by changing the drift of the Brownian motion. In models with jumps, if the Gaussian component is absent, this is no longer possible, but a much greater variety of equivalent measures can be obtained by altering the distribution of jumps. The following proposition describes the possible measure changes under which a Lévy process remains a Lévy process.

**Proposition 9** (see Sato [59], Theorems 33.1 and 33.2). *Let  $(X, \mathbb{P})$  be a Lévy process on  $\mathbb{R}^d$  with characteristic triplet  $(A, \nu, \gamma)$ ; choose  $\eta \in \mathbb{R}^d$  and  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  with*

$$\int_{\mathbb{R}^d} (e^{\phi(x)/2} - 1)^2 \nu(dx) < \infty. \quad (33)$$

and define

$$U_t := \eta \cdot X^c + \int_0^t \int_{\mathbb{R}^d} (e^{\phi(x)} - 1) \tilde{J}_X(ds dx),$$

where  $X^c$  denotes the continuous martingale (Brownian motion) part of  $X$ , and  $\tilde{J}_X$  is the compensated jump measure of  $X$ .

Then  $\mathcal{E}(U)_t$  is a positive martingale such that the probability measure  $\mathbb{P}'$  defined by

$$\frac{d\mathbb{P}'|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = \mathcal{E}(U)_t, \quad (34)$$

is equivalent to  $\mathbb{P}$  and under  $\mathbb{P}'$ ,  $X$  is a Lévy process with characteristic triplet  $(A, \nu', \gamma')$  where  $\nu' = e^\phi \nu$  and

$$\gamma' = \gamma + \int_{|x| \leq 1} x(\nu' - \nu)(dx) + A\eta. \quad (35)$$

A useful example, which will be the basis of our construction of an equivalent martingale measure is provided by the Esscher transform. Let  $X$  be a Lévy process on  $\mathbb{R}^d$  with characteristic triplet  $(A, \nu, \gamma)$ , and let  $\theta \in \mathbb{R}^d$  be such that  $\int_{|x| > 1} e^{\theta \cdot x} \nu(dx) < \infty$ . Applying a measure transformation of Proposition 9 with  $\eta = \theta$  and  $\phi(x) = \theta \cdot x$ , we obtain an equivalent probability under which  $X$  is a Lévy process with Lévy measure  $\tilde{\nu}(dx) = e^{\theta \cdot x} \nu(dx)$  and third component of the characteristic triplet  $\tilde{\gamma} = \gamma + A\theta + \int_{|x| \leq 1} x(e^{\theta \cdot x} - 1)\nu(dx)$ . Using Proposition 9, the Radon-Nikodym derivative corresponding to this measure change is found to be

$$\frac{d\mathbb{P}'|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = \frac{e^{\theta \cdot X_t}}{E[e^{\theta \cdot X_t}]} = \exp(\theta \cdot X_t - \kappa(\theta)t), \quad (36)$$

where  $\kappa(\theta) := \ln E[\exp(\theta \cdot X_1)] = \psi(-i\theta)$ .

Although the two definitions of an exponential Lévy model, via the ordinary exponential (27) or via the stochastic exponential (26), are equivalent, the set of Lévy processes that lead to arbitrage-free models of the form (27) does not necessarily coincide with the set that yields arbitrage-free models of the form (26). In particular, we shall see that the no-arbitrage conditions for multidimensional stochastic and ordinary exponentials are considerably different. It will be more convenient to find these conditions for models of type (26) first and then deduce the conditions for ordinary exponentials using the transformation  $X_t := \ln \mathcal{E}(Y)_t$ . In the multidimensional case, this transformation must be applied to each component.

In an exponential Lévy model of type (26), the absence of arbitrage is tantamount to the existence of a probability  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  such that  $\mathcal{E}(X)$  is a  $\mathbb{Q}$ -martingale. We will see that when this is the case, it is always possible to find a martingale probability  $\mathbb{Q} \sim \mathbb{P}$  under which  $X$  remains a Lévy process, which means that  $X$  itself must be a  $\mathbb{Q}$ -martingale (cf. Proposition 8.23 in [20]).



## 7.2 One-dimensional models

We start with the one-dimensional case. In the sequel,  $\text{cc}(A)$  denotes the smallest convex cone containing  $A$  and  $\text{ri}(A)$  denotes the relative interior of the set  $A$ , that is, the interior of  $A$  in the smallest linear subspace containing  $A$ . In particular,  $\text{ri}(\{0\}) = \{0\}$ .

**Theorem 2** (Absence of arbitrage in models based on stochastic exponentials, one-dimensional case).

*Let  $(X, \mathbb{P})$  be a real-valued Lévy process on  $[0, T]$  with characteristic triplet  $(\sigma^2, \nu, \gamma)$ . The following statements are equivalent:*

1. *There exists a probability  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  such that  $(X, \mathbb{Q})$  is a Lévy process and a martingale.*
2. *Either  $X \equiv 0$  or  $(X, \mathbb{P})$  is not a.s. monotone.*
3. *One of the following conditions is satisfied:*

- (i)  $\sigma > 0$ .
- (ii)  $\sigma = 0$  and  $\int_{|x| \leq 1} |x| \nu(dx) = \infty$ .
- (iii)  $\sigma = 0$ ,  $\int_{|x| \leq 1} |x| \nu(dx) < \infty$  and  $-b \in \text{ri}(\text{cc}(\text{supp } \nu))$ , where  $b = \gamma - \int_{|x| \leq 1} x \nu(dx)$  is the drift of  $X$ .

Condition 2. implies that if an exponential Lévy model admits an arbitrage, it can be realized by a buy-and-hold strategy (if  $X$  is increasing) or a corresponding short sale (if  $X$  is decreasing).

It is easy to see that condition (iii) above is satisfied if and only if  $\sigma = 0$ ,  $\int_{|x| \leq 1} |x| \nu(dx) < \infty$  and one of the following is true:

- $\nu((-\infty, 0)) > 0$  and  $\nu((0, \infty)) > 0$ .
- $\nu((-\infty, 0)) > 0$  and  $b > 0$ .
- $\nu((0, \infty)) > 0$  and  $b < 0$ .
- The trivial case of a constant process:  $\nu = 0$  and  $b = 0$ .

In other words, when a finite-variation Lévy process has one-sided jumps, it is arbitrage-free if the jumps and the drift point in opposite directions.

Before proceeding with the proof of theorem 2, we will show that for *one-dimensional* exponential Lévy models of the form (27), the no-arbitrage conditions are actually the same as for stochastic exponentials.

**Corollary 3** (Absence of arbitrage in models based on ordinary exponential, one-dimensional case). *Let  $(X, \mathbb{P})$  be a real-valued Lévy process on  $[0, T]$  with characteristic triplet  $(\sigma^2, \nu, \gamma)$ . The following statements are equivalent:*

1. *There exists a probability  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  such that  $(X, \mathbb{Q})$  is a Lévy process and  $e^X$  is a martingale.*
2. *Either  $X \equiv 0$  or  $(X, \mathbb{P})$  is not a.s. monotone.*
3. *One of the following conditions is satisfied:*
  - (i)  $\sigma > 0$ .
  - (ii)  $\sigma = 0$  and  $\int_{|x| \leq 1} |x| \nu(dx) = \infty$ .
  - (iii)  $\sigma = 0$ ,  $\int_{|x| \leq 1} |x| \nu(dx) < \infty$  and  $-b \in \text{ri}(\text{cc}(\text{supp } \nu))$ .

*Proof.* It suffices to show that  $\ln \mathcal{E}(X)$  is monotone if and only if  $X$  is monotone. From [20, Proposition 8.22] it is easy to see that  $\ln \mathcal{E}(X)$  is a finite variation process if and only if  $X$  is a finite variation process. In the finite-variation case, the stochastic exponential has a simple form:

$$\mathcal{E}(X)_t = e^{bt} \prod_{s \leq t} (1 + \Delta X_s),$$

and it is readily seen that the monotonicity properties of  $X$  and  $\log \mathcal{E}(X)$  are the same.  $\square$

*Proof of theorem 2.* We exclude the trivial case  $X \equiv 0$  a.s. which clearly does not constitute an arbitrage opportunity (every probability is a martingale measure).

The equivalence 2  $\iff$  3 follows from [20, Proposition 3.10].

3  $\implies$  1. Define a probability  $\tilde{\mathbb{P}}$  equivalent to  $\mathbb{P}$  by

$$\frac{d\tilde{\mathbb{P}}|_{\mathcal{F}_T}}{d\mathbb{P}|_{\mathcal{F}_T}} = \mathcal{E} \left( \int_0^\cdot \int_{\mathbb{R}} (e^{-x^2} - 1) \tilde{J}_X(ds dx) \right)_T$$

Under  $\tilde{\mathbb{P}}$ ,  $X$  has characteristic triplet  $(\sigma^2, \tilde{\nu}, \tilde{\gamma})$  with  $\tilde{\nu} = e^{-x^2} \nu$  and  $\tilde{\gamma} = \gamma + \int_{|x| \leq 1} x(e^{-x^2} - 1) \nu(dx)$ . It is easy to see that  $E^{\tilde{\mathbb{P}}}[e^{\lambda X_t}] < \infty$  for all  $\lambda \in \mathbb{R}$  and all  $t > 0$ .

Suppose that the convex function  $\lambda \mapsto E^{\tilde{\mathbb{P}}}[e^{\lambda X_1}]$  has a finite minimizer  $\lambda^*$ . Then, using the dominated convergence theorem,  $E^{\tilde{\mathbb{P}}}[X_1 e^{\lambda^* X_1}] = 0$  which implies that  $X$  is a  $\mathbb{Q}$ -martingale with

$$\frac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\tilde{\mathbb{P}}|_{\mathcal{F}_t}} = \frac{e^{\lambda^* X_t}}{E[e^{\lambda^* X_t}]} \quad (\text{Essher transform})$$

To show the existence of a finite minimizer  $\lambda^*$ , it is sufficient to prove that  $E^{\tilde{\mathbb{P}}}[e^{\lambda X_1}] \rightarrow \infty$  as  $\lambda \rightarrow \infty$ , or, equivalently, that the function

$$f(\lambda) = \log E^{\tilde{\mathbb{P}}}[e^{\lambda X_1}] = \frac{\sigma^2}{2} \lambda^2 + \tilde{\gamma} \lambda + \int_{\mathbb{R}} (e^{\lambda x} - 1 - \lambda x 1_{|x| \leq 1}) e^{-x^2} \nu(dx).$$

goes to infinity as  $\lambda \rightarrow \infty$ . In case (i),  $f''(\lambda) \geq \sigma^2$  which means that  $f(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ . In case (ii),

$$f'(\lambda) = \tilde{\gamma} + \int_{|x| > 1} x e^{-x^2} \nu(dx) + \int_{\mathbb{R}} x (e^{\lambda x} - 1) e^{-x^2} \nu(dx),$$

and it is not difficult to check that  $\lim_{\lambda \rightarrow +\infty} f'(\lambda) = +\infty$  and  $\lim_{\lambda \rightarrow -\infty} f'(\lambda) = -\infty$  which means that  $f(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ . In case (iii),

$$f'(\lambda) = b + \int_{\mathbb{R}} x e^{\lambda x} e^{-x^2} \nu(dx),$$

and it is easy to see, by examining one by one the different mutually exclusive cases listed after the statement of the Theorem, that in each of these cases  $f'$  is bounded from below on  $\mathbb{R}$  and therefore once again,  $f(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ .

1  $\Rightarrow$  2. It is clear that a process cannot be a martingale under one probability and a.s. monotone under an equivalent probability, unless it is constant.  $\square$

### 7.3 Multidimensional models

In the multidimensional case, the no arbitrage conditions for ordinary and stochastic exponentials are different. We start with the simpler case of stochastic exponentials.

Let  $(X, \mathbb{P})$  be an  $\mathbb{R}^d$ -valued Lévy process on  $[0, T]$  with characteristic triplet  $(A, \nu, \gamma)$ . To describe the no-arbitrage conditions, we need to separate the finite and infinite variation components of  $X$ . We therefore introduce the

linear subspace  $\mathcal{L} \subseteq \mathbb{R}^d$  containing all vectors  $w \in \mathbb{R}^d$  such that  $w.X$  is a finite variation process. From proposition 3.8 and theorem 4.1 in [20], it follows that

$$\mathcal{L} = \mathcal{N}(A) \cap \left\{ w \in \mathbb{R}^d : \int_{|x| \leq 1} |w.x| \nu(dx) < \infty \right\},$$

where  $\mathcal{N}(A) := \{w \in \mathbb{R}^d : Aw = 0\}$ . Further, denote by  $X^\mathcal{L}$  the projection of  $X$  on  $\mathcal{L}$ .  $X^\mathcal{L}$  is a finite variation Lévy process with triplet  $(0, \nu^\mathcal{L}, \gamma^\mathcal{L})$ , and we denote its drift by

$$b^\mathcal{L} := \gamma^\mathcal{L} - \int_{\mathcal{L} \cap \{x: |x| \leq 1\}} x \nu^\mathcal{L}(dx).$$

**Theorem 3** (Absence of arbitrage in models based on stochastic exponential, multidimensional case). *Let  $(X, \mathbb{P})$  be an  $\mathbb{R}^d$ -valued Lévy process on  $[0, T]$  with characteristic triplet  $(A, \nu, \gamma)$ . The following statements are equivalent:*

1. *There exists a probability  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  such that  $(X, \mathbb{Q})$  is a Lévy process and  $(X^i)$  is a  $\mathbb{Q}$ -martingale for all  $i$ .*
2. *For every  $w \in \mathbb{R}^d$ , the process  $w.X$  satisfies one of the equivalent conditions 2. or 3. of Theorem 2.*
3.  *$-b^\mathcal{L} \in \text{ri}(\text{cc}(\text{supp } \nu^\mathcal{L}))$ .*

Let us comment on the equivalent conditions of the above theorem.

To understand condition 2., assume that for some  $w \in \mathbb{R}^d$ , the process  $w.X$  does not satisfy the equivalent conditions of Theorem 2, meaning that it is either strictly increasing or strictly decreasing. Consider a portfolio where the relative proportions of different assets are kept constant and equal to  $w_i$ . The proportions may, of course, change when the underlying assets jump, but it is assumed that they are readjusted to their constant values immediately after the jump. Such a strategy is called a fixed-mix strategy. The discounted value  $\hat{V}_t$  of such a portfolio satisfies the SDE  $d\hat{V}_t = \hat{V}_{t-} w.dX_t$ , and therefore either this portfolio constitutes an arbitrage strategy or an arbitrage strategy can be obtained by shorting this portfolio. Condition 2 thus implies that a multidimensional exponential Lévy model is arbitrage-free if and only if there are no fixed-mix arbitrage strategies.

The third condition is a concise characterization of arbitrage-free exponential Lévy models in terms of their characteristic triplets. This condition

is always satisfied if the process  $X$  has no finite-variation components: in this case  $\mathcal{L} = \{0\}$  and condition 3. reduces to  $0 \in \{0\}$ . If the process is of finite variation, this condition reduces to  $-b \in \text{ri}(\text{cc}(\text{supp } \nu))$ , that is, the drift and the finite variation jumps must point in opposite directions.

*Proof of theorem 3.*  $1 \Rightarrow 2$  is readily obtained by an application of Theorem 2 to the process  $w.X$ .

$2 \Rightarrow 1$ . By an argument similar to the one in the proof of Theorem 2, we can suppose without loss of generality that for all  $\lambda \in \mathbb{R}^d$ ,  $E[e^{\lambda \cdot X_1}] < \infty$ . The function  $f : \lambda \mapsto E[e^{\lambda \cdot X_1}] < \infty$  is then a proper convex differentiable function on  $\mathbb{R}^d$  and if  $\lambda^*$  is a minimizer of this function,  $E[X_1^i e^{\lambda^* \cdot X_1}] = 0$  for all  $i = 1, \dots, d$  and we can define an equivalent martingale measure  $\mathbb{Q}$  using the Esscher transform

$$\frac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} := \frac{e^{\lambda^* \cdot X_1}}{E[e^{\lambda^* \cdot X_1}]}.$$

Suppose that  $w.X$  is a Lévy process satisfying conditions 2 or 3 of Theorem 2. Then it follows from the proof of this theorem that  $w.X$  is either constant or  $\lim_{\lambda \rightarrow \infty} E[e^{\lambda w \cdot X_1}] = \infty$ . Hence, the function  $f$  is constant along every recession direction, which implies that  $f$  attains its minimum (Theorem 27.1 in [55]).

$2 \Rightarrow 3$ . Suppose  $-b^\mathcal{L} \notin \text{ri}(\text{cc}(\text{supp } \nu^\mathcal{L}))$ . Then  $-b^\mathcal{L}$  can be weakly separated from  $\text{cc}(\text{supp } \nu^\mathcal{L})$  by a hyperplane contained in  $\mathcal{L}$ , passing through the origin, and which does not contain  $-b^\mathcal{L}$  or  $\text{cc}(\text{supp } \nu^\mathcal{L})$  completely (theorems 11.3 and 11.7 in [55]). This means that there exists  $w \in \mathcal{L}$  such that

$$b^\mathcal{L} \cdot w \geq 0 \quad \text{and} \quad x \cdot w \geq 0, \quad \forall x \in \text{supp } \nu^\mathcal{L}$$

with either  $b^\mathcal{L} \cdot w > 0$  or  $x \cdot w > 0$  for some  $x \in \text{supp } \nu^\mathcal{L}$ . In this case,  $\text{ri}(\text{cc}(\text{supp } \nu^w))$  is either  $\{0\}$  or  $(0, \infty)$ , where the measure  $\nu^w$  is defined by  $\nu^w(A) := \nu^\mathcal{L}(\{x \in \mathcal{L} : w \cdot x \in A\})$ . If  $b^\mathcal{L} \cdot w > 0$ , this implies that  $-b^\mathcal{L} \cdot w \notin \text{ri}(\text{cc}(\text{supp } \nu^w))$ . If  $b^\mathcal{L} \cdot w = 0$  then necessarily  $x \cdot w > 0$  for some  $x \in \text{supp } \nu^\mathcal{L}$  which means that in this case  $\text{ri}(\text{cc}(\text{supp } \nu^w)) = (0, \infty)$  and once again  $-b^\mathcal{L} \cdot w \notin \text{ri}(\text{cc}(\text{supp } \nu^w))$ . In both cases, we have obtained a contradiction with 2.

$3 \Rightarrow 2$ . Assume that  $-b^\mathcal{L} \in \text{ri}(\text{cc}(\text{supp } \nu^\mathcal{L}))$  and let  $w \in \mathbb{R}^d$ . If  $w \notin \mathcal{L}$  then  $w.X$  has infinite variation and the claim is shown. Assume that  $w \in \mathcal{L}$  and let  $R^w := \text{ri}(\text{cc}(\text{supp } \nu^w))$ .  $R^w$  can be equal to  $\mathbb{R}$ , half-axis or a single point  $\{0\}$ . If  $R^w = \mathbb{R}$ , there is nothing to prove. In the two other cases,

$-b.w \notin R^w$  means that  $w$  weakly separates  $-b^\mathcal{L}$  from  $\text{cc}(\text{supp } \nu^\mathcal{L})$  in such a way that either  $b^\mathcal{L}.w > 0$  or  $x.w > 0$  for some  $x \in \text{supp } \nu^\mathcal{L}$ , which is a contradiction with 3.  $\square$

**Case of models based on ordinary exponentials** In multidimensional models of type (27), contrary to the one-dimensional case, the no-arbitrage conditions are not the same as in models of type (26), as the following example illustrates. Let  $N$  be a standard Poisson process with intensity  $\lambda$  and define

$$\begin{aligned} X_t^1 &= N_t - \lambda(e - 1)t; & S_t^1 &= S_0^1 e^{X_t^1}. \\ X_t^2 &= -N_t - \lambda(e^{-1} - 1)t; & S_t^2 &= S_0^2 e^{X_t^2}. \end{aligned}$$

The linear combination  $X^1 + X^2$  is nonconstant and monotone, however the model is arbitrage-free since  $S^1$  and  $S^2$  are easily seen to be martingales.

To check whether a model of type (27) based on an  $\mathbb{R}^d$ -valued Lévy process  $X$  is arbitrage-free, one should construct the equivalent model of type (26) by computing  $Y_t^i = \ln \mathcal{E}(X^i)_t$  for  $i = 1, \dots, d$ , and then check the conditions of Theorem 3 for the process  $Y$ . The following remarks can facilitate this task in some commonly encountered cases:

- The space  $\mathcal{L}$  of finite variation components is invariant under the mapping  $\ln \mathcal{E}$ ; therefore, if the process  $X$  does not have finite variation components, the model is arbitrage-free.
- If the Lévy measure  $\nu^X$  of  $X$  has full support then the Lévy measure  $\nu^Y$  of  $Y$  satisfies  $\text{cc}(\text{supp } \nu^Y) = \mathbb{R}^d$ , which implies that the model is arbitrage-free.
- If an orthant is contained in the support of  $\nu^X$ , this orthant will also be contained in  $\text{cc}(\text{supp } \nu^Y)$ .

**Exercise 18.** Let  $(X, P)$  be a Lévy process with characteristic triple  $(A, \nu, \gamma)$ , and let  $Q$  be a probability measure defined by

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_t} = \frac{e^{\theta X_t}}{E[e^{\theta X_t}]}$$

for  $\theta \in \mathbb{R}$  such that  $E[e^{\theta X_t}] < \infty$ . Compute the characteristic triple of  $X$  under  $Q$ .

**Exercise 19.**

- Let  $X$  be a Lévy process and let  $f : \mathbb{R} \rightarrow (0, \infty)$ . What condition must be imposed on the function  $f$  for the sum

$$\sum_{t \in [0,1]: \Delta X_t \neq 0} f(\Delta X_t)$$

to converge a.s.?

- Let  $(X, P)$  be a Lévy process with Lévy measure  $\frac{1}{|x|^{1+\alpha}}$  and let  $(X, Q)$  be a Lévy process with Lévy measure  $\frac{1}{|x|^{1+\alpha'}}$  (with  $\alpha > 0$  and  $\alpha' > 0$ ). Use the previous question to show that  $P \sim Q$  implies  $\alpha = \alpha'$ . Check your result using Proposition 9.

## 8 European options in exp-Lévy models

Given the results of section 7, in any “reasonable” exponential Lévy model we can assume that there exists a probability measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  such that the discounted prices of all assets are  $\mathbb{Q}$ -martingales. In practice, this measure is usually found by calibrating the exponential Lévy model to market quoted prices of European options [7, 21], and the first step in using the model is therefore to obtain fast pricing algorithms for European calls and puts.

Prices of European options in exponential Lévy models can be computed directly from the characteristic function of  $X$  which is explicitly known from the Lévy-Khintchine formula. This idea was first introduced to finance by Carr and Madan [15] (for European calls and puts) and later extended and generalized by many authors including [54, 47, 46, 27]. The result given below is a slight generalization of the one in [27], allowing both discontinuous payoff functions and Lévy processes without a bounded density, such as variance gamma.

We start with a one-dimensional risk-neutral exponential Lévy model in the form (27). Under the risk-neutral probability, the process  $e^X$  must therefore be a martingale, a condition which can be expressed in terms of the characteristic triplet of  $X$ :

$$\gamma + \frac{A}{2} + \int_{\mathbb{R}} (e^y - 1 - y1_{|y| \leq 1}) \nu(dy) = 0.$$

We consider a European option with pay-off  $G(S_T) = G(\hat{S}_T)$  at time  $T$  and denote by  $g$  its log-payoff function:  $G(e^x) \equiv g(x)$ . As above, we denote by  $\Phi_t$  the characteristic function of  $X_t$ .

**Proposition 10.** *Suppose that there exists  $R \neq 0$  such that*

$$g(x)e^{-Rx} \text{ has finite variation on } \mathbb{R}, \quad (37)$$

$$g(x)e^{-Rx} \in L^1(\mathbb{R}), \quad (38)$$

$$E[e^{RX_{T-t}}] < \infty \quad \text{and} \quad \int_{\mathbb{R}} \frac{|\Phi_{T-t}(u - iR)|}{1 + |u|} du < \infty. \quad (39)$$

Then the price at time  $t$  of the European option with pay-off function  $G$  satisfies

$$\begin{aligned} P(t, S_t) &:= e^{-r(T-t)} E[G(S_T) | \mathcal{F}_t] \\ &= \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR) \Phi_{T-t}(-u - iR) \hat{S}_t^{R-iu} du, \end{aligned} \quad (40)$$

where

$$\hat{g}(u) := \int_{\mathbb{R}} e^{iux} g(x) dx.$$

*Proof.* By integration by parts for Stieltjes integrals,

$$\hat{g}(u + iR) = \int_{\mathbb{R}} g(x) e^{ix(u+iR)} dx = \frac{i}{u + iR} \int_{\mathbb{R}} e^{ix(u+iR)} dg(x). \quad (41)$$

This implies in particular that

$$|\hat{g}(u + iR)| \leq \frac{C}{|u + iR|}, \quad u \in \mathbb{R}. \quad (42)$$

Suppose that  $R > 0$  (the case  $R < 0$  can be treated in a similar manner) and consider the function

$$f(x) = e^{Rx} \int_x^\infty p(dz),$$

where  $p$  denotes the distribution of  $X_{T-t}$ . From the assumption (39) it follows

$$\int_{\mathbb{R}} e^{Rx} p(dx) < \infty$$



and therefore  $\lim_{x \rightarrow \infty} f(x) = 0$ . Clearly also  $\lim_{x \rightarrow -\infty} f(x) = 0$ . By integration by parts,

$$\int_{-N}^N f(x) dx = \frac{1}{R} \int_{-N}^N e^{Rx} p(dx) + \frac{1}{R} (f(N) - f(-N)).$$

This shows that  $f \in L^1(\mathbb{R})$  and it follows that

$$\int_{\mathbb{R}} e^{-iux} f(x) dx = \frac{\Phi_{T-t}(-u - iR)}{R - iu}.$$

From condition (39) it follows that  $f$  can be recovered by Fourier inversion (cf. [57, Theorem 9.11]):

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iux} \frac{\Phi_{T-t}(-u - iR)}{R - iu} du. \quad (43)$$

Let us now turn to the proof of (40). From (41), (43) and Fubini's theorem,

$$\frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR) \Phi_{T-t}(-u - iR) \hat{S}_t^{R-iu} du \quad (44)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} dg(x) e^{Rx} \int_{\mathbb{R}} du \frac{e^{iux} \Phi_{T-t}(-u - iR) e^{(R-iu) \log \hat{S}_t}}{R - iu} \quad (45)$$

$$= \int_{\mathbb{R}} dg(x) e^{-R(x - \log \hat{S}_t)} f(x - \log \hat{S}_t) = \int_{\mathbb{R}} dg(x) \int_{x - \log \hat{S}_t}^{\infty} p(dz) \quad (46)$$

$$= \int_{\mathbb{R}} g(x + \log \hat{S}_t) p(dx) = E^Q[G(\hat{S}_T) | \mathcal{F}_t] = E^Q[G(S_T) | \mathcal{F}_t]. \quad (47)$$

□

*Example 3.* The *digital* option has pay-off  $G(S_T) = 1_{S_T \geq K}$ . In this case for all  $R > 0$  conditions (37) and (38) are satisfied and

$$\hat{g}(u + iR) = \frac{K^{iu-R}}{R - iu}.$$

*Example 4.* The European call option has pay-off  $G(S_T) = (S_T - K)^+$ . Therefore, conditions (37) and (38) are satisfied for all  $R > 1$ ,

$$\hat{g}(u + iR) = \frac{K^{iu+1-R}}{(R - iu)(R - 1 - iu)}.$$

and the price of a call option can be written as an inverse Fourier transform:

$$\begin{aligned} C(t, S_t) &= \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \frac{K^{iu+1-R} \hat{S}_t^{R-iu} \Phi_{T-t}(-u-iR)}{(R-iu)(R-1-iu)} du \\ &= \frac{S_t}{2\pi} \int_{\mathbb{R}} \frac{e^{k^f(iu+1-R)} \Phi_{T-t}(-u-iR)}{(R-iu)(R-1-iu)} du \end{aligned} \quad (48)$$

where  $k^f$  is the log forward moneyness defined by  $k^f = \ln(K/S_t) - r(T-t)$ . This property allows to compute call option prices for many values of  $k^f$  in a single computation using the FFT algorithm as explained below.

## 8.1 Numerical Fourier inversion

In this paragraph we describe an efficient implementation of Formula 48 using the Fast Fourier Transform (FFT). Let

$$\zeta(u) = \frac{\Phi_{T-t}(-u-iR)}{(R-iu)(R-1-iu)}$$

Option prices can be computed by evaluating numerically the inverse Fourier transform of  $\zeta$ :

$$C(k^f) = S_t e^{k^f(1-R)} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk^f} \zeta(v) dv \quad (49)$$

This integral can be efficiently computed using the Fast Fourier transform, an algorithm due to Cooley and Tukey [26] which allows to compute  $F_0, \dots, F_{N-1}$ , given by,

$$F_n = \sum_{k=0}^{N-1} f_k e^{-2\pi i n k / N}, \quad n = 0 \dots N-1,$$

using only  $O(N \ln N)$  operations.

To approximate option prices, we truncate and discretize the integral (49) as follows:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk^f} \zeta(v) dv &= \frac{1}{2\pi} \int_{-A/2}^{A/2} e^{-ivk^f} \zeta(v) dv + \varepsilon_T \\ &= \frac{A}{2\pi N} \sum_{m=0}^{N-1} w_m \zeta(v_m) e^{-ik^f v_m} + \varepsilon_T + \varepsilon_D, \end{aligned} \quad (50)$$

where  $\varepsilon_T$  is the truncation error,  $\varepsilon_D$  is the discretization error,  $v_m = -A/2 + m\Delta$ ,  $\Delta = A/(N - 1)$  is the discretization step and  $w_m$  are weights, corresponding to the chosen integration rule (for instance, for the trapezoidal rule  $w_0 = w_{N-1} = 1/2$  and all other weights are equal to 1). Now, setting  $k_n^f = k_0^f + \frac{2\pi n}{N\Delta}$  we see that the sum in the last term becomes a discrete Fourier transform:

$$\frac{A}{2\pi N} e^{ik_0^f A/2} \sum_{m=0}^{N-1} w_m e^{-ik_0^f v_m} f(k_m^f) e^{-2\pi i n m/N}$$

The FFT algorithm allows to compute option prices for the log strikes  $k_n^f = k_0^f + \frac{2\pi n}{N\Delta}$ . The log strikes are thus equidistant with the step  $d$  satisfying

$$d\Delta = \frac{2\pi}{N}.$$

Typically,  $k_0^f$  is chosen so that the grid is centered around the money,  $\Delta$  is fixed to keep the discretization error low, and  $N$  is adjusted to keep the truncation error low and have a sufficiently small step between strikes (increasing  $N$  reduces the truncation error and the distance between consecutive strikes at the same time). The option prices for the strikes not on the grid must be computed by interpolation. It should be noted that the FFT method should only be used when option prices for many strikes must be computed simultaneously (such as for calibration). If only a small number of strikes is needed, integration methods with variable step size usually have a better performance.

## 9 Integro-differential equations for exotic options

In exponential Lévy models, the price of some options whose pay-off depends on the trajectory of the underlying asset price, such as barriers, can be expressed as the solution of an equation similar to the Black-Scholes PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC - rS \frac{\partial C}{\partial S}.$$

Let  $X$  be a Lévy process such that  $e^X$  is a martingale under  $Q$  (risk-neutral probability) and the price of the underlying asset is given by  $S_t = S_0 e^{rt+X_t}$ . Then the price at time  $t$  of the European option

$$P_t = e^{-r(T-t)} E[(S_T - K)^+ | \mathcal{F}_t]$$

can be expressed as deterministic function of  $t$  and  $S_t$ :  $P_t = P(t, S_t)$  and in addition  $e^{-rt} P(t, S_t)$  is a martingale.

In a similar manner, for the up-and-out option we get

$$P_t^B = e^{-r(T-t)} E[(S_T - K)^+ 1_{\max_{0 \leq s \leq T} S_s < B} | \mathcal{F}_t].$$

The process  $e^{-rt} P_t^B$  is therefore a martingale. Define now  $P^B(t, S)$  as the deterministic function given by

$$P^B(t, S) = e^{-r(T-t)} E[g(S_{T \wedge \tau_t}) | S_t = S],$$

where  $g(S) = (S - K)^+ 1_{S < B}$  and  $\tau_t = \inf\{s \geq t : X_s \geq B\}$ . Then,  $P_t^B = P^B(t, S_t)$  if the barrier has not been reached yet.

Under the assumption that  $e^X$  is a martingale, the dynamics of  $S$  is

$$dS_t = rS_t dt + S_t \sigma dW_t + \int_{\mathbb{R}} S_{t-} (e^x - 1) \tilde{J}_X(dt \times dx),$$

where  $J_X$  is the compensated jump measure of  $X$  and  $\sigma$  is the volatility of its diffusion component. Let the function  $\tilde{P}(t, S_t)$  be sufficiently regular. Applying the Itô formula to this function

$$\begin{aligned} d\tilde{P}(t, S_t) = & \left\{ \frac{\partial \tilde{P}}{\partial t} + rS \frac{\partial \tilde{P}}{\partial S} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 \tilde{P}}{\partial S^2} \right. \\ & \left. + \int_{\mathbb{R}} \left( \tilde{P}(t, S_t e^x) - \tilde{P}(t, S_t) - S_t (e^x - 1) \frac{\partial \tilde{P}}{\partial S} \right) \nu(dx) \right\} \\ & + \frac{\partial \tilde{P}}{\partial S} \sigma S_t dW_t + \int_{\mathbb{R}} (\tilde{P}(t, S_t e^x) - \tilde{P}(t, S_t)) \tilde{J}_X(dt \times dx). \end{aligned}$$

We can then state the following result (results of this type are known as verification theorems):

**Proposition 11.** *Let  $P(t, S)$  be a function which is differentiable with respect to  $t$  once and with respect to  $S$  twice, such that the derivative  $\frac{\partial P}{\partial S}$  is bounded, and assume that  $P$  satisfies the equation*

$$\frac{\partial P}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 P}{\partial S^2} + \int_{\mathbb{R}} \left( P(t, Se^x) - P(t, S) - S(e^x - 1) \frac{\partial P}{\partial S} \right) \nu(dx) = rP - rS \frac{\partial P}{\partial S}$$

*with the terminal condition  $P(T, S) = (S - K)^+$ . Then the price at time  $t$  of a European call option with maturity  $T$  and strike  $K$  is given by  $P(t, S_t)$ .*

The price of a European option in a model with jumps therefore solves an equation which is similar to the Black-Scholes PDE, but contains a correction term depending on the Lévy measure. Equations of this type are called integro-differential equations. It remains to prove that the above equation admits smooth solutions, and for European options it is a relatively easy task [24]. One could also prove formally that the price of an up-and-out option with barrier  $B$  solves the same equation with the additional condition  $P^B(t, S) = 0$  for  $S \geq B$  but in this case the problem of regularity is really difficult and it may become necessary to relax the notion of solution [25].

## 10 Gap options

The gap options are a class of exotic equity derivatives offering protection against rapid downside market moves (gaps). These options have zero delta, allowing to make bets on large downside moves of the underlying without introducing additional sensitivity to small fluctuations, just as volatility derivatives allow to make bets on volatility without going short or long delta. The market for gap options is relatively new, and they are known under many different names: gap options, crash notes, gap notes, daily cliquets, gap risk swaps etc. The gap risk often arises in the context of constant proportion portfolio insurance (CPPI) strategies [23] and other leveraged products such as the leveraged credit-linked notes. The sellers of gap options (who can be seen as the buyers of the protection against gap risk) are typically major banks who want to get off their books the risk associated to CPPI or other leveraged products. The buyers of gap options and the sellers of the protection are usually hedge funds looking for extra returns.

The pay-off of a gap option is linked to the occurrence of a *gap event*, that is, a 1-day downside move of sufficient size in the underlying. The following

single-name gap option was commercialized by a big international bank in 2007 under the name of *gap risk swap*:

*Example 5* (Single-name gap option).

- The protection seller pays the notional amount  $N$  to the protection buyer at inception and receives Libor + spread monthly until maturity or the first occurrence of the gap event, whichever comes first, plus the notional at maturity if no gap event occurs.
- The gap event is defined as a downside move of over 10% in the DJ Euro Stoxx 50 index within 1 day (close to close).
- If a gap event occurs between dates  $t - 1$  and  $t$ , the protection seller immediately receives the reduced notional  $N(1 - 10 * (0.9 - R))^+$ , where  $R = \frac{S_t}{S_{t-1}}$  is the index performance at gap, after which the product terminates.

The gap options are therefore similar to equity default swaps, with a very important difference, that in EDS, the price change from the inception date of the contract to a given date is monitored, whereas in gap options, only 1-day moves are taken into account.

The pay-off of a multi-name gap option depends of the total number of gap events occurring in a basket of underlyings during a reference period. We are grateful to Zareer Dadachanji from Credit Suisse for the following example.

*Example 6* (Multiname gap option).

- As before, the protection seller pays the notional amount  $N$  to the protection buyer and receives Libor + spread monthly until maturity. If no gap event occurs, the protection seller receives the full notional amount at the maturity of the contract.
- A gap event is defined as a downside move of over 20% during one business day in any underlying from a basket of 10 names.
- If a gap event occurs, the protection seller receives at maturity a reduced notional amount  $kN$ , where the reduction factor  $k$  is determined from the number  $M$  of gap events using the following table:

$M$	0	1	2	3	$\geq 4$
$k$	1	1	1	0.5	0

The gap options are designed to capture stock jumps, and clearly cannot be priced within a diffusion model with continuous paths, since any such model will largely underestimate the gap risk. For instance, for a stock with a 25% volatility, the probability of having an 10% gap on any one day during one year is  $3 \times 10^{-8}$ , and the probability of a 20% gap is entirely negligible.

## 10.1 Single asset gap options

Suppose that the time to maturity  $T$  of a gap option is subdivided onto  $N$  periods of length  $\Delta$  (e.g. days):  $T = N\Delta$ . The return of the  $k$ -th period will be denoted by  $R_k^\Delta = \frac{S_{k\Delta}}{S_{(k-1)\Delta}}$ . For the analytic treatment, we formalize the single-asset gap option as follows.

**Definition 10** (Gap option). Let  $\alpha$  denote the return level which triggers the gap event and  $k^*$  be the time of first gap expressed in the units of  $\Delta$ :  $k^* := \inf\{k : R_k^\Delta \leq \alpha\}$ . The gap option is an option which pays to its holder the amount  $f(R_{k^*}^\Delta)$  at time  $\Delta k^*$ , if  $k^* \leq N$  and nothing otherwise.

Supposing that the interest rate is deterministic and equal to  $r$ , it is easy to see that the pay-off structure of example 5 can be expressed as a linear combination of pay-offs of definition 10.

We first treat the case where the log-returns are independent and stationary.

**Proposition 12.** *Let the log-returns  $(R_k^\Delta)_{k=1}^N$  be i.i.d. and denote the distribution of  $\log R_1^\Delta$  by  $p_\Delta(dx)$ . Then the price of a gap option as of definition 10 is given by*

$$G_\Delta = e^{-r\Delta} \int_{-\infty}^{\beta} f(e^x) p_\Delta(dx) \frac{1 - e^{-rT} \left( \int_{\beta}^{\infty} p_\Delta(dx) \right)^N}{1 - e^{-r\Delta} \int_{\beta}^{\infty} p_\Delta(dx)}, \quad (51)$$

with  $\beta := \log \alpha < 0$ .

*Proof.*

$$\begin{aligned}
G_\Delta &= E \left[ e^{-\Delta k^* r} f(R_{k^*}^\Delta) 1_{k^* \leq N} \right] \\
&= \sum_{n=1}^N \mathbb{P}[k^* = n] E[f(R_n^\Delta) | k^* = n] e^{-\Delta n r} \\
&= \sum_{n=1}^N \mathbb{P}[R_n^\Delta \leq \alpha] E[f(R_n^\Delta) | R_n^\Delta \leq \alpha] e^{-\Delta n r} \prod_{l=1}^{n-1} \mathbb{P}[R_l^\Delta > \alpha] \\
&= e^{-r\Delta} \int_{-\infty}^{\beta} f(e^x) p_\Delta(dx) \frac{1 - e^{-rT} \left( \int_{\beta}^{\infty} p_\Delta(dx) \right)^N}{1 - e^{-r\Delta} \int_{\beta}^{\infty} p_\Delta(dx)}.
\end{aligned}$$

□

**Numerical evaluation of prices** Formula (51) allows to compute gap option prices by Fourier inversion. For this, we need to be able to evaluate the cumulative distribution function  $F_\Delta(x) := \int_{-\infty}^x p_\Delta(d\xi)$  and the integral

$$\int_{-\infty}^{\beta} f(e^x) p_\Delta(dx). \tag{52}$$

Let  $\phi_\Delta$  be the characteristic function of  $p_\Delta$ , and suppose that  $p_\Delta$  satisfies  $\int |x| p_\Delta(dx) < \infty$  and  $\int_{\mathbb{R}} \frac{|\phi_\Delta(u)|}{1+|u|} du < \infty$ . Let  $F'$  be the CDF and  $\phi'$  the characteristic function of a Gaussian random variable with zero mean and standard deviation  $\sigma' > 0$ . Then by Lemma 1 in [23],

$$F_\Delta(x) = F'(x) + \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \frac{\phi'(u) - \phi_\Delta(u)}{iu} du. \tag{53}$$

The Gaussian random variable is only needed to obtain an integrable expression in the right hand side and can be replaced by any other well-behaved random variable.

The integral (52) is nothing but the price of a European option with payoff function  $f$  and maturity  $\Delta$ . For arbitrary  $f$  it can be evaluated using the Fourier transform method as described in section 8. For the numerical evaluation, the integrals must be truncated to a finite interval  $[-L, L]$ . Since  $\Delta$  is small, the characteristic function  $\phi_\Delta(u)$  decays slowly at infinity, which means that  $L$  must be sufficiently large (for example, in a jump-diffusion



model with volatility  $\sigma$ ,  $L \sim \frac{C}{\sigma\sqrt{\Delta}}$ , where  $C$  is a constant which depends on the desired precision, such as  $C = 5$  — see [67]). The computation of the integrals will therefore be rather costly. For this reason, we do not recommend to use the exact formula, and propose an approximation, which is based on an expansion of  $G_\Delta$  around the value  $\Delta = 0$ . In other words, instead of using a numerical method whose computational complexity increases when  $\Delta$  is small, we suggest an explicit formula whose precision improves when  $\Delta \rightarrow 0$ .

**Approximate pricing formula** Suppose that  $S_t = S_0 e^{X_t}$ , where  $X$  is a Lévy process. This means that  $p_\Delta$  as defined above is the distribution of  $X_\Delta$ .

Since  $r\Delta \sim 10^{-4}$  and the probability of having a gap on a given day  $\int_{-\infty}^{\beta} p_\Delta(dx)$  is also extremely small, with very high precision,

$$G_\Delta \approx \int_{-\infty}^{\beta} f(e^x) p_\Delta(dx) \frac{1 - e^{-rT - N \int_{-\infty}^{\beta} p_\Delta(dx)}}{r\Delta + \int_{-\infty}^{\beta} p_\Delta(dx)}. \quad (54)$$

Our second approximation is less trivial. From [58], we know that for all Lévy processes and under very mild hypotheses on the function  $f$ , we have

$$\int_{-\infty}^{\beta} g(x) p_\Delta(dx) \sim \Delta \int_{-\infty}^{\beta} g(x) \nu(dx),$$

as  $\Delta \rightarrow 0$ , where  $\nu$  is the Lévy measure of  $X$ . Consequently, when  $\Delta$  is nonzero but small, we can replace the integrals with respect to the density with the integrals with respect to the Lévy measure in formula (54), obtaining an approximate but explicit expression for the gap option price:

$$G_\Delta \approx G_0 = \lim_{\Delta \rightarrow 0} G_\Delta = \int_{-\infty}^{\beta} f(e^x) \nu(dx) \frac{1 - e^{-rT - T \int_{-\infty}^{\beta} \nu(dx)}}{r + \int_{-\infty}^{\beta} \nu(dx)}. \quad (55)$$

This approximation is obtained by making the time interval at which returns are monitored (a priori, one day), go to zero. In other words,  $G_0$  is the zero-order term of the Taylor expansion of  $G_\Delta$  around the point  $\Delta = 0$ . The error of this approximation will therefore decay proportionally to  $\Delta$  when  $\Delta \rightarrow 0$ .

**A modified gap option** For a better understanding of the risks of a gap option, it is convenient to interpret the pricing formula (55) as an exact price

of a modified gap option rather than the true price of the original option. From now on, we define the single-asset gap option as follows.

**Definition 11** (Modified gap option). Let  $\tau = \inf\{t : \Delta X_t \leq \beta\}$  be the time of the first jump of  $X$  smaller than  $\beta$ . The gap option as a product which pays to its holder the amount  $f\left(\frac{S_\tau}{S_{\tau-}}\right) = f(e^{\Delta X_\tau})$  if  $\tau \leq T$  and zero otherwise.

The price of this product is given by

$$G = E^Q[e^{-r\tau} f(e^{\Delta X_\tau}) 1_{\tau \leq T}]$$

which is easily seen to be equal to  $G_0$ :

**Proposition 13.** *Suppose that the underlying follows an exponential Lévy model:  $S_t = S_0 e^{X_t}$ , where  $X$  is a Lévy process with Lévy measure  $\nu$ . Then the price of the gap option as of definition 11, or, equivalently, the approximate price of the gap option as of definition 10 is given by*

$$G = \int_{-\infty}^{\beta} f(e^x) \nu(dx) \frac{1 - e^{-rT - T \int_{-\infty}^{\beta} \nu(dx)}}{r + \int_{-\infty}^{\beta} \nu(dx)}$$

with  $\beta := \log \alpha$ .

The gap option then arises as a pure jump risk product, which is only sensitive to negative jumps larger than  $\beta$  in absolute value, but not to small fluctuations of the underlying. In particular, it has zero delta.

## 10.2 Multi-asset gap options and Lévy copulas

As explained earlier in this section, a multiline (basket) gap option is a product where one monitors the total number of gap events in a basket of underlyings over the lifetime of the option  $[0, T]$ . A gap event is defined as a negative return of size less than  $\alpha$  between consecutive closing prices (close-to-close) in any of the underlyings of the basket. The pay-off of the product at date  $T$  is determined by the total number of gap events in the basket over the reference period. To compute the price of a multiline gap option, we suppose that  $M$  underlying assets  $S^1, \dots, S^M$  follow an  $M$ -dimensional exponential Lévy model, that is,  $S_t^i = S_0^i e^{X_t^i}$  for  $i = 1, \dots, M$ ,

where  $(X^1, \dots, X^M)$  is an  $M$ -dimensional Lévy process with Lévy measure  $\nu$ . We make the same simplifying hypothesis as in definition 11, that is, we define a gap event as a negative jump smaller than a given value  $\beta$  in any of the assets, rather than a negative daily return. From now on, we define a multiname gap option as follows.

**Definition 12.** For a given  $\beta < 0$ , let

$$N_t = \sum_{i=1}^M \#\{(s, i) : s \leq t, 1 \leq i \leq M \text{ and } \Delta X_s^i \leq \beta\} \quad (56)$$

be the process counting the total number of gap events in the basket before time  $t$ . The multiname gap option is a product which pays to its holder the amount  $f(N_T)$  at time  $T$ .

The pay-off function  $f$  for a typical multiname gap option is given in example 6. Notice that the single-name gap option stops at the first gap event, whereas in the multiname case the gap events are counted up to the maturity of the product.

The biggest difficulty in the multidimensional case, is that now we have to model simultaneous jumps in the prices of different underlyings. The multidimensional Lévy measures can be conveniently described using their tail integrals. The tail integral  $U$  describes the intensity of *simultaneous jumps in all components* smaller than the components of a given vector. Given an  $M$ -dimensional Lévy measure  $\nu$ , we define the *tail integral* of  $\nu$  by

$$U(z_1, \dots, z_M) = \nu(\{x \in \mathbb{R}^M : x_1 \leq z_1, \dots, x_M \leq z_M\}), \quad z_1, \dots, z_M < 0. \quad (57)$$

The tail integral can also be defined for positive  $z$  (see [38]), but we do not introduce this here since we are only interested in jumps smaller than a given negative value.

To describe the intensity of simultaneous jumps of a subset of the components of  $X$ , we define the *marginal* tail integral: for  $m \leq M$  and  $1 \leq i_1 < \dots < i_m \leq M$ , the  $(i_1, \dots, i_m)$ -marginal tail integral of  $\nu$  is defined by

$$U_{i_1, \dots, i_m}(z_1, \dots, z_m) = \nu(\{x \in \mathbb{R}^M : x_{i_1} \leq z_1, \dots, x_{i_m} \leq z_m\}), \quad z_1, \dots, z_m < 0. \quad (58)$$

The process  $N$  counting the total number of gap events in the basket is clearly a piecewise constant increasing integer-valued process which moves

only by jumps of integer size. The jump sizes can vary from 1 (in case of a gap event affecting a single component) to  $M$  (simultaneous gap event in all components). The following lemma describes the structure of this process via the tail integrals of  $\nu$ .

**Lemma 2.** *The process  $N$  counting the total number of gap events is a Lévy process with integer jump sizes  $1, \dots, M$  occurring with intensities  $\lambda_1, \dots, \lambda_M$  given by*

$$\lambda_m = \sum_{k=m}^M (-1)^{k-m} \sum_{1 \leq i_1 < \dots < i_k \leq M} C_m^k U_{i_1, \dots, i_k}(\beta, \dots, \beta), \quad 1 \leq m \leq M, \quad (59)$$

where  $C_m^k$  denotes the binomial coefficient and the second sum is taken over all possible sets of  $k$  integer indices satisfying  $1 \leq i_1 < \dots < i_k \leq M$ .

*Proof.* Since  $X$  is a process with stationary and independent increments, it follows from formula (56) that  $N$  has stationary and independent increments as well. A jump of size  $m$  in  $N$  occurs if and only if exactly  $m$  components of  $X$  jump by an amount smaller or equal to  $\beta$ . Therefore,

$$\lambda_m = \sum_{1 \leq i_1 < \dots < i_m \leq M} \nu(\{x_i \leq \beta \forall i \in \{i_1, \dots, i_m\}; x_i > \beta \forall i \notin \{i_1, \dots, i_m\}\}) \quad (60)$$

The expression under the sum sign can be written as

$$\begin{aligned} & \nu(\{x_i \leq \beta \forall i \in \{i_1, \dots, i_m\}; x_i > \beta \forall i \notin \{i_1, \dots, i_m\}\}) \\ &= \nu(\{x_i \leq \beta \forall i \in \{i_1, \dots, i_m\}\}) \\ &+ \sum_{p=1}^{M-m} \sum_{\substack{1 \leq j_1 < \dots < j_p \leq M \\ \{j_1, \dots, j_p\} \cap \{i_1, \dots, i_m\} = \emptyset}} (-1)^p \nu(\{x_i \leq \beta \forall i \in \{i_1, \dots, i_m\} \cup \{j_1, \dots, j_p\}\}) \\ &= U_{i_1, \dots, i_m}(\beta, \dots, \beta) + \sum_{p=1}^{M-m} \sum_{\substack{1 \leq j_1 < \dots < j_p \leq M \\ \{j_1, \dots, j_p\} \cap \{i_1, \dots, i_m\} = \emptyset}} (-1)^p U_{i_1, \dots, i_m, j_1, \dots, j_p}(\beta, \dots, \beta) \end{aligned}$$

Combining this equation with (60) and gathering the terms with identical tail integrals, one obtains (59).  $\square$

The process  $N$  can equivalently be represented as

$$N_t = \sum_{m=1}^M m N_t^{(m)},$$

where  $N^{(1)}, \dots, N^{(M)}$  are independent Poisson processes with intensities  $\lambda_1, \dots, \lambda_M$ . Since these processes are independent, the expectation of any functional of  $N_T$  (the price of a gap option) can be computed as

$$E[f(N_T)] = e^{-\lambda T} \sum_{n_1, \dots, n_M=0}^{\infty} f\left(\sum_{k=1}^M k n_k\right) \prod_{i=1}^M \frac{(\lambda_i T)^{n_i}}{n_i!}, \quad (61)$$

where  $\lambda := \sum_{i=1}^M \lambda_i$ . In practice, after a certain number of gap events, the gap option has zero pay-off and the sum in (61) reduces to a finite number of terms. In example 6,  $f(n) \equiv 0$  for  $n \geq 4$  and

$$E[f(N_T)] = e^{-\lambda T} \left\{ 1 + \lambda_1 T + \frac{(\lambda_1 T)^2}{2} + \lambda_2 T \right. \quad (62)$$

$$\left. + \frac{(\lambda_1 T)^3}{12} + \frac{\lambda_1 \lambda_2 T^2}{2} + \frac{\lambda_3 T}{2} \right\}. \quad (63)$$

The price of the protection (premium over the risk-free rate received by the protection seller) is given by the discounted expectation of  $1 - f(N_T)$ , that is,

$$e^{-rT} E[1 - f(N_T)]. \quad (64)$$

To make computations with the formula (61), one needs to evaluate the tail integral of  $\nu$  and all its marginal tail integrals. These objects are determined both by the individual gap intensities of each component and by the dependence among the components of the multidimensional process. For modeling purposes, the dependence structure can be separated from the behavior of individual components via the notion of Lévy copula [20, 38], which is parallel to the notion of copula but defined at the level of jumps of a Lévy process. More precisely we will use the *positive Lévy copulas* which describe the one-sided (in this case, only downward) jumps of a Lévy process, as opposed to general Lévy copulas which are useful when both upward and downward jumps are of interest.

**Positive Lévy copulas** Let  $\overline{\mathbb{R}} := (-\infty, \infty]$  denote the extended real line, and for  $a, b \in \overline{\mathbb{R}}^d$  let us write  $a \leq b$  if  $a_k \leq b_k$ ,  $k = 1, \dots, d$ . In this case,  $(a, b]$  denotes the interval

$$(a, b] := (a_1, b_1] \times \dots \times (a_d, b_d].$$

For a function  $F$  mapping a subset  $D \subset \overline{\mathbb{R}}^d$  into  $\overline{\mathbb{R}}$ , the  $F$ -volume of  $(a, b]$  is defined by

$$V_F((a, b]) := \sum_{u \in \{a_1, b_1\} \times \dots \times \{a_d, b_d\}} (-1)^{N(u)} F(u),$$

where  $N(u) := \#\{k : u_k = a_k\}$ . In particular,  $V_F((a, b]) = F(b) - F(a)$  for  $d = 1$  and  $V_F((a, b]) = F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2)$  for  $d = 2$ . If  $F(u) = \prod_{i=1}^d u_i$ , the  $F$ -volume of any interval is equal to its Lebesgue measure.

A function  $F : D \rightarrow \overline{\mathbb{R}}$  is called  $d$ -increasing if  $V_F((a, b]) \geq 0$  for all  $a, b \in D$  such that  $a \leq b$ . The distribution function of a random vector is one example of a  $d$ -increasing function.

A function  $F : [0, \infty]^d \rightarrow [0, \infty]$  is called a *positive Lévy copula* if it satisfies the following conditions:

1.  $F(u_1, \dots, u_d) = 0$  if  $u_i = 0$  for at least one  $i \in \{1, \dots, d\}$ ,
2.  $F$  is  $d$ -increasing,
3.  $F_i(u) = u$  for any  $i \in \{1, \dots, d\}$ ,  $u \in \mathbb{R}$ , where  $F_i$  is the one-dimensional margin of  $F$ , obtained from  $F$  by replacing all arguments of  $F$  except the  $i$ -th one with  $\infty$ :

$$F_i(u) = F(u_1, \dots, u_d)_{u_i=u, u_j=\infty \forall j \neq i}.$$

The positive Lévy copula has the same properties as ordinary copula but is defined on a different domain ( $[0, \infty]^d$  instead of  $[0, 1]^d$ ). Higher-dimensional margins of a positive Lévy copula are defined similarly:

$$F_{i_1, \dots, i_m}(u_1, \dots, u_m) = F(v_1, \dots, v_d)_{v_{i_k}=u_k, k=1, \dots, m; v_j=\infty, j \notin \{i_1, \dots, i_m\}}.$$

The Lévy copula links the tail integral to one-dimensional margins; the following result is a direct corollary of Theorem 3.6 in [38].

**Proposition 14.**

- Let  $X = (X^1, \dots, X^d)$  be a  $\mathbb{R}^d$ -valued Lévy process, and let the (one-sided) tail integrals and marginal tail integrals of  $X$  be defined by (57) and (58). Then there exists a positive Lévy copula  $F$  such that the tail integrals of  $X$  satisfy

$$U_{i_1, \dots, i_m}(x_1, \dots, x_m) = F_{i_1, \dots, i_m}(U_{i_1}(x_1), \dots, U_{i_m}(x_m)) \quad (65)$$

for any nonempty index set  $\{i_1, \dots, i_m\} \subseteq \{1, \dots, d\}$  and any  $(x_1, \dots, x_m) \in (-\infty, 0)^m$ .

- Let  $F$  be an  $M$ -dimensional positive Lévy copula and  $U_i, i = 1, \dots, d$  tail integrals of real-valued Lévy processes. Then there exists a  $\mathbb{R}^d$ -valued Lévy process  $X$  whose components have tail integrals  $U_i, i = 1, \dots, d$  and whose marginal tail integrals satisfy equation (65) for any nonempty index set  $\{i_1, \dots, i_m\} \subseteq \{1, \dots, d\}$  and any  $(x_1, \dots, x_m) \in (-\infty, 0)^m$ .

In terms of the Lévy copula  $F$  of  $X$  and its marginal tail integrals, formula (59) can be rewritten as

$$\lambda_m = \sum_{k=m}^M (-1)^{k-m} \sum_{1 \leq i_1 < \dots < i_k \leq M} C_m^k F_{i_1, \dots, i_k}(U_{i_1}(\beta), \dots, U_{i_k}(\beta))$$

To compute the intensities  $\lambda_i$  and price the gap option, it is therefore sufficient to know the individual gap intensities  $U_i(\beta)$  ( $M$  real numbers), which can be estimated from 1-dimensional gap option prices or from the prices of short-term put options and the Lévy copula  $F$ . This Lévy copula will typically be chosen in some suitable parametric family. One convenient choice is the Clayton family of (positive) Lévy copulas defined by

$$F^\theta(u_1, \dots, u_M) = (u_1^{-\theta} + \dots + u_M^{-\theta})^{-1/\theta}. \quad (66)$$

The dependence structure in the Clayton family is determined by a single parameter  $\theta > 0$ . The limit  $\theta \rightarrow +\infty$  corresponds to complete dependence (all components jump together) and  $\theta \rightarrow 0$  produces independent components. The Clayton family has the nice property of being margin-stable: if  $X$  has

Clayton Lévy copula then all lower-dimensional margins also have Clayton Lévy copula:

$$F_{i_1, \dots, i_m}^\theta(u_1, \dots, u_m) = (u_1^{-\theta} + \dots + u_m^{-\theta})^{-1/\theta}.$$

For the Clayton Lévy copula, equation (59) simplifies to

$$\lambda_m = \sum_{k=m}^M (-1)^{k-m} \sum_{1 \leq i_1 < \dots < i_k \leq M} C_m^k (U_{i_1}(\beta)^{-\theta} + \dots + U_{i_k}(\beta)^{-\theta})^{-1/\theta}.$$

This formula can be used directly for baskets of reasonable size (say, less than 20 names). For very large baskets, one can make the simplifying assumption that all individual stocks have the same gap intensity:  $U_k(\beta) = U_1(\beta)$  for all  $k$ . In this case, formula (59) reduces to the following simple result:

**Proposition 15.** *Suppose that the prices of  $M$  underlyings follow an  $M$ -dimensional exponential Lévy model with Lévy measure  $\nu$ . If the individual components of the basket are identically distributed and the dependence structure is described by the Clayton Lévy copula with parameter  $\theta$ , the price of a basket gap option as of definition 12 is given by*

$$E[f(N_T)] = e^{-\lambda T} \sum_{n_1, \dots, n_M=0}^{\infty} f\left(\sum_{k=1}^M kn_k\right) \prod_{i=1}^M \frac{(\lambda_i T)^{n_i}}{n_i!},$$

where

$$\lambda_m = U_1(\beta) C_m^M \sum_{j=0}^{M-m} \frac{(-1)^j C_j^{M-m}}{(m+j)^{1/\theta}} \quad (67)$$

Figure 1 shows the behavior of the intensities  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_{10}$  as a function of the dependence parameter  $\theta$  in a basket of 10 names, with a single-name gap probability of 1%. Note that formula (67) implies

$$\lim_{\theta \rightarrow \infty} \lambda_m = \begin{cases} 0, & m < M \\ U_1(\beta), & m = M. \end{cases}$$

$$\lim_{\theta \rightarrow 0} \lambda_m = \begin{cases} 0, & m > 1 \\ MU_1(\beta), & m = 1. \end{cases},$$



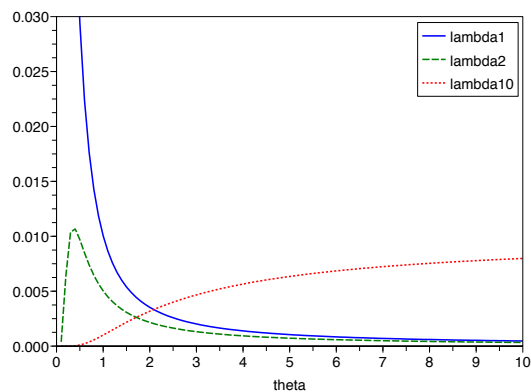


Figure 1: The intensities  $\lambda_i$  of different jump sizes of the gap counting process as a function of  $\theta$  for  $M = 10$  names and a single-name loss probability of 1%.

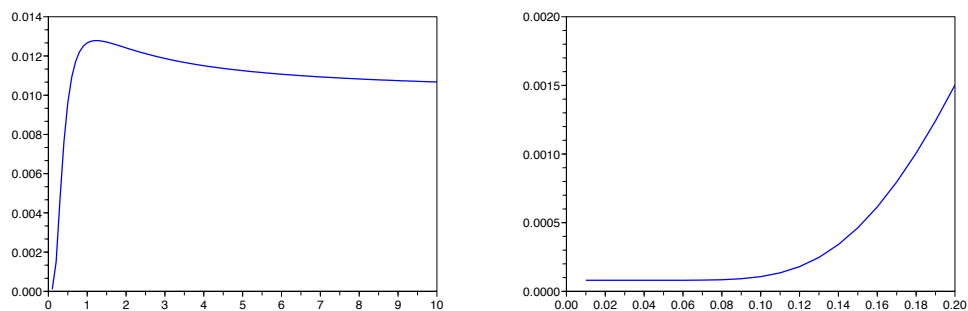


Figure 2: Left: Expected loss of a multi-name gap option in the Credit Suisse example as a function of the dependence parameter  $\theta$ . The single-name loss probability is 1%. Right: zoom of the right graph for small values of  $\theta$ .

in agreement with the behavior observed in Figure 1.

Figure 2 shows the price of the multiline gap option of example 6 computed using the formula (64). The price achieves a maximum for a finite nonzero value of  $\theta$ . This happens because for this particular payoff structure, the protection seller does not lose money if only 1 or 2 gap events occur during the lifetime of the product, and only starts to pay after 3 or more gap events. The probability of having 3 or more gap events is very low with independent components.

## 11 Implied volatility

Recall the well-known Black-Scholes formula for call option prices:

$$C^{BS}(t, S_t, T, K, \sigma) = S_t N(d_1) - K e^{-r(T-t)} N(d_2) \quad (68)$$

with  $d_{1,2} = \frac{\log(\frac{S_t}{K e^{-r\tau}}) \pm \tau \sigma^2 / 2}{\sigma \sqrt{\tau}}$  and  $N(u) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-\frac{z^2}{2}} dz$ ,

where  $\tau = T - t$ . If all other parameters are fixed, (68) is an increasing continuous function of  $\sigma$ , mapping  $(0, \infty)$  into  $((S_t - K e^{-r\tau})^+, S_t)$ . The latter interval is the greatest interval allowed by arbitrage bounds on call option prices. Therefore, given the market price  $C_t^*(T, K)$  of a call option, one can always invert (68) and find the value of volatility parameter which, when substituted into the Black-Scholes formula, gives the correct option price:

$$\exists! \Sigma_t(T, K) > 0 : \quad C^{BS}(t, S_t, T, K, \Sigma_t(T, K)) = C_t^*(K, T). \quad (69)$$

This value is called the (Black-Scholes) implied volatility of the option. For fixed  $(T, K)$ , the implied volatility  $\Sigma_t(T, K)$  is in general a stochastic process and, for fixed  $t$ , its value depends on the characteristics of the option such as the maturity  $T$  and the strike level  $K$ : the function  $\Sigma_t : (T, K) \rightarrow \Sigma_t(T, K)$  is called the *implied volatility surface* at date  $t$  (see Figure 3). Using the log moneyness  $k = \log(K/S_t)$  of the option, one can also represent the implied volatility surface as a function of  $k$  and time to maturity:  $I_t(\tau, k) = \Sigma_t(t + \tau, S_t e^k)$ . From the independence and stationarity of increments of  $X$ , it follows that the definition of implied volatility (69) is equivalent to

$$E[(e^{X_\tau} - e^{k-r\tau})^+] = E[(e^{IW_\tau - \frac{I^2 \tau}{2}} - e^{k-r\tau})^+].$$

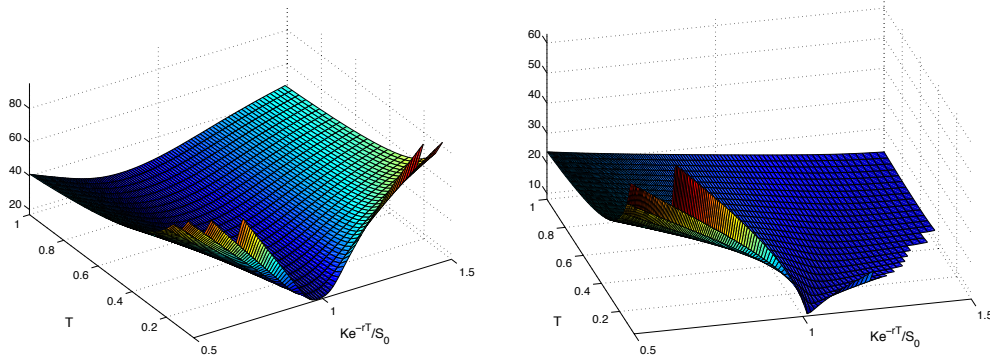


Figure 3: Left: Profile of the implied volatility surface as a function of time to maturity and moneyness for the Merton jump-diffusion model with  $\sigma = 15\%$ ,  $\delta = 1$  and  $\lambda = 0.1$ . Right: Implied volatility surface as a function of time to maturity and moneyness for the variance gamma model using parameters taken from [48]. Note the flattening of the skew with maturity.

Since each side depends only on  $(\tau, k)$  and not on  $t$  one concludes that in exponential Lévy models, the implied volatility for a given log moneyness  $k$  and time to maturity  $\tau$  does not evolve in time:  $I_t(\tau, k) = I_0(\tau, k) := I(\tau, k)$ . This property is known as the *floating smile property*.

In exponential Lévy models, the properties of the implied volatility surfaces can be characterized in terms of the asymptotic behavior of the surface for large and small values of strike and maturity. We start with the large and small strike behavior which was first analyzed by Roger Lee [45]; this analysis was subsequently extended and made more precise by Benaim and Friz [32, 31]. Their results, reviewed below, take a particularly simple form in the case of Lévy processes, because the critical exponents do not depend on time. Next, we study the short maturity asymptotics, where it turns out that the behavior of the implied volatility is very different for out of the money (OTM) and at the money (ATM) options. Below, we present some original results for the two cases. Finally, the long-maturity asymptotics were recently studied by Tehranchi [69, 68] and Rogers and Tehranchi [56]. We review their results in the case of Lévy processes, where once again, the formulation is particularly simple and interesting links to the large deviations theory and Cramér's theorem can be made.

## 11.1 Large/small strikes

The limiting slope of time-rescaled implied variance as a function of log-strike turns out to be related to the critical exponents of the moment generating function of the log-price process  $X$ , defined by

$$q_t^* = -\inf\{u : E[e^{uX_t}] < \infty\}, \quad r_t^* = \sup\{u : E[e^{uX_t}] < \infty\}.$$

It is clear that the interval  $[-q_t^*, r_t^*]$  is nonempty, because  $E[e^{uX_t}] < \infty$  at least for all  $u \in [0, 1]$  by the martingale condition.

**Proposition 16** (Implied volatility asymptotics at extreme strikes [31]). *Fix  $\tau > 0$  and suppose that  $r_\tau^* \in (0, \infty)$  and  $q_\tau^* \in (0, \infty)$  and that the moment generating function blows up in a regularly varying way around its critical exponents (see [31] for a precise definition). Then the implied volatility  $I(\tau, k)$  satisfies*

$$\frac{I^2(\tau, -k)\tau}{|k|} \sim \xi(q_\tau^*) \quad \text{and} \quad \frac{I^2(\tau, k)\tau}{k} \sim \xi(r_\tau^* - 1), \quad \text{as } k \rightarrow +\infty,$$

where the function  $\xi$  is defined by  $\xi(x) = 2 - 4(\sqrt{x^2 + x} - x)$ .

This proposition extends in a natural way to the case of infinite critical exponents:

$$\frac{I^2(\tau, -k)\tau}{|k|} \xrightarrow{k \rightarrow +\infty} 0 \quad \text{if } q_\tau^* = \infty \quad \text{and} \quad \frac{I^2(\tau, k)\tau}{k} \xrightarrow{k \rightarrow +\infty} 0 \quad \text{if } r_\tau^* = \infty.$$

This was shown already in the original work of Roger Lee [45].

For Lévy processes, the exponents  $q^*$  and  $r^*$  do not depend on  $t$  and are particularly easy to compute, since the moment generating function is known from the Lévy-Khintchine formula. In particular, the models with exponential tail decay of the Lévy measure such as variance gamma, normal inverse Gaussian and Kou satisfy the necessary conditions for the proposition 16 and their critical exponents coincide with the inverse decay lengths:  $q^* = \lambda_-$  and  $r^* = \lambda_+$ . Figure 4 shows that the asymptotic linear slope of the implied variance as a function of log strike can be observed for values of  $k$  which are not so far from zero, especially for short maturity options.

In Merton model, the tails of the Lévy measure are thinner than exponential and the critical exponents  $q^*$  and  $r^*$  are infinite. The remark after

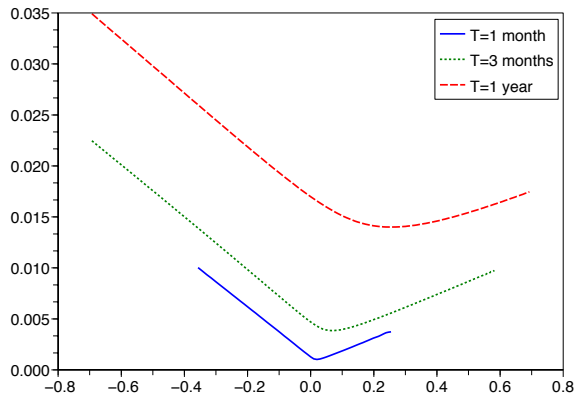


Figure 4: Smile asymptotics: implied variance multiplied by square root of maturity as function of log-strike in the variance gamma model with parameters taken from [48]:  $\sigma = 0.1213$ ,  $\theta = -0.1436$ ,  $\kappa = 0.1686$ .

Proposition 16 then only tells us that the limiting slope of the implied variance is zero, but other results in [32] allow to compute the exact asymptotics: for the right tail we have

$$I^2(\tau, k)\tau \underset{k \rightarrow \infty}{\sim} \delta \times \frac{k}{2\sqrt{2 \log k}}, \quad \text{when } \delta > 0$$

and

$$I^2(\tau, k)\tau \underset{k \rightarrow \infty}{\sim} \mu \times \frac{k}{2 \log k}, \quad \text{when } \delta = 0,$$

where  $\delta$  is the standard deviation of the jump size and  $\mu$  is the mean jump.

## 11.2 Short maturity asymptotics

The short maturity behavior of implied volatility in exponential Lévy models is very different from that observed in stochastic / local volatility models with continuous paths. While in continuous models the implied volatility usually converges to a finite nonzero value as  $\tau \rightarrow 0$ , in models with jumps the implied volatility of out of the money or in the money options blows up. On the other hand, the implied volatility of at-the-money options converges to the volatility of the diffusion component as  $\tau \rightarrow 0$ ; in particular it converges

to zero for pure jump models. This leads to very pronounced smiles for short maturity options (in agreement with market-quoted smiles). The intuitive explanation of this effect is that in most continuous models, the stock returns at short time scales become close to Gaussian; in particular, the skewness and excess kurtosis converge to zero as  $\tau \rightarrow 0$ . By contrast, in models with jumps, the distribution of stock returns at short time scales shifts further away from the Gaussian law; the skewness and kurtosis explode as  $\frac{1}{\sqrt{\tau}}$  and  $\frac{1}{\tau}$  respectively.

The short maturity asymptotics of implied volatility smile in exponential Lévy models can be computed by comparing the option price asymptotics in the Black-Scholes model to those in the exponential Lévy model (many results in this direction can be found in Carr and Wu [17]). To simplify the developments, we suppose that the interest rate is zero. Then the normalized Black-Scholes price satisfies

$$c_{BS}(\tau, k, \sigma) = N(d_1) - e^k N(d_2), \quad d_{1,2} = \frac{-k}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}.$$

Using the asymptotic expansion of the function  $N$  [1], we get, for the ATM options ( $k = 0$ ):

$$c_{BS}(\tau, k, \sigma) \sim \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} \quad (70)$$

and for other options

$$c_{BS}(\tau, k, \sigma) \sim \frac{e^{k/2}}{k^2\sqrt{2\pi}}\sigma^3\tau^{3/2}e^{-\frac{k^2}{2\sigma^2\tau}}, \quad (71)$$

where the notation  $f \sim g$  signifies  $\frac{f}{g} \rightarrow 1$  as  $\tau \rightarrow 0$ .

In every exponential Lévy model satisfying the martingale condition, we have [58]

$$E[(e^{X_\tau} - e^k)^+] \sim \tau \int (e^x - e^k)^+ \nu(dx), \quad \text{for } k > 0 \quad (72)$$

$$E[(e^k - e^{X_\tau})^+] \sim \tau \int (e^k - e^x)^+ \nu(dx), \quad \text{for } k < 0 \quad (73)$$

From these estimates, the following universal result can be deduced: it confirms the numerical observation of smile explosion in exponential Lévy models and gives the exact rate at which this explosion takes place.

**Proposition 17** (Short maturity asymptotics: OTM options). *Let  $X$  be a Lévy process with Lévy measure  $\nu$  satisfying  $\text{supp } \nu = \mathbb{R}$ . Then, for a fixed log moneyness  $k \neq 0$ , the implied volatility  $I(\tau, k)$  in the exponential Lévy model  $S_t = S_0 e^{X_t}$  satisfies*

$$\lim_{\tau \rightarrow 0} \frac{2I^2(\tau, k)\tau \log \frac{1}{\tau}}{k^2} = 1. \quad (74)$$

*Proof.* Suppose first that  $k > 0$ . It is clear that  $I(\tau, k)\sqrt{\tau} \rightarrow 0$  as  $\tau \rightarrow 0$  because otherwise the option price would not converge to 0. We then have, from OTM Black-Scholes asymptotics (71):

$$\lim_{\tau \rightarrow 0} \frac{c_{BS}(\tau, k, I(\tau, k))}{C_1 I(\tau, k)^3 \tau^{3/2} e^{-\frac{k^2}{2I^2(\tau, k)\tau}}} = 1,$$

where  $C_1 > 0$  does not depend on  $\tau$ . Denote the (normalized) call price in the exponential Lévy model by  $c(\tau, k)$ . Under the full support hypothesis,  $c(\tau, k) \sim C_2 \tau$  with  $C_2 > 0$  which once again does not depend on  $\tau$ . By definition of the implied volatility we then have

$$\lim_{\tau \rightarrow 0} \frac{C_2 \tau}{C_1 I(\tau, k)^3 \tau^{3/2} e^{-\frac{k^2}{2I^2(\tau, k)\tau}}} = 1.$$

Taking the logarithm gives

$$\lim_{\tau \rightarrow 0} \left\{ \log(C_2/C_1) + 3 \log I(\tau, k) + \frac{1}{2} \log \tau - \frac{k^2}{2I^2(\tau, k)\tau} \right\} = 0.$$

Now, knowing that  $I^2(\tau, k)\tau \rightarrow 0$ , we can multiply all terms by  $I^2(\tau, k)\tau$ :

$$\lim_{\tau \rightarrow 0} \left\{ I^2 \tau \log(C_2/C_1) + \frac{3}{2} I^2 \tau \log(I^2 \tau) - I^2 \tau \log \tau - \frac{k^2}{2} \right\} = 0.$$

Since the first two terms disappear in the limit, this completes the proof in the case  $k > 0$ . The case  $k < 0$  can be treated in a similar manner using put options.  $\square$

For ATM options, the situation is completely different, from estimate (70) we will deduce that the implied volatility does not explode but converges to the volatility of the diffusion component.

**Proposition 18** (Short maturity asymptotics: ATM options).

1. Let  $X$  be a Lévy process without diffusion component and with Lévy measure  $\nu$  satisfying  $\int_{|x|\leq 1} |x|\nu(dx) < \infty$ . Then, the ATM implied volatility  $I(\tau, 0)$  in the exponential Lévy model  $S_t = S_0 e^{X_t}$  falls as  $\sqrt{\tau}$  for short maturities:

$$\lim_{\tau \rightarrow 0} \frac{I(\tau, 0)}{\sqrt{2\pi\tau} \max(\int(e^x - 1)^+\nu(dx), \int(1 - e^x)^+\nu(dx))} = 1.$$

2. Let  $X$  be a Lévy process with characteristic exponent

$$\psi(u) = i\gamma u - |u|^\alpha f(u)$$

for  $1 < \alpha < 2$  and some continuous bounded function  $f$  satisfying

$$\lim_{u \rightarrow +\infty} f(u) = c_+, \quad \lim_{u \rightarrow -\infty} f(u) = c_-, \quad 0 < c_1, c_2 < \infty.$$

This includes in particular stable and tempered stable processes with  $1 < \alpha < 2$ . Then, the ATM implied volatility  $I(\tau, 0)$  in the exponential Lévy model  $S_t = S_0 e^{X_t}$  falls as  $\tau^{1/\alpha-1/2}$  for short maturities:

$$\lim_{\tau \rightarrow 0} \frac{I(\tau, 0)}{C\tau^{1/\alpha-1/2}\sqrt{2\pi}} = 1.$$

with  $C = \Gamma(1 - 1/\alpha)(c_+^{1/\alpha} + c_-^{1/\alpha})$ .

3. Let  $X$  be a Lévy process with a diffusion component with volatility  $\sigma$  and Lévy measure satisfying  $\int x^2\nu(dx) < \infty$ . Then the ATM implied volatility  $I(\tau, 0)$  in the exponential Lévy model  $S_t = S_0 e^{X_t}$  converges to  $\sigma$  as  $\tau \rightarrow 0$ .

The short-maturity smile asymptotics are illustrated in Figure 5: the ATM implied volatility converges to the value of  $\sigma$  and the out of the money and in the money volatilities eventually become very large as  $\tau$  approaches zero.

*Proof.*



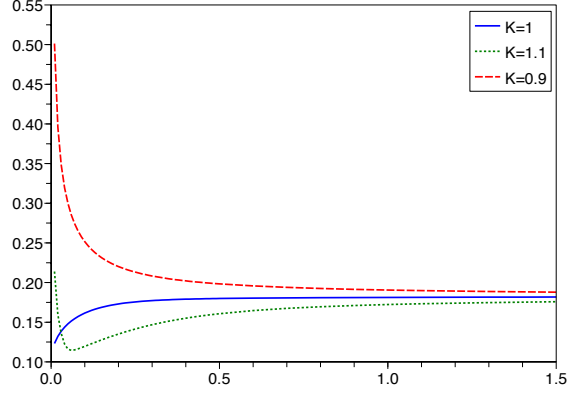


Figure 5: Smile asymptotics: term structure of implied volatility for different strikes in Merton's model with parameters  $\sigma = 0.1$ ,  $\lambda = 2$ ,  $\mu = -0.1$  and  $\delta = 0.5$ .

1. Let  $b$  denote the drift of  $X$ . Since  $X$  is a finite-variation process, the Itô-Tanaka formula applied to the function  $(1 - e^{X_\tau})^+$  does not yield a local time term, and we obtain

$$\begin{aligned} & E[(1 - e^{X_\tau})^+] \\ &= E \left[ b \int_0^\tau e^{X_t} 1_{X_t \leq 0} dt + \int_0^\tau \int_{\mathbb{R}} \nu(dx) \{(1 - e^{X_t+x})^+ - (1 - e^{X_t})^+\} \right] dt. \end{aligned}$$

By L'Hopital's rule,

$$\begin{aligned} & \lim_{\tau \rightarrow 0} \frac{1}{\tau} E[(1 - e^{X_\tau})^+] \\ &= b \lim_{\tau \rightarrow 0} E[e^{X_\tau} 1_{X_\tau \leq 0}] + \lim_{\tau \rightarrow 0} E \left[ \int_{\mathbb{R}} \nu(dx) \{(1 - e^{X_\tau+x})^+ - (1 - e^{X_\tau})^+\} \right]. \end{aligned}$$

From Theorem 43.20 in [59],  $\frac{X_t}{t} \rightarrow b$  almost surely as  $t \rightarrow 0$ . From this we deduce that  $\lim_{\tau \rightarrow 0} E[e^{X_\tau} 1_{X_\tau \leq 0}] = 1_{b \leq 0}$ . Using the dominated convergence for the second term above, we finally obtain

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau} E[(1 - e^{X_\tau})^+] = b 1_{b \leq 0} + \int_{\mathbb{R}} \nu(dx) (1 - e^x)^+.$$

Since by the martingale condition,

$$b + \int_{\mathbb{R}} (e^x - 1)\nu(dx) = 0,$$

this limit can be rewritten as

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau} E[(1 - e^{X_\tau})^+] = \max\left(\int (e^x - 1)^+ \nu(dx), \int (1 - e^x)^+ \nu(dx)\right).$$

Comparing this expression with the Black-Scholes ATM asymptotics, we obtain the desired result.

2. Let  $p_t$  denote the density of  $X_t$  (which exists and is square integrable under the hypotheses of this part). The ATM call option price is given by

$$c(\tau, 0) = \int (e^x - 1)^+ p_t(x) dx.$$

Let us fix a constant  $\beta < -1$  and define

$$\tilde{c}(\tau, 0) = \int (e^x - 1)^+ e^{\beta x} p_t(x) dx.$$

Then it follows from results in [58] that

$$|c(\tau, 0) - \tilde{c}(\tau, 0)| = \int (e^x - 1)^+ (1 - e^{\beta x}) p_t(dx) = O(\tau)$$

as  $\tau \rightarrow 0$ . This means that it is sufficient to study the decay properties of  $\tilde{c}$ . This function is a scalar product of the square integrable function  $p_t$  by the square integrable function  $(e^x - 1)^+ e^{\beta x}$ , whose Fourier transform is given by

$$\int_0^\infty e^{iux} (e^{(1+\beta)x} - e^{\beta x}) dx = \frac{1}{iu + \beta} - \frac{1}{iu + \beta + 1} = -\frac{1}{(u - i\beta)(u - i\beta - i)}.$$

By the Plancherel theorem we then have

$$\tilde{c}(\tau, 0) = -\frac{1}{2\pi} \int \frac{e^{\tau\psi(u)}}{(u - i\beta)(u - i - i\beta)} du = -\frac{1}{2\pi} \int \frac{e^{i\tau\gamma u - \tau|u|^\alpha f(u)}}{(u - i\beta)(u - i - i\beta)} du.$$

On the other hand, direct computation using Cauchy's integral formula (for  $\gamma \neq 0$ ) or elementary calculus (for  $\gamma = 0$ ) shows that

$$\int \frac{e^{i\gamma u\tau} du}{(u - i\beta)(u - i - i\beta)} = 2\pi(e^{\gamma\tau\beta} - e^{\gamma\tau(\beta+1)})1_{\gamma < 0} = O(\tau)$$

as  $\tau \rightarrow 0$ . Then, changing the variable of integration, we obtain

$$\tilde{c}(\tau, 0) = \frac{\tau^{1/\alpha}}{2\pi} \int \frac{e^{i\gamma z\tau^{1-1/\alpha}}(1 - e^{-|z|^\alpha f(\tau^{-1/\alpha}z)})}{(z - i\tau^{1/\alpha}\beta)(z - i\tau^{1/\alpha}(1 + \beta))} dz + O(\tau).$$

The dominated convergence theorem then yields

$$\tau^{-1/\alpha}\tilde{c}(\tau, 0) \rightarrow \frac{1}{2\pi} \int_{-\infty}^0 \frac{1 - e^{-c-|z|^\alpha}}{z^2} dz + \frac{1}{2\pi} \int_0^\infty \frac{1 - e^{-c+|z|^\alpha}}{z^2} dz$$

as  $\tau \rightarrow \infty$ . This result generalizes the findings of Carr and Wu [17]. Computing the integrals and comparing the result to the Black-Scholes at the money asymptotics, we obtain the final result.

3. Under the conditions of this part, we can write the characteristic exponent of  $X$  as  $\psi(u) = i\gamma u - f(u)u^2$  for a continuous bounded function  $f$  satisfying  $\lim_{u \rightarrow \infty} f(u) = \frac{\sigma^2}{2}$ . Then, exactly as in the previous part, the dominated convergence theorem yields

$$\frac{\tilde{c}(\tau, 0)}{\sqrt{\tau}} \rightarrow \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1 - e^{-\frac{\sigma^2 u^2}{2}}}{u^2} du = \frac{\sigma}{\sqrt{2\pi}},$$

which is equal to the Black-Scholes ATM asymptotics.

□

### 11.3 Flattening of the smile far from maturity

As the time to maturity  $\tau$  goes to infinity, the implied volatility  $I(\tau, k)$  in an exponential Lévy model converges to a constant value  $I(\infty)$  which does not depend on  $k$  (see Figures 5 and 3). As a result, the implied volatility smile flattens for long maturities, a phenomenon which is also observed in the options markets, although with a slower rate. This flattening has been often attributed (e.g. [16]) to the central limit theorem, according to which,

for a Lévy process with finite variance, the distribution of increments  $(X_\tau - E[X_\tau])/\sqrt{\tau}$  becomes approximately Gaussian as  $\tau$  goes to infinity. However, contrary to this intuition, the flattening of the smile is not a consequence of the central limit theorem, but, rather, of a “large deviation” principle which governs the tail behavior of the sample average of  $n$  i.i.d. random variables. In fact, as observed by Rogers and Tehranchi [56], the implied volatility flattens even in models where log-returns have infinite variance such as the finite moment log-stable process of [16].

To understand this, consider a Lévy process  $X$  with  $E[X_1] < \infty$ . Since in a risk-neutral model  $E[e^{X_t}] = 1$ , the Jensen inequality implies that  $E[X_t] < 0$  for all  $t$ . Therefore, by the law of large numbers,  $X_t \rightarrow -\infty$  almost surely as  $t \rightarrow \infty$ , which means that  $e^{X_t} \rightarrow 0$  a.s. The exercise of a long-dated call option is thus an event with a very small probability. The probability of such rare events is given by Cramér’s theorem, which is the cornerstone of the theory of large deviations, rather than by the CLT.

The normalized price of a call option with log-moneyness  $k$  can be written as

$$c(\tau, k) = E(e^{X_\tau} - e^k)^+ = \tilde{\mathbb{P}}[X_\tau \geq k] - e^k \mathbb{P}[X_\tau \geq k],$$

where we introduce the new probability  $\tilde{\mathbb{P}}$  via the Esscher transform:

$$\frac{d\tilde{\mathbb{P}}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} := e^{X_t}.$$

Denote  $\alpha = E[X_1]$  and  $\tilde{\alpha} = \tilde{E}[X_1]$ . An easy computation using Proposition 9 shows that

$$\begin{aligned} \alpha &= -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^x - x - 1)\nu(dx) < 0 \\ \tilde{\alpha} &= \frac{\sigma^2}{2} + \int_{\mathbb{R}} (xe^x - e^x + 1)\nu(dx) > 0. \end{aligned}$$

To make the probability of a rare event appear, we rewrite the option price as

$$c(\tau, k) = 1 - \tilde{\mathbb{P}} \left[ -\frac{X_\tau - \tilde{\alpha}\tau}{\tau} > \tilde{\alpha} - \frac{k}{\tau} \right] - e^k \mathbb{P} \left[ \frac{X_\tau - \alpha\tau}{\tau} \geq -\alpha + \frac{k}{\tau} \right].$$

These probabilities can be estimated with the help of the famous Cramér’s theorem which gives the exact convergence rate in the law of large numbers.

**Theorem 4** (Cramér). *Let  $\{X_i\}_{i \geq 1}$  be an i.i.d. sequence of random variables with  $E[X_i] = 0$  for all  $i$ . Then for all  $x \geq 0$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P \left[ \frac{1}{n} \sum_{i=1}^n X_i \geq x \right] = -I(x),$$

where  $I(x)$  is the Fenchel transform of the log-Laplace transform of  $X_1$ :

$$I(x) = \sup_{\theta} (\theta x - l(\theta)), \quad l(\theta) = \log E[e^{\theta X_1}].$$

Suppose that the Lévy measure  $\nu$  is such that<sup>1</sup>

$$\int_{|x|>1} x \nu(dx) < \infty \quad \text{and} \quad \int_{|x|>1} x e^x \nu(dx) < \infty \quad (75)$$

and define the log-Laplace transforms by

$$\tilde{l}(\theta) := \log \tilde{E}[e^{-\theta(X_1 - \tilde{\alpha})}] \quad \text{and} \quad l(\theta) := \log E[e^{\theta(X_1 - \alpha)}],$$

and the respective Fenchel transforms by  $\tilde{I}$  and  $I$ . A direct computation then shows that

$$\tilde{I}(\tilde{\alpha}) = I(-\alpha) = \sup_{\theta} \left\{ \frac{\sigma^2}{2} (\theta - \theta^2) - \int_{\mathbb{R}} (e^{\theta x} - \theta e^x - 1 + \theta) \nu(dx) \right\},$$

and that the functions  $\tilde{I}$  and  $I$  are finite and hence, continuous, in the neighborhood of, respectively,  $\tilde{\alpha}$  and  $-\alpha$ . Hence the sup above can be restricted to the interval  $\theta \in [0, 1]$ , since the function being maximized is concave and equal to 0 for  $\theta = 0$  and  $\theta = 1$ . Using Cramér's theorem and the continuity of  $\tilde{I}$  and  $I$ , we then obtain

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log(1 - c(\tau, k)) = \sup_{\theta \in [0, 1]} \left\{ \frac{\sigma^2}{2} (\theta - \theta^2) - \int_{\mathbb{R}} (e^{\theta x} - \theta e^x - 1 + \theta) \nu(dx) \right\}. \quad (76)$$

Note that this formula is valid for any  $k$ , we can even take  $k$  to be a function of  $\tau$  as long as  $k = o(\tau)$  as  $\tau \rightarrow \infty$ . Specializing this formula to the Black-Scholes model, where  $\nu \equiv 0$  and the sup can be computed explicitly, we get

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log(1 - c_{BS}(\tau, k, \sigma)) = \frac{\sigma^2}{8}.$$

---

<sup>1</sup>The finite moment log stable process of Carr and Wu [17] satisfies these hypotheses although the variance of the log-price is infinite in this model.

From Equation (76), it follows in particular that the implied volatility satisfies  $\tau I^2(\tau, k) \rightarrow \infty$  as  $\tau \rightarrow \infty$  (otherwise the call option price would not converge to 1). Since in the Black-Scholes model the option price depends only on  $\tau\sigma^2$  but not on  $\tau$  or  $\sigma$  separately, we can write

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau I^2(\tau, k)} \log(1 - c_{BS}(\tau I^2(\tau, k), k, 1)) = \frac{1}{8},$$

and combining this with (76), we obtain the final result:

**Proposition 19** ([69]). *Let  $X$  be a Lévy process with Lévy measure satisfying (75). Then the implied volatility  $I(\tau, k)$  in the exponential Lévy model  $S_t = S_0 e^{X_t}$  satisfies*

$$\lim_{\tau \rightarrow \infty} I^2(\tau, k) = 8 \sup_{\theta} \left\{ \frac{\sigma^2}{2} (\theta - \theta^2) - \int_{\mathbb{R}} (e^{\theta x} - \theta e^x - 1 + \theta) \nu(dx) \right\}. \quad (77)$$

The exact formula (77) for the limiting long-term implied volatility in an exponential Lévy model is difficult to use in practice: even if for some models such as variance gamma it yields a closed form expression, it is rather cumbersome. However, for small jump sizes, Taylor expansion shows that this expression is not very different from the total variance of the Lévy process:

$$I^2(\infty, k) \approx \sigma^2 + \int x^2 \nu(dx).$$

The smile flattening in exponential Lévy models has thus little to do with the so called aggregational normality of stock returns. One may think that the implied volatility converges to its limiting value faster for Lévy processes to which the central limit theorem applies. However, the results of Rogers and Tehranchi [56] suggest otherwise: they give the following upper bound, valid in exponential Lévy models as soon as  $E[|X_t|] < \infty$ , for the rate of convergence of the implied volatility skew to zero:

$$\limsup_{\tau \rightarrow \infty} \sup_{k_1, k_2 \in [-M, M]} \tau \left| \frac{I(\tau, k_2)^2 - I(\tau, k_1)^2}{k_2 - k_1} \right| \leq 4, \quad 0 < M < \infty.$$

See also [68] for explicit asymptotics of the the derivative  $\frac{\partial I(\tau, k)}{\partial k}$  as  $\tau \rightarrow \infty$ .

## 12 Hedging in exponential Lévy models

Exponential Lévy models generally correspond to incomplete markets, making exact replication impossible. Hedging must therefore be interpreted as approximation of the terminal pay-off with an admissible portfolio. The usual practice is to minimize the expected squared deviation of the hedging portfolio from the contingent claim, an approach known as quadratic hedging. The resulting strategies are often explicitly computable and, more importantly, they are linear, because the hedging portfolios can be interpreted as orthogonal projections of contingent claims onto the closed linear subspace of hedgeable portfolios. To hedge a book of options written on the same underlying, a trader can therefore compute the hedge ratio for every option in the book and then add them up, just like this is typically done with delta hedging. This greatly reduces the computational cost of hedging and is an important advantage of quadratic hedging compared to other, e.g., utility-based approaches.

To define the criterion to be minimized in a mean square sense, two approaches are possible. In the first approach [12, 52, 39], the hedging strategy is supposed to be self-financing, and one minimizes the quadratic hedging error at maturity, that is, the expected squared difference between the terminal value of the hedging portfolio and the option's pay-off:

$$\inf_{V_0, \phi} E[|V_T(\phi) - H|^2] \quad \text{where} \quad V_T(\phi) = V_0 + \int_0^T \phi_t^0 dS_t^0 + \int_0^T \phi_t dS_t, \quad (78)$$

where  $S^0$  is the risk-free asset. If the interest rate is constant, we can choose the zero-coupon bond with maturity  $T$  as the risk-free asset:  $S_t^0 = e^{-r(T-t)}$  and after discounting this problem becomes:

$$\inf_{\hat{V}_0, \phi} E[|V_T(\phi) - H|^2], \quad \text{where} \quad V_T = \hat{V}_0 + \int_0^T \phi_t d\hat{S}_t.$$

In the second approach [30, 29, 61, 64], strategies that are not self-financing are allowed, but they are required to replicate the option's pay-off exactly:  $V_T(\phi) = H$ . In an incomplete market, this means that the option's seller will have to continuously inject / withdraw money from the hedging portfolio. The cumulative amount of funds injected or withdrawn is called the *cost process*. It is given by

$$C_t(\phi) = V_t(\phi) - G_t(\phi),$$

where

$$V_t(\phi) = \phi_t^0 S_t^0 + \phi_t S_t$$

and  $G$  is the *gain process* given by

$$G_t = \int_0^t \phi_s^0 dS_s^0 + \int_0^t \phi_s dS_s.$$

The discounted cost process is then given by

$$\hat{C}_t = \phi_t^0 + \phi_t \hat{S}_t - \int_0^t \phi_s d\hat{S}_s.$$

The risk-minimizing strategy, as introduced by Föllmer and Sondermann [30], is a strategy which replicates the option's pay-off, and has the cost process which varies as little as possible, that is, this strategy minimizes, at each date  $t$ , the residual cost given by

$$E[(\hat{C}_T - \hat{C}_t)^2 | \mathcal{F}_t]. \quad (79)$$

over all admissible continuations of the strategy from date  $t$  onwards. The risk-minimizing strategy always exists in the martingale case (when the discounted stock price is a martingale), but in the general case, it may fail to exist even in the most simple examples [61]. Motivated by this difficulty, Föllmer and Schweizer [29] introduced the notion of *locally risk minimizing strategy*, which corresponds to finding the extremum of (79) with respect to suitably defined small perturbations of the strategy, or, in other words, measuring the riskiness of the cost process locally in time. Local risk minimization is discussed in detail in section 12.2.

The expectations in (79) and (78) are taken with respect to some probability which we have to specify. To begin, let us assume that we have chosen a martingale measure  $\mathbb{Q}$  and the expectations in (78) and (79) are taken with respect to  $\mathbb{Q}$ . In particular,  $\hat{S}$  is a martingale under  $\mathbb{Q}$ . Assume now that  $H \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$  and  $\hat{S}$  is also square-integrable. If we consider portfolios of the form:

$$\mathcal{S} = \left\{ \phi \text{ caglad predictable and } E \left| \int_0^T \phi_t d\hat{S}_t \right|^2 < \infty \right\} \quad (80)$$

then the set  $\mathbb{A}$  of attainable pay-offs is a closed linear subspace of  $L^2(\Omega, \mathcal{F}, \mathbb{Q})$ , and the quadratic hedging problem becomes an orthogonal projection:

$$\inf_{\hat{V}_0, \phi} E |V_T(\phi) - H|^2 = \inf_{A \in \mathbb{A}} \|H - A\|_{L^2(\mathbb{Q})}^2. \quad (81)$$



The solution is then given by the well-known Galtchouk-Kunita-Watanabe decomposition [43, 33], which states that any random variable  $H \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$  can be represented as

$$H = E[H] + \int_0^T \phi_t^H d\hat{S}_t + N_T^H, \quad (82)$$

where  $(N_t^H)$  is a square integrable martingale orthogonal to  $\hat{S}$ . The optimal hedging strategy is then given by  $\phi^H$  and the initial cost of the hedging portfolio is  $V_0 = e^{-r(T-t)}E[H]$ .

Introducing the martingale  $\hat{H}_t := E[H|\mathcal{F}_t]$  generated by  $H$ , we have

$$\hat{H}_t = E[H] + \int_0^t \phi_s^H d\hat{S}_s + N_t^H,$$

and the orthogonality implies

$$\langle \hat{H} - \int_0^\cdot \phi_s^H d\hat{S}_s, \hat{S} \rangle \equiv 0,$$

which means that the optimal hedge ratio may be expressed more explicitly using the predictable covariation of the option price and the stock price:

$$\phi_t^H = \frac{d\langle \hat{H}, \hat{S} \rangle_t}{d\langle \hat{S}, \hat{S} \rangle_t}. \quad (83)$$

In the martingale setting, optimizing the *global* hedging error (78) we obtain a strategy which is also risk minimizing in the sense of equation (79). For any strategy  $\phi$ , we have

$$\begin{aligned} E[(\hat{C}_T - \hat{C}_t)^2 | \mathcal{F}_t] &= (\hat{H}_t - \hat{V}_t)^2 + E \left[ \left( H - \hat{H}_t - \int_t^T \phi_s d\hat{S}_s \right)^2 \middle| \mathcal{F}_t \right] \\ &= (\hat{H}_t - \hat{V}_t)^2 + E[(N_T - N_t)^2 | \mathcal{F}_t] + E \left[ \left( \int_t^T (\phi_s - \phi_s^H) d\hat{S}_s \right)^2 \middle| \mathcal{F}_t \right]. \end{aligned}$$

To minimize this expression, we clearly need to take  $\phi = \phi^H$  and choose  $\phi^0$  such that  $\hat{V}_t = \hat{H}_t$  for all  $t$ . In this case, the discounted cost process is given by

$$\hat{C}_t = \hat{V}_t - \int_0^t \phi_s^H d\hat{S}_s = E[H] + N_t^H.$$

We shall see in section 12.2 that in the martingale setting, the strategy  $\phi^H$  which minimizes the terminal hedging error also coincides with the locally risk minimizing strategy of Föllmer and Schweizer [29]. Moreover, it is often easy to compute in terms of option prices. This is no longer true if  $\hat{S}$  is not a martingale. However using the *risk-neutral* second moment of the hedging error as a criterion for measuring risk is not very natural:  $\mathbb{Q}$  represents a pricing rule and not a statistical description of market events, so the profit and loss (P&L) of a portfolio may have a large variance while its “risk neutral” variance can be small. Nevertheless, to estimate the expected return of a stock, and therefore, to distinguish it from a martingale, one needs historical stock return observations covering an extended period of time, often exceeding the lifetime of the option. Option hedging, on the other hand, is a “local” business, where one tries to cancel out the daily movements of option prices with the daily movements of the underlying and locally, every stock behaves like a martingale. Without contributing to this ongoing argument, we review both approaches in the next two sections.

## 12.1 Quadratic hedging in exponential-Lévy models under the martingale measure

Although the quadratic hedging problem is “solved” by the Galtchouk-Kunita-Watanabe decomposition, from a practical point of view the problem is of course to compute the risk minimizing hedge  $\phi_t^H$ . Formulas for  $\phi_t^H$  with various degrees of explicitness and under various assumptions on the driving process  $X$  and on the pay-off  $G$  were given in [12, 28, 9, 36] and several other papers. In particular [24] provide the expressions for hedge ratios in the case when the hedging portfolio itself contains options. In the case of European pay-offs and exponential Lévy models, the problem was solved in [39] using Fourier analysis techniques. Their method, reviewed in section 12.2 covers the general case as well as the martingale case. In this section, we provide another Fourier-based result, which is specialized to the martingale setting but works under different regularity assumptions on the pay-off than in [39], which include, for instance, digital options.

**Proposition 20** (Quadratic hedge in exponential-Lévy models, martingale case). *Let  $X$  be a Lévy process with Lévy measure  $\nu$ , diffusion coefficient  $\sigma$ , and characteristic function  $\Phi$ , such that  $e^X$  is a martingale and assume:*

- i. The log-payoff function satisfies the conditions (37) and (38).*

ii. The integrability condition (39) holds for all  $t < T$ .

iii. The Lévy measure of  $X$  satisfies

$$\int_{|x|>1} e^{2(x \vee Rx)} \nu(dx) < \infty. \quad (84)$$

Then the optimal quadratic hedging for a European option with pay-off  $G(S_T)$  at date  $T$  in an exponential Lévy model  $S_t = S_0 e^{rt+X_t}$  amounts to holding a position in the underlying

$$\phi_t = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR) \Phi_{T-t}(-u - iR) \hat{S}_{t-}^{R-iu-1} \Upsilon(R - iu) du \quad (85)$$

$$\text{where } \Upsilon(y) = \frac{\kappa(y+1) - \kappa(y) - \kappa(1)}{\kappa(2) - 2\kappa(1)}, \quad \text{and } \kappa(z) := \log E[e^{zX_1}], \quad (86)$$

or, equivalently,  $\phi_t = \phi(t, S_{t-})$  where:

$$\phi(t, S) = \frac{\sigma^2 \frac{\partial P}{\partial S}(t, S) + \frac{1}{S} \int \nu(dz) (e^z - 1) [P(t, S e^z) - P(t, S)]}{\sigma^2 + \int (e^z - 1)^2 \nu(dz)} \quad (87)$$

with  $P(t, S) = e^{-r(T-t)} E^{\mathbb{Q}}[G(S_T) | S_t = S]$  the option price at date  $t$  when the underlying is at the level  $S$ .

*Remark 1.* Condition (84), which is the only assumption imposed in addition to those of Proposition 10, guarantees that both the price process  $S_t$  and the option pay-off  $G(S_T)$  are square integrable.

*Proof.* By Itô formula, the discounted stock price dynamics is given by

$$\hat{S}_T = \hat{S}_0 + \int_0^T \hat{S}_t \sigma dW_t + \int_0^T \int_{\mathbb{R}} \hat{S}_t (e^x - 1) \tilde{J}_X(dt \times dx). \quad (88)$$

To prove the proposition using the formula (83), we now need to obtain a similar integral representation for the option's discounted price function  $\hat{P}(t, S_t) = e^{r(T-t)} P(t, S_t)$ .

Let  $t < T$ . Applying the Itô formula under the integral sign in (40), we find

$$\begin{aligned} \hat{P}(t, S_t) - \hat{P}(0, S_0) &= \frac{1}{2\pi} \int_{\mathbb{R}} du \hat{g}(u + iR) \int_0^t \Phi_{T-s}(-u - iR) (R - iu) \hat{S}_s^{R-iu} \sigma dW_s \\ &+ \frac{1}{2\pi} \int_{\mathbb{R}} du \hat{g}(u + iR) \int_0^t \Phi_{T-s}(-u - iR) \hat{S}_{s-}^{R-iu} \int_{\mathbb{R}} (e^{(R-iu)z} - 1) \tilde{J}_X(ds \times dz). \end{aligned} \quad (89)$$

Let us first assume that  $\sigma > 0$  and study the first term in the right-hand side of (89), which can be written as

$$\int_{\mathbb{R}} \mu(du) \int_0^t H_s^u dW_s$$

where

$$\mu(du) = |\hat{g}(u + iR)\Phi_{T-t}(-u - iR)|du \quad (90)$$

is a finite positive measure on  $\mathbb{R}$  and

$$H_s^u = \frac{\sigma \hat{g}(u + iR)\Phi_{T-s}(-u - iR)}{2\pi |\hat{g}(u + iR)\Phi_{T-t}(-u - iR)|} (R - iu) \hat{S}_s^{R-iu}$$

By the Fubini theorem for stochastic integrals (see [53, page 208]), we can interchange the two integrals in (90) provided that

$$E \int_0^t \mu(du) |H_s^u|^2 ds < \infty \quad (91)$$

Under the assumption (39) it is easy to check that

$$\frac{\Phi_{T-s}(-u - iR)}{|\Phi_{T-t}(-u - iR)|} \leq C$$

for all  $s \leq t \leq T$  for some constant  $C > 0$  which does not depend on  $s$  and  $t$ . To prove (91) it is then sufficient to check

$$E \int_0^t \int_{\mathbb{R}} |\hat{g}(u + iR)\Phi_{T-t}(-u - iR)| |\hat{S}_s^{2(R-iu)}|^2 (R - iu)^2 dudt < \infty$$

which holds because

$$|\Phi_{T-t}(-u - iR)| \leq C e^{-(T-t)\frac{\sigma^2 u^2}{2}} \quad (92)$$

Therefore, the first term on the right-hand side of (89) is equal to

$$\int_0^t \tilde{\sigma}_s dW_s, \quad \tilde{\sigma}_s = \frac{\sigma}{2\pi} \int_{\mathbb{R}} du \hat{g}(u + iR)\Phi_{T-s}(-u - iR)(R - iu) \hat{S}_s^{R-iu}. \quad (93)$$

This also shows that  $\tilde{\sigma}_s = \sigma S_s \frac{\partial P(s, S_s)}{\partial S}$ .

Let us now turn to the second term in the right-hand side of (89). Here we need to apply the Fubini theorem for stochastic integrals with respect to a compensated Poisson random measure [4, Theorem 5] and the applicability condition boils down to

$$E \int_0^t \int_{\mathbb{R}} |\hat{g}(u+iR)\Phi_{T-t}(-u-iR)| |\hat{S}_s^{2(R-iu)}|^2 \int_{\mathbb{R}} |e^{(R-iu)z} - 1|^2 \nu(dz) du ds < \infty$$

If  $\sigma > 0$ , this is once again guaranteed by (92), and when  $\sigma = 0$ ,

$$\int_{\mathbb{R}} |e^{(R-iu)z} - 1|^2 \nu(dz) = \psi(-2iR) - 2\Re\psi(-u-iR).$$

Since, for some  $C < \infty$ ,

$$|\Re\psi(-u-iR)\Phi_{T-t}(-u-iR)| = |\Re\psi(-u-iR)e^{(T-t)\Re\psi(-u-iR)}| \leq Ce^{\frac{T-t}{2}\Re\psi(-u-iR)},$$

the integrability condition is satisfied and we conclude that

$$\hat{P}(t, S_t) - \hat{P}(0, S_0) = \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \int_{\mathbb{R}} \tilde{\gamma}_s(z) \tilde{J}_X(ds \times dz) \quad (94)$$

for all  $t < T$  with  $\tilde{\sigma}$  as above and

$$\begin{aligned} \tilde{\gamma}_s(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} du \hat{g}(u+iR)\Phi_{T-s}(-u-iR) \hat{S}_{s-}^{R-iu} (e^{(R-iu)z} - 1) \\ &= \hat{P}(s, S_{s-}e^z) - \hat{P}(s, S_{s-}). \end{aligned} \quad (95)$$

The optimal (risk-minimizing) hedge is obtained from formula (83):

$$\hat{\phi}_t = \frac{\sigma \hat{S}_t \tilde{\sigma}_t + \hat{S}_t \int_{\mathbb{R}} \nu(dz) (e^z - 1) \tilde{\gamma}_t(z)}{\hat{S}_t^2 (\sigma^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz))}.$$

Substituting the expressions for  $\tilde{\sigma}$  and  $\tilde{\gamma}$  in terms of option prices into the above expression, we obtain (87) directly. On the other hand, the Fourier representations (93) and (95) and an application of Fubini's theorem yield (85).  $\square$

As a by-product, the martingale representation (94) also yields the expression for the residual risk of a hedging strategy:

$$\begin{aligned} E[\epsilon(\phi)^2] &= E \left[ \int_0^T dt \int_{\mathbb{R}} \nu(dz) \left( \hat{P}(t, S_{t-}e^z) - \hat{P}(t, S_{t-}) - \hat{S}_{t-} \phi_t (e^z - 1) \right)^2 \right] \\ &+ E \left[ \int_0^T \hat{S}_{t-}^2 \left( \phi_t - \frac{\partial P}{\partial S}(t, S_{t-}) \right)^2 \sigma^2 dt \right]. \end{aligned} \quad (96)$$

This allows us to examine whether there are any cases where the hedging error can be reduced to zero, i.e., where one can achieve a perfect hedge for every option and the market is complete. Hedging error is zero if and only if, for almost all  $t$ , there exists  $k \in \mathbb{R}$  with:

$$\left(\sigma S_t \frac{\partial P}{\partial S}, (P(t, S_t e^z) - P(t, S_t))_{z \in \text{supp } \nu}\right) = k(\sigma S_t, (S_t(e^z - 1))_{z \in \text{supp } \nu})$$

This is only true in two (trivial) cases:

- The Lévy process  $X$  is a Brownian motion with drift:  $\nu = 0$  and we retrieve the Black-Scholes delta hedge

$$\phi_t = \Delta^{BS}(t, S_t) = \frac{\partial P}{\partial S}(t, S_t).$$

- The Lévy process  $X$  is a Poisson process with drift:  $\sigma = 0$  and there is a single possible jump size:  $\nu = \delta_{x_0}(x)$ . In this case the hedging error equals

$$E \left[ \int_0^T dt \left( \hat{P}(t, S_{t-} e^{x_0}) - \hat{P}(t, S_{t-}) - \hat{S}_{t-} \phi_t (e^{x_0} - 1) \right)^2 \right]$$

so by choosing

$$\phi_t = \frac{P(t, S_{t-} e^{x_0}) - P(t, S_{t-})}{S_{t-} (e^{x_0} - 1)}$$

we obtain a replication strategy.

In other cases, the market is incomplete (an explicit counter-example may be constructed using power option with pay-off  $H_T = (S_T)^\alpha$ ).

**Delta-hedging vs. optimal strategy** We see that the optimal strategy (87) can be represented as a weighted average of the delta hedging  $\frac{\partial P}{\partial S}$  and a certain integral involving the sensitivities of the option price to various possible jumps. But how far is the optimal strategy from the pure delta hedging? To answer this question, if option prices are regular (e.g. when  $\sigma > 0$ ) and jumps are small, we can perform a Taylor expansion with respect to the jump size in equation (87), obtaining

$$\Delta(t, S) = \frac{\partial P}{\partial S} + \frac{S}{2\Sigma^2} \frac{\partial^2 P}{\partial S^2} \int \nu(dz) (e^z - 1)^3.$$

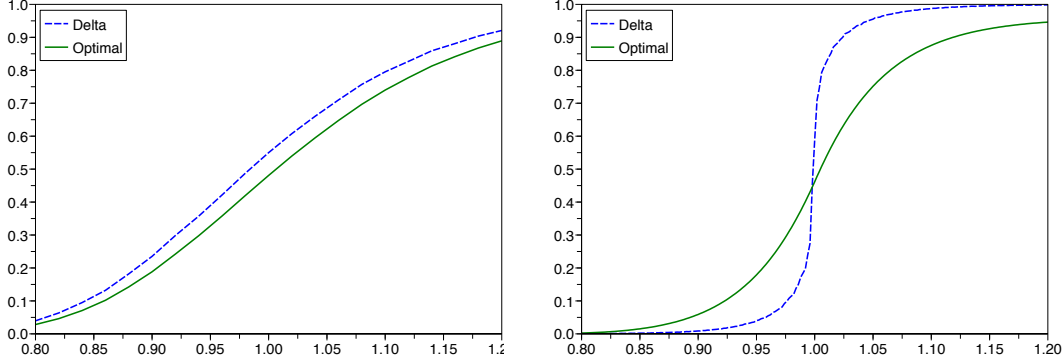


Figure 6: Hedge ratios for the optimal strategy of proposition 20 and the delta hedging strategy as function of stock price  $S$ . Left: hedging with stock in Kou model: the optimal strategy introduces a small asymmetry correction to delta hedging. Right: variance gamma model close to maturity (2 days): the optimal strategy is very far from delta hedging.

where

$$\Sigma^2 = \sigma^2 + \int (e^z - 1)^2 \nu(dz).$$

Typically in equity markets the jumps are negative and small, therefore  $\Delta(t, S) < \frac{\partial P}{\partial S}$  and the optimal strategy represents a small (of the order of third power of jump size) asymmetry correction. This situation is represented in Figure 6, left graph. On the other hand, for pure-jump processes such as variance gamma, we cannot perform the Taylor expansion, because the second derivative  $\frac{\partial^2 P}{\partial S^2}$  may not even exist, and the correction may therefore be quite large (see Figure 6, right graph).

**How big is the hedging error?** To answer this question, we simulated the terminal value of the hedging portfolio and that of the option's payoff over 10000 trajectories for different strategies and different parameter sets.

In the first case study, Kou model with parameters estimated from market data (MSFT) during a calm period was used, and the option to hedge was a European put with strike  $K = 90\%$  of the spot price and time to maturity  $T = 1$  year. The hedging errors are given in Table 1 and the left graph in Figure 7 shows the P&L histograms. For this parameter set, the optimal

<b>Strategy</b>	<b>Root of mean squared error</b>
Delta hedging	0.0133
Optimal quadratic	0.0133
Delta hedging in Black-Scholes model (error due to discrete hedging)	0.0059
No hedging	0.107

Table 1: Hedging errors for different strategies in Kou model expressed in percentage of the initial stock price. Model parameters were estimated from MSFT time series. The “Black-Scholes” strategy corresponds to delta-hedging in the Black-Scholes model with equivalent volatility.

strategy is very close to delta hedging, and consequently, the hedging error is the same for delta hedging as for the optimal strategy. On the other hand, this error is very low, it is only twice as big as what we would get in the Black and Scholes model with equivalent volatility (this error in the Black-Scholes model is due to the fact that in the simulations, the portfolio is only rebalanced once a day and not continuously).

In the second case study, Kou model with unfrequent large negative jumps (10%) was used, and we wanted once again to hedge an OTM European put ( $K = 90\%$ ,  $T = 1$ ). The hedging errors are given in Table 12.1 and the P&L histograms in Figure 7, right graph. Here we see that first, the optimal strategy has a much better performance than delta-hedging, and second, even this performance may not be sufficient, since the residual error is still of order of 4% of the initial stock price. This means that in this context, the market is “strongly incomplete” and hedging with stock only does not allow to make the risk at terminal date sufficiently small. In this case, to improve the hedging performance, one can include additional liquid assets, such as options on the same underlying, or variance swaps, into the hedging portfolio.

## 12.2 Quadratic hedging in exponential Lévy models under the historical measure

Throughout this section, to simplify notation, we suppose that the interest rate is equal to zero; the formulas for the general case can be obtained by working with discounted claims. Let  $S$  be the price process of the underlying,



Strategy	Root of mean squared error
Delta-hedging	0.051
Optimal quadratic	0.041
No hedging	0.156

Table 2: Hedging errors for different strategies in Kou model expressed in percentage of the initial stock price. A parameter set ensuring the presence of large negative jumps was taken.

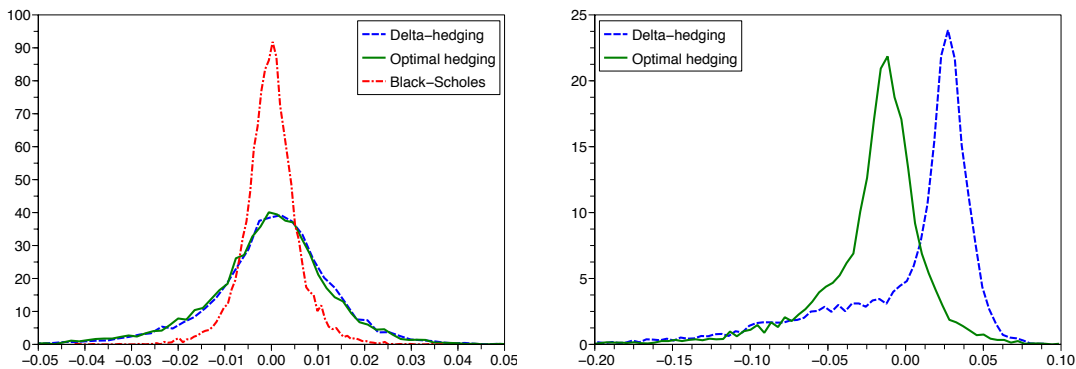


Figure 7: Histograms of the residual hedging error in Kou model. Left: parameters estimated from MSFT time series. This graph shows the residual hedging error in Kou model with the optimal quadratic strategy (solid line), in Kou model with the delta-hedging strategy (dashed line) and in the Black-Scholes model with the delta-hedging strategy (dash-dot line). In the latter case, the error is only due to discrete-time hedging, and this curve was include to assess the magnitude of the discretization error for other tests. Right: strong negative jumps.

and suppose that it can be written in the form

$$S_t = S_0 + M_t + \int_0^t \alpha_s d\langle M \rangle_s \quad (97)$$

for some square integrable martingale  $M$  and some predictable process  $\alpha$ . If  $S$  is an exponential of a Lévy process  $X$  with Lévy measure  $\nu$  satisfying  $\int_{|x|>1} e^{2x}\nu(dx) < \infty$  and diffusion coefficient  $\sigma$ , which can be written as

$$S_t = S_0 + \int_0^t \gamma S_u du + \int_0^t S_u \sigma dW_u + \int_0^t \int_{\mathbb{R}} S_{u-} (e^z - 1) \tilde{J}_X(du \times dz), \quad (98)$$

then the representation (97) holds with

$$\begin{aligned} M_t &= \int_0^t S_u \sigma dW_u + \int_0^t \int_{\mathbb{R}} S_{u-} (e^z - 1) \tilde{J}_X(du \times dz) \\ \langle M \rangle_t &= \int_0^t S_u^2 \left( \sigma^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz) \right) du \\ \alpha_t &= \frac{\gamma}{S_t \left( \sigma^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz) \right)} \end{aligned}$$

We then introduce the so-called *mean-variance tradeoff* process

$$K_t := \int_0^t \alpha_s^2 d\langle M \rangle_s.$$

In an exponential Lévy model, the mean-variance tradeoff is deterministic:

$$K_t = \frac{\gamma^2 t}{\sigma^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz)}.$$

**Local risk minimization** The *locally risk minimizing* strategy [29, 62] is a (not necessarily self-financing) trading strategy whose discounted cost process  $\hat{C}$  is a martingale orthogonal to  $M$ . This strategy is optimal in the sense that we eliminate all the risk associated to the underlying with hedging, and the only part of risk that remains in the cost process is the risk which is orthogonal to the fluctuations of the underlying, and hence, cannot be hedged with it. If the market is complete, then all risk is explained by the underlying and the cost process of a locally minimizing strategy becomes

constant, that is, the strategy becomes self-financing. As already mentioned, the locally risk minimizing strategy also has the interpretation of minimizing the residual risk (79) with respect to suitably defined small perturbations of the strategy [62]. Since the cost process is nonconstant, the locally risk minimizing strategy is not a self-financing strategy in general however since  $C$  is a martingale with mean zero this strategy is self-financing *on average*.

The locally risk minimizing strategy is closely related to an extension of the Kunita-Watanabe decomposition to semimartingale setting, known as the Föllmer-Schweizer decomposition [29, 61, 64, 65].

**Definition 13.** Let  $H \in L^2(\mathbb{P})$  be a contingent claim. A sum  $H = H_0 + \int_0^T \phi_u^H dS_u + L_T^H$  is called the *Föllmer-Schweizer decomposition* of  $H$  if  $H_0$  is  $\mathcal{F}_0$ -measurable,  $\phi^H$  is an admissible trading strategy and  $L^H$  is a square integrable martingale with  $L_0^H = 0$ , orthogonal to  $M$ .

Given a Föllmer-Schweizer decomposition for the claim  $H$ , the locally risk minimizing strategy for  $H$  can be constructed by taking  $\phi_t = \phi_t^H$  for all  $t$ , and choosing  $\phi^0$  such that the cost process is  $C_t = H_0 + L_t^H$  for all  $t$ , which amounts to  $\phi_t^0 = H_0 + L_t^H - \phi_t^H S_t - \int_0^t \phi_u^H dS_u$ .

**Relationship with the minimal martingale measure** Define a process  $Z$  via  $Z := \mathcal{E}(-\int_0^\cdot \alpha_s dM_s)$  and assume that  $Z$  is a strictly positive square integrable martingale. Then we can define a new measure  $\mathbb{Q}^M$  by  $\frac{d\mathbb{Q}^M|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} := Z_t$ . By Girsanov-Meyer theorem ([53], theorem 36 in chapter 3), we have that (i)  $\mathbb{Q}^M$  is a martingale measure, that is,  $S$  becomes a martingale under  $\mathbb{Q}$  and (ii) any square integrable martingale which is orthogonal to  $M$  under  $\mathbb{P}$  remains a martingale under  $\mathbb{Q}$  (although it may no longer be orthogonal to  $M$ ). This measure is known as the minimal martingale measure [2, 64].

The minimal martingale measure allows to express the Föllmer-Schweizer decomposition in a more explicit form. First, compute the process  $L^H$ :

$$L_t^H = E^{\mathbb{Q}^M}[L_T^H | \mathcal{F}_t] = E^{\mathbb{Q}^M}[H | \mathcal{F}_t] - H_0 - \int_0^t \phi_u^H dS_u.$$

Since  $L_0^H = 0$ , the initial capital for the Föllmer-Schweizer strategy is  $H_0 = E^{\mathbb{Q}^M}[H]$ . Let  $H_t^M := E^{\mathbb{Q}^M}[H | \mathcal{F}_t]$ . The orthogonality condition under  $\mathbb{P}$  then yields an analogue of formula (83):

$$\phi_t^H = \frac{d\langle H^M, S \rangle_t^{\mathbb{P}}}{d\langle S, S \rangle_t^{\mathbb{P}}}.$$

In models with jumps, the minimal martingale measure does not always exist as a probability measure (but may turn out to be a signed measure). In an exponential-Lévy model of the form (98), the density of the minimal martingale measure simplifies to  $Z = \mathcal{E}(U)$  with

$$U_t = -\frac{\gamma}{\sigma^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz)} \left\{ \sigma W_t + \int_0^t \int_{\mathbb{R}} (e^z - 1) \tilde{J}(ds \times dz) \right\}.$$

By Proposition 9, this yields a probability change if

$$\frac{\gamma(e^x - 1)}{\sigma^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz)} < 1 \quad \forall x \in \text{supp } \nu,$$

which imposes a strong restriction on the drift parameter  $\gamma$ . If this condition is not satisfied, the Föllmer-Schweizer decomposition may still exist, but the interpretation using the minimal martingale measure is no longer valid, and the initial capital may turn out to be negative.

The existence of a Föllmer-Schweizer decomposition has been studied by many authors (see for example [2, 64]), and in particular it was shown that the decomposition always exists in the case of exponential Lévy models. For these models, explicit formulas for the coefficients of this decomposition for European options are given in [39]:

**Proposition 21** (Föllmer-Schweizer decomposition for European options in exponential Lévy models [39]).

- *Case of exponential pay-offs.* Let  $z \in \mathbb{C}$  with  $S_T^z \in L^2(\mathbb{P})$ . Then the contingent claim  $H(z) = S_T^z$  admits a Föllmer-Schweizer decomposition with

$$\begin{aligned} \phi(z)_t &= \Upsilon(z) e^{\eta(z)(T-t)} S_{t-}^{z-1} \\ L(z)_t &= e^{\eta(z)(T-t)} S_t^z - e^{\eta(z)T} S_0^z - \int_0^t \phi(z)_u dS_u, \end{aligned}$$

where the coefficients  $\Upsilon$  and  $\eta$  are given by

$$\Upsilon(z) = \frac{k(z+1) - k(z) - k(1)}{k(2) - 2k(1)}, \quad \eta(z) = k(z) - k(1)\Upsilon(z),$$

and  $k(z) = \log E[e^{zX_1}]$  is the Laplace exponent of  $X$ .

- *Case of arbitrary payoffs.* Let the option payoff be  $H = f(S_T)$  with  $f$  of the form

$$f(s) = \int s^z \Pi(dz)$$

for some finite complex measure  $\Pi$  on a strip  $\{z \in \mathbb{C} : R' \leq \Re z \leq R\}$ , where  $R', R \in \mathbb{R}$  are chosen such that  $E[e^{2RX_1}] < \infty$  and  $E[e^{2R'X_1}] < \infty$ . Then  $H$  admits a Föllmer-Schweizer decomposition with coefficients

$$\begin{aligned} \phi_t^H &= \int \phi(z)_t \Pi(dz) \\ L_t^H &= \int L(z)_t \Pi(dz). \end{aligned}$$

*Example 7.* Let  $N^1$  and  $N^2$  be two standard Poisson processes with intensity 1 under  $\mathbb{P}$  and suppose that the stock price is given by

$$S_t = \gamma t + 2N_t^1 + N_t^2 - 3t,$$

and that the contingent claim to be hedged is

$$H = 5N_T^1.$$

Define

$$L_t = N_t^1 - 2N_t^2 + t$$

Then  $L$  is a  $\mathbb{P}$ -martingale and

$$[L, S]_t = 2N_t^1 - 2N_t^2$$

which means that  $L$  is orthogonal to the martingale part of  $S$  under  $P$ . It is now easy to check that the Föllmer-Schweizer decomposition for  $H$  is given by

$$H = (5 - 2\gamma)T + L_T + 2S_T.$$

The locally risk-minimizing strategy therefore consists in

- Buying 2 units of the risky asset at date  $t = 0$  (at zero price) and holding them until maturity.
- Placing  $(5-2\gamma)T$  at the bank account and dynamically adding/withdrawing money according to the value of  $L$ .

The initial cost of this strategy is thus equal to  $H_0 = (5 - 2\gamma)T$ , which can be both positive and negative (if  $\gamma > \frac{5}{2}$ ), and therefore cannot be interpreted as the price of the claim  $H$ . An intuitive explanation is that when the stock returns are very high, one can obtain a terminal pay-off which is (on average) positive even with a negative initial capital.

The minimal martingale measure in this setting is defined by

$$\frac{d\mathbb{Q}^M|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = Z_t, \quad \frac{dZ_t}{Z_{t-}} = -\frac{\gamma}{5}(2dN_t^1 + dN_t^2 - 3dt).$$

From Proposition 9, we deduce that  $\mathbb{Q}^M$  is a probability measure if and only if  $\gamma < \frac{5}{2}$ , in which case  $N^1$  and  $N^2$  are independent Poisson processes under  $\mathbb{Q}^M$ , with intensities

$$\lambda_1 = 1 - \frac{2\gamma}{5} \quad \text{and} \quad \lambda_2 = 1 - \frac{\gamma}{5}.$$

Easy calculations show that

- The martingale property of  $L$  is preserved under  $\mathbb{Q}^M$ , and in particular, we can compute

$$H_t^M = E^{\mathbb{Q}^M}[H|\mathcal{F}_t] = 5\lambda_1 T + 5(N_t^1 - \lambda_1 t)$$

and 
$$\phi_t^H = \frac{d\langle H^M, S \rangle_t^{\mathbb{P}}}{d\langle S, S \rangle_t^{\mathbb{P}}} = 2.$$

- On the other hand, the orthogonality of  $S$  and  $L$  is not preserved under  $\mathbb{Q}^M$ : this would require  $[L, S]_t = 2N_t^1 - 2N_t^2$  to be a  $\mathbb{Q}^M$ -martingale, which holds if and only if  $\lambda_1 = \lambda_2$ .

**Variance-optimal hedging** An alternative approach is to choose a self-financing strategy  $\phi$  and the initial capital  $V_0$  such as to minimize

$$E^{\mathbb{P}} \left[ (V_0 + G_T(\phi) - H)^2 \right].$$

under the statistical measure  $\mathbb{P}$ . This approach, known as mean-variance hedging or variance optimal hedging, is described in many papers including [12, 52, 65, 11, 39, 18]. The general results concerning existence of optimal

strategies are given in [18]. Schweizer [63] studies the case where the mean-variance tradeoff process  $K$  is deterministic and shows that in this case, the variance-optimal hedging strategy is also linked to the Föllmer-Schweizer decomposition. Hubalek et al. [39] exploit these results to derive explicit formulas for the hedging strategy in the case of Lévy processes. The following proposition uses the notation of Proposition 21.

**Proposition 22** (Mean variance hedging in exponential Lévy models [39]). *Let the contingent claim  $H$  be as in the second part of Proposition 21. Then the variance optimal initial capital and the variance optimal hedging strategy are given by*

$$\begin{aligned} V_0 &= H_0 \\ \phi_t &= \phi_t^H + \frac{\lambda}{S_{t-}}(H_{t-} - V_0 - G_{t-}(\phi)), \end{aligned} \quad (99)$$

where  $\lambda = \frac{\kappa(1)}{\kappa(2) - 2\kappa(1)}$  and

$$H_t = \int S_t^z e^{\eta(z)(T-t)} \Pi(dz).$$

In the case of exponential Lévy models, and in all models with deterministic mean-variance tradeoff, the variance optimal initial wealth is therefore equal to the initial value of the locally risk minimizing strategy. This allows to interpret the above result as a “stochastic target” approach to hedging, where the locally risk minimizing portfolio  $H_t$  plays the role of a “stochastic target” which we would like to follow because it allows to approach the option’s pay-off with the least fluctuations. Since the locally risk-minimizing strategy is not self-financing, if we try to follow it with a self-financing strategy, our portfolio may deviate from the locally risk minimizing portfolio upwards or downwards. The strategy (99) measures this deviation at each date and tries to compensate it by investing more or less in the stock, depending on the sign of the expected return ( $\lambda$  is the expected excess return divided by the square of the volatility).

## 13 Calibration of exp-Lévy models

In the Black-Scholes setting, the only model parameter to choose is the volatility  $\sigma$ , originally defined as the annualized standard deviation of logarithmic stock returns. The notion of model calibration does not exist, since

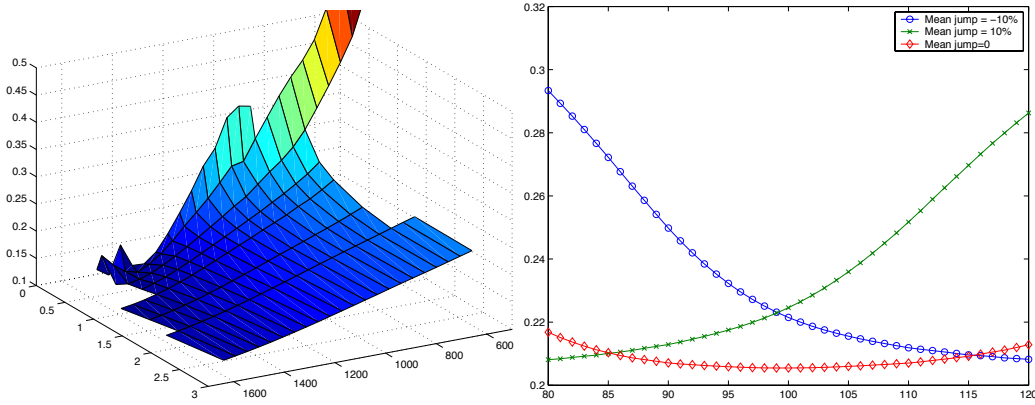


Figure 8: Left: implied volatilities of options on S&P 500 index as a function of their strikes and maturities. Right: implied volatilities as a function of strike for different values of the mean jump size in Merton jump diffusion model. Other parameters: volatility  $\sigma = 0.2$ , jump intensity  $\lambda = 1$ , jump standard deviation  $\delta = 0.05$ , option maturity  $T = 1$  month.

after observing a trajectory of the stock price, the pricing model is completely defined. On the other hand, since the pricing model is defined by a single volatility parameter, this parameter can be reconstructed from a single option price (by inverting the Black-Scholes formula). This value is known as the implied volatility of this option.

If the real markets obeyed the Black-Scholes model, the implied volatility of all options written on the same underlying would be the same and equal to the standard deviation of returns of this underlying. However, empirical studies show that this is not the case: implied volatilities of options on the same underlying depend on their strikes and maturities (figure 8, left graph).

Jump-diffusion models provide an explanation of the implied volatility smile phenomenon since in these models the implied volatility is both different from the historical volatility and changes as a function of strike and maturity. Figure 8, right graph shows possible implied volatility patterns (as a function of strike) in the Merton jump-diffusion model.

The results of calibration of the Merton model to S&P index options are presented in figure 9. The calibration was carried out separately for each maturity using the routine [8] from Premia software. In this program, the vector of unknown parameters  $\theta$  is found by minimizing numerically the



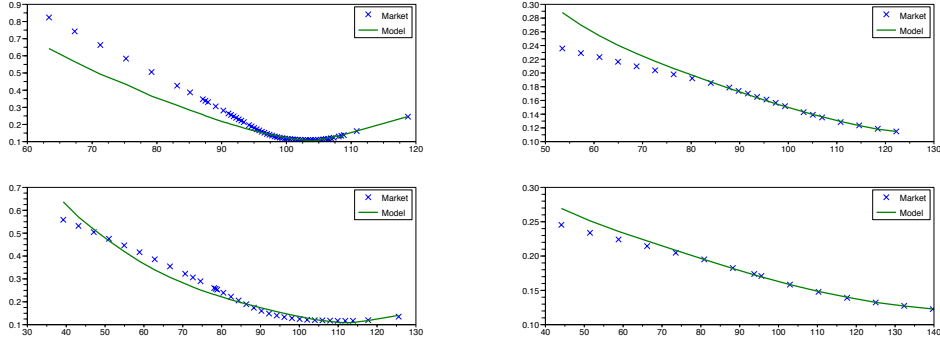


Figure 9: Calibration of Merton jump-diffusion model to market data separately for each maturity. Top left: maturity 1 month. Bottom left: maturity 5 months. Top right: maturity 1.5 years. Bottom right: maturity 3 years.

squared norm of the difference between market and model prices:

$$\theta^* = \arg \inf \|P^{obs} - P^\theta\|^2 \equiv \arg \inf \sum_{i=1}^N w_i (P_i^{obs} - P^\theta(T_i, K_i))^2, \quad (100)$$

where  $P^{obs}$  denotes the prices observed in the market and  $P^\theta(T_i, K_i)$  is the Merton model price computed for parameter vector  $\theta$ , maturity  $T_i$  and strike  $K_i$ . Here, the weights  $w_i := \frac{1}{(P_i^{obs})^2}$  were chosen to ensure that all terms in the minimization functional are of the same order of magnitude. The model prices were computed simultaneously for all strikes present in the data using the FFT-based algorithm described in section 8. The functional in (100) was then minimized using a quasi-newton method (LBFGS-B described in [13]). In the case of Merton model, the calibration functional is sufficiently well behaved, and can be minimized using this convex optimization algorithm. In more complex jump-diffusion models, in particular, when no parametric shape of the Lévy measure is assumed, a penalty term must be added to the distance functional in (100) to ensure convergence and stability. This procedure is described in detail in [21, 22, 67].

The calibration for each individual maturity is quite good, however, although the options of different maturities correspond to the same trading day and the same underlying, the parameter values for each maturity are different, as seen from table 3. In particular, the behavior for short (1 to 5 months) and long (1 to 3 years) maturities is qualitatively different, and

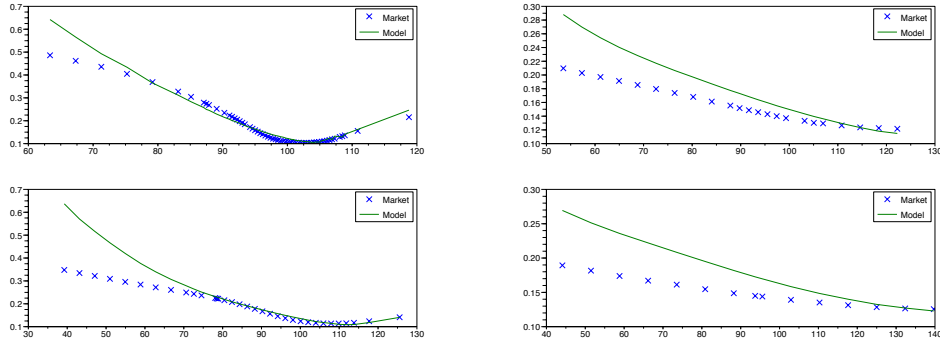


Figure 10: Calibration of Merton jump-diffusion model simultaneously to 4 maturities. Calibrated parameter values:  $\sigma = 9.0\%$ ,  $\lambda = 0.39$ , jump mean  $-0.12$  and jump standard deviation  $0.15$ . Top left: maturity 1 month. Bottom left: maturity 5 months. Top right: maturity 1.5 years. Bottom right: maturity 3 years.

for longer maturities the mean jump size tends to increase while the jump intensity decreases with the length of the holding period.

Figure 10 shows the result of simultaneous calibration of Merton model to options of 4 different maturities, ranging from 1 month to 3 years. As we see, the calibration error is much bigger than in figure 9. This happens because for processes with independent and stationary increments (and the log-price in Merton model is an example of such process), the law of the entire process is completely determined by its law at any given time  $t$  (this follows from the Lévy-Khintchine formula — equation 10). If we have calibrated the model parameters for a single maturity  $T$ , this fixes completely the risk-neutral stock price distribution for all other maturities. A special kind of maturity dependence is therefore hard-wired into every Lévy jump diffusion model, and table 3 shows that it does not always correspond to the term structures of market option prices.

To calibrate a jump-diffusion model to options of several maturities at the same time, the model must have a sufficient number of degrees of freedom to reproduce different term structures. This is possible for example in the Bates model (101), where the smile for short maturities is explained by the presence of jumps whereas the smile for longer maturities and the term structure of implied volatility is taken into account using the stochastic volatility process. Figure 11 shows the calibration of the Bates model to the same data set as

<b>Maturity</b>	<b><math>\sigma</math></b>	<b><math>\lambda</math></b>	<b>jump mean</b>	<b>jump std. dev.</b>
1 month	9.5%	0.097	-1.00	0.71
2 months	9.3%	0.086	-0.99	0.63
5 months	10.8%	0.050	-0.59	0.41
11 months	7.1%	0.70	-0.13	0.11
17 months	8.2%	0.29	-0.25	0.12
23 months	8.2%	0.26	-0.27	0.15
35 months	8.8%	0.16	-0.38	0.19

Table 3: Calibrated Merton model parameters for different times to maturity.

above. As we see, the calibration quality has improved and is now almost as good as when each maturity was calibrated separately. The calibration was once again carried out using the tool [8] from Premia.

## 14 Limits and extensions of Lévy processes

Despite the fact that Lévy processes reproduce the implied volatility smile for a single maturity quite well, when it comes to calibrating several maturities at the same time, the calibration by Lévy processes becomes much less precise. This is clearly seen from the three graphs of Figure 12. The top graph shows the market implied volatilities for four maturities and different strikes. The bottom left graphs depicts implied volatilities, computed in an exponential Lévy model calibrated using a nonparametric algorithm to the first maturity present in the market data. One can see that while the calibration quality is acceptable for the first maturity, it quickly deteriorates as the time to maturity increases: the smile in an exponential Lévy model flattens too fast. The same effect can be observed in the bottom right graph: here, the model was calibrated to the last maturity, present in the data. As a result, the calibration quality is poor for the first maturity: the smile in an exponential Lévy model is more pronounced and its shape does not resemble that of the market.

It is difficult to calibrate an exponential Lévy model to options of several maturities because due to independence and stationarity of their increments, Lévy processes have a very rigid term structure of cumulants. In particular, the skewness of a Lévy process is proportional to the inverse square root of time and the excess kurtosis is inversely proportional to time [49]. A

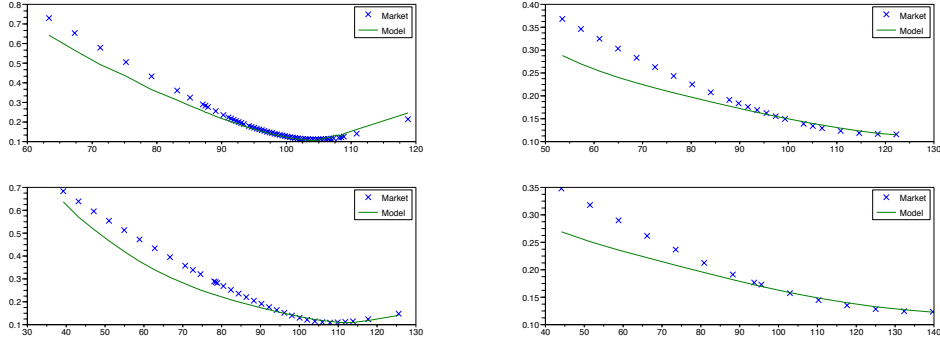


Figure 11: Calibration of the Bates stochastic volatility jump-diffusion model simultaneously to 4 maturities. Top left: maturity 1 month. Bottom left: maturity 5 months. Top right: maturity 1.5 years. Bottom right: maturity 3 years. Calibrated parameters (see equation (101)): initial volatility  $\sqrt{V_0} = 12.4\%$ , rate of volatility mean reversion  $\xi = 3.72$ , long-run volatility  $\sqrt{\eta} = 11.8\%$ , volatility of volatility  $\theta = 0.501$ , correlation  $\rho = -48.8\%$ , jump intensity  $\lambda = 0.038$ , mean jump size  $-1.14$ , jump standard deviation  $0.73$ .

number of empirical studies have compared the term structure of skewness and kurtosis implied in market option prices to the skewness and kurtosis of Lévy processes. Bates [6], after an empirical study of implicit kurtosis in  $\$/DM$  exchange rate options concludes that “while implicit excess kurtosis does tend to increase as option maturity shrinks, . . . , the magnitude of maturity effects is not as large as predicted [by a Lévy model]”. For stock index options, Madan and Konikov [49] report even more surprising results: both implied skewness and kurtosis actually decrease as the length of the holding period becomes smaller. It should be mentioned, however, that implied skewness/kurtosis cannot be computed from a finite number of option prices with high precision.

A second major difficulty arising while trying to calibrate an exponential Lévy model is the time evolution of the smile. Exponential Lévy models belong to the class of so called “sticky moneyness” models, meaning that in an exponential Lévy model, the implied volatility of an option with given moneyness (strike price to spot ratio) does not depend on time. This can be seen from the following simple argument. In an exponential Lévy model  $Q$ , the implied volatility  $\sigma$  of a call option with moneyness  $m$ , expiring in  $\tau$

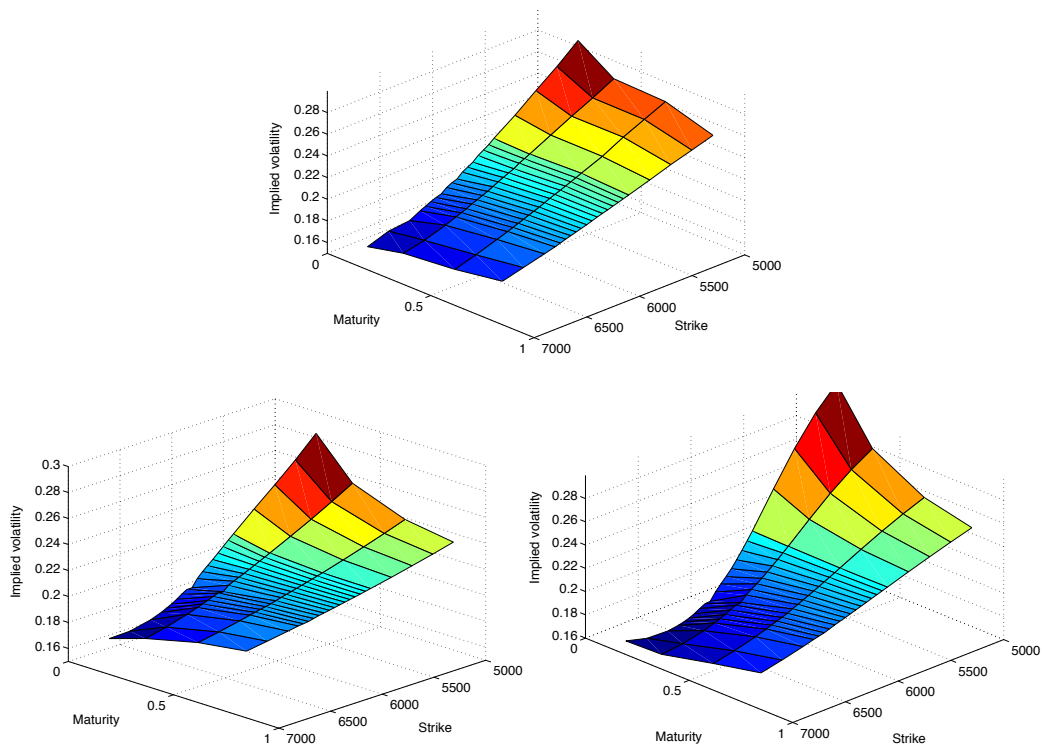


Figure 12: Top: Market implied volatility surface. Bottom left: implied volatility surface in an exponential Lévy model, calibrated to market prices of the first maturity. Bottom right: implied volatility surface in an exponential Lévy model, calibrated to market prices of the last maturity.

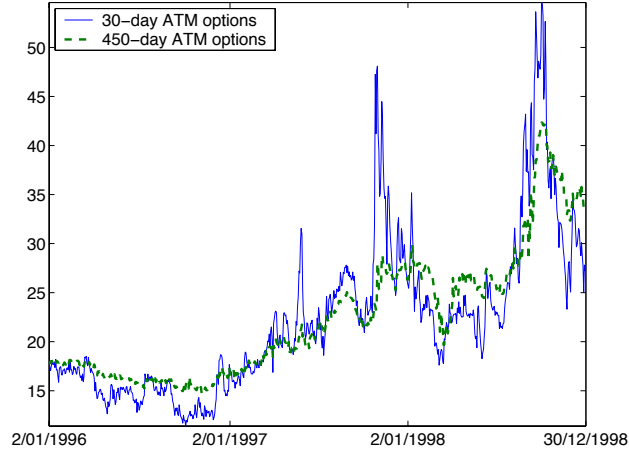


Figure 13: Implied volatility of at the money European options on CAC40 index.

years, satisfies:

$$e^{-r\tau} E^Q[(S_t e^{r\tau + X_\tau} - mS_t)^+ | \mathcal{F}_t] = e^{-r\tau} E[(S_t e^{r\tau + \sigma W_\tau - \frac{\sigma^2}{2}\tau} - mS_t)^+ | \mathcal{F}_t]$$

Due to the independent increments property,  $S_t$  cancels out and we obtain an equation for the implied volatility  $\sigma$  which does not contain  $t$  or  $S_t$ . Therefore, in an exp-Lévy model this implied volatility does not depend on date  $t$  or stock price  $S_t$ . This means that once the smile has been calibrated for a given date  $t$ , its shape is fixed for all future dates. Whether or not this is true in real markets can be tested in a model-free way by looking at the implied volatility of at the money options with the same maturity for different dates. Figure 13 depicts the behavior of implied volatility of two at the money options on the CAC40 index, expiring in 30 and 450 days. Since the maturities of available options are different for different dates, to obtain the implied volatility of an option with fixed maturity  $T$  for each date, we have taken two maturities, present in the data, closest to  $T$  from above and below:  $T_1 \leq T$  and  $T_2 > T$ . The implied volatility  $\Sigma(T)$  of the hypothetical option with maturity  $T$  was then interpolated using the following formula:

$$\Sigma^2(T) = \Sigma^2(T_1) \frac{T_2 - T}{T_1 - T} + \Sigma^2(T_2) \frac{T - T_1}{T_2 - T_1}.$$

As we have seen, in an exponential Lévy model the implied volatility of an option which is at the money and has fixed maturity must not depend on

time or stock price. Figure 13 shows that in reality this is not so: both graphs are rapidly varying random functions.

This simple test shows that real markets do not have the “sticky money-ness” property: arrival of new information can alter the form of the smile. The exponential Lévy models are therefore “not random enough” to account for the time evolution of the smile. Moreover, models based on additive processes, that is, time-inhomogeneous processes with independent increments, although they perform well in calibrating the term structure of implied volatilities for a given date [20], are not likely to describe the time evolution of the smile correctly since in these models the future form of the smile is still a deterministic function of its present shape [20]. To describe the time evolution of the smile in a consistent manner, one may need to introduce additional stochastic factors (e.g. stochastic volatility).

Several models combining jumps and stochastic volatility appeared in the literature. In the Bates [5] model, one of the most popular examples of the class, an independent jump component is added to the Heston stochastic volatility model:

$$\begin{aligned} dX_t &= \mu dt + \sqrt{V_t} dW_t^X + d\mathbf{Z}_t, & S_t &= S_0 e^{X_t}, \\ dV_t &= \xi(\eta - V_t)dt + \theta \sqrt{V_t} dW_t^V, & d\langle W^V, W^X \rangle_t &= \rho dt, \end{aligned} \quad (101)$$

where  $Z$  is a compound Poisson process with Gaussian jumps. Although  $X_t$  is no longer a Lévy process, its characteristic function is known in closed form [20, chapter 15] and the pricing and calibration procedures are similar to those used for Lévy processes.

## References

- [1] Abramowitz, M. and Stegun, I., eds., *Handbook of Mathematical Functions*, Dover: New York, 1968.
- [2] Ansel, J. P. and Stricker, C., *Lois de martingale, densités et décomposition de Föllmer-Schweizer*, *Annales de l’Institut Henri Poincaré*, 28, pp. 375–392.
- [3] Appelbaum, D., *Lévy Processes and Stochastic Calculus*, Cambridge University Press, 2004.

- [4] Appelbaum, D., *Martingale-valued measures, Ornstein-Uhlenbeck processes with jumps and operator self-decomposability in Hilbert space*, in In Memoriam Paul-André Meyer, Seminaire de Probabilités XXXIX, Springer, 2006, pp. 171–196.
- [5] D. BATES, *Jumps and stochastic volatility: the exchange rate processes implicit in Deutschemark options*, Rev. Fin. Studies, 9 (1996), pp. 69–107.
- [6] D. S. BATES, *Testing option pricing models*, in Statistical Methods in Finance, vol. 14 of Handbook of Statistics, North-Holland, Amsterdam, 1996, pp. 567–611.
- [7] Belomestny, D. and Reiss, M., *Spectral calibration of exponential Lévy models*, Finance and Stochastics, 10 (2006), pp. 449–474.
- [8] A. BEN HAJ YEDDER, *Calibration of stochastic volatility model with jumps*. A computer program, part of Premia software. See [www.premia.fr](http://www.premia.fr), 2004.
- [9] Benth, F. E., Di Nunno, G., Løkka, A., Øksendal, B., and Proske, F., *Explicit representation of the minimal variance portfolio in markets driven by Lévy processes*, Mathematical finance, 13 (2003), pp. 55–72.
- [10] Bertoin, J., *Lévy Processes*, Cambridge University Press: Cambridge, 1996.
- [11] Biagini, F. and Guasoni, P., *Mean-variance hedging with random volatility jumps*, Stochastic Analysis and Applications, 20 (2002).
- [12] Bouleau, N. and Lamberton, D., *Residual risks and hedging strategies in Markovian markets*, Stochastic Process. Appl., 33 (1989), pp. 131–150.
- [13] R. BYRD, P. LU, AND J. NOCEDAL, *A limited memory algorithm for bound constrained optimization*, SIAM Journal on Scientific and Statistical Computing, 16 (1995), pp. 1190–1208.
- [14] Carr, P., Geman, H., Madan, D., and Yor, M., *The fine structure of asset returns: An empirical investigation*, Journal of Business, 75 (2002), pp. 305–332.



- [15] Carr, P. and Madan, D., *Option valuation using the fast Fourier transform*, J. Comput. Finance, 2 (1998), pp. 61–73.
- [16] Carr, P. and Wu, L., *The finite moment logstable process and option pricing*, Journal of Finance, 58 (2003), pp. 753–778.
- [17] Carr, P. and Wu, L., *What type of process underlies options? A simple robust test*, J. Finance, 58 (2003).
- [18] Černý, A. and Kallsen, J., *On the structure of general mean-variance hedging strategies*, The Annals of Probability, 35 (2007), pp. 1479–1531.
- [19] Cherny, A. S. and Shiryaev, A. N., *Change of time and measure for Lévy processes*. MaPhySto lecture notes series 2002-13, 2002.
- [20] Cont, R. and Tankov, P., *Financial Modelling with Jump Processes*, Chapman & Hall / CRC Press, 2004.
- [21] Cont, R. and Tankov, P., *Non-parametric calibration of jump-diffusion option pricing models*, J. Comput. Finance, 7 (2004), pp. 1–49.
- [22] R. CONT AND P. TANKOV, *Retrieving Lévy processes from option prices: Regularization of an ill-posed inverse problem*, SIAM Journal on Control and Optimization, 45 (2006), pp. 1–25.
- [23] ———, *Constant proportion portfolio insurance in the presence of jumps in asset prices*, Mathematical Finance, 19 (2009), pp. 379–401.
- [24] Cont, R., Tankov, P., and Voltchkova, E., *Hedging with options in models with jumps*, in Stochastic Analysis and Applications - the Abel Symposium 2005, Springer, 2007.
- [25] R. CONT AND E. VOLTCHKOVA, *Integro-differential equations for option prices in exponential Lévy models*, Finance and Stochastics, 9 (2005), pp. 299–325.
- [26] J. W. COOLEY AND J. W. TUKEY, *An algorithm for the machine calculation of complex Fourier series*, Math. Comp., 19 (1965), pp. 297–301.

- [27] Eberlein, E., Glau, K., and Papapantoleon, A., *Analysis of Fourier transform valuation formulas and applications*, Appl. Math. Finance, 13 (2010), pp. 211–240.
- [28] Föllmer, H. and Elliott, R., *Orthogonal martingale representation*, in Stochastic Analysis, Mayer-Wolf, E., Merzbach, E., and Schwartz, A., eds., Academic Press: Boston, 1991, pp. 139–152.
- [29] Föllmer, H. and Schweizer, M., *Hedging of contingent claims under incomplete information*, in Applied Stochastic Analysis, Davis, M. H. A. and Elliott, R. J., eds., Gordon and Breach, 1991, pp. 389–414.
- [30] Föllmer, H. and Sondermann, D., *Hedging of non-redundant contingent claims*, in Contributions to Mathematical Economics, Hildenbrand, W. and Mas-Colell, A., eds., North Holland: Amsterdam, 1986, pp. 205–223.
- [31] Friz, P. and Benaim, S., *Smile asymptotics II: Models with known MGF*, Journal of Applied Probability, 45 (2008), pp. 16–32.
- [32] Friz, P. and Benaim, S., *Regular variation and smile asymptotics*, Mathematical Finance, 19 (2009), pp. 1–12.
- [33] Galtchouk, L., *Représentation des martingales engendrées par un processus à accroissements indépendants (cas des martingales de carré intégrable)*, Ann. Inst. H. Poincaré Sect. B (N.S.), 12 (1976), pp. 199–211.
- [34] H. GEMAN, D. MADAN, AND M. YOR, *Asset prices are Brownian motion: Only in business time*, in Quantitative Analysis in Financial Markets, M. Avellaneda, ed., World Scientific, River Edge, NJ, 2001, pp. 103–146.
- [35] Goll, T. and Kallsen, J., *Optimal portfolios for logarithmic utility*, Stochastic Processes and their applications, 89 (2000), pp. 31–48.
- [36] Jacod, J., Méléard, S., and Protter, P., *Explicit form and robustness of martingale representations*, Annals of Probability, 28 (2000), pp. 1747–1780.

- [37] Jakubenas, P., *On option pricing in certain incomplete markets*, Proceedings of the Steklov Mathematical Institute, 237 (2002).
- [38] J. KALLSEN AND P. TANKOV, *Characterization of dependence of multi-dimensional Lévy processes using Lévy copulas*, Journal of Multivariate Analysis, 97 (2006), pp. 1551–1572.
- [39] Kallsen, J., Hubalek, F., and Krawczyk, L., *Variance-optimal hedging for processes with stationary independent increments*, The Annals of Applied Probability, 16 (2006), pp. 853–885.
- [40] Kardaras, C., *No-free-lunch equivalences for exponential Lévy models under convex constraints on investment*, Math. Finance, 19 (2009), pp. 161–187.
- [41] Koponen, I., *Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process.*, Physical Review E, 52 (1995), pp. 1197–1199.
- [42] Kou, S., *A jump-diffusion model for option pricing*, Management Science, 48 (2002), pp. 1086–1101.
- [43] Kunita, H. and Watanabe, S., *On square integrable martingales*, Nagoya Math. J., 30 (1967), pp. 209–245.
- [44] Kyprianou, A., *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer, 2006.
- [45] Lee, R., *The moment formula for implied volatility at extreme strikes*, Mathematical Finance, 14 (2004), pp. 469–480.
- [46] Lee, R. W., *Option pricing by transform methods: extensions, unification and error control*, J. Comput. Finance, 7 (2004).
- [47] Lewis, A., *A simple option formula for general jump-diffusion and other exponential Lévy processes.* available from <http://www.optioncity.net>, 2001.
- [48] Madan, D., Carr, P., and Chang, E., *The variance gamma process and option pricing*, European Finance Review, 2 (1998), pp. 79–105.

- [49] D. MADAN AND M. KONIKOV, *Option pricing using variance gamma Markov chains*, Rev. Derivatives Research, 5 (2002), pp. 81–115.
- [50] D. MADAN AND F. MILNE, *Option pricing with variance gamma martingale components*, Math. Finance, 1 (1991), pp. 39–55.
- [51] Merton, R., *Option pricing when underlying stock returns are discontinuous*, J. Financial Economics, 3 (1976), pp. 125–144.
- [52] Pham, H., *Hedging and optimization problems in continuous financial models*, in Mathematical Finance: Theory and Practice, Cont, R. and Yong, J., eds., Vol. 1 of Series in Contemporary Applied Mathematics, Higher Education Press: Beijing, 2000, pp. 353–381.
- [53] Protter, P., *Stochastic integration and differential equations*, Springer: Berlin, second ed., 2004.
- [54] Raible, S., *Lévy Processes in Finance: Theory, Numerics and Empirical Facts*, PhD thesis, Freiburg University, 2000.
- [55] Rockafellar, R. T., *Convex Analysis*, Princeton University Press, 1970.
- [56] Rogers, L. C. G. and Tehranchi, M., *Can the implied volatility surface move by parallel shifts?*, Finance Stoch., 14 (2010), 235–248.
- [57] Rudin, W., *Real and complex analysis*, McGraw-Hill, 3 ed., 1987.
- [58] Rüschemdorf, L. and Woerner, J. H., *Expansion of transition distributions of Lévy processes in small time*, Bernoulli, 8 (2002), pp. 81–96.
- [59] Sato, K., *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press: Cambridge, UK, 1999.
- [60] Schoutens, W., *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley: New York, 2003.
- [61] Schweizer, M., *Risk-minimality and orthogonality of martingales*, Stochastics Stochastics Rep., 30 (1990), pp. 123–131.
- [62] Schweizer, M., *Option hedging for semimartingales*, Stochastic Process. Appl., 37 (1991), pp. 339–363.

- [63] Schweizer, M., *Approximating random variables by stochastic integrals*, Annals of Probability, 22 (1994), pp. 1536–1575.
- [64] Schweizer, M., *On the minimal martingale measure and the Föllmer-Schweizer decomposition*, Stochastic Anal. Appl., 13 (1995), pp. 573–599.
- [65] Schweizer, M., *A guided tour through quadratic hedging approaches*, Cambridge University Press: Cambridge, 2001, pp. 538–574.
- [66] Selivanov, A. V., *On the martingale measures in exponential Lévy models*, Theory of Probability and its Applications, 49 (2005), pp. 261–274.
- [67] P. TANKOV, *Lévy Processes in Finance: Inverse Problems and Dependence Modelling*, PhD thesis, Ecole Polytechnique, France, 2004.
- [68] Tehranchi, M. R., *Asymptotics of implied volatility far from maturity*, J. Appl. Proba., 46 (2009), 261–274.
- [69] Tehranchi, M. R., *Implied volatility: long maturity behavior*, in Encyclopedia of Quantitative Finance, Wiley, 2010.