

MINICOURSE: *BANACH REPRESENTATIONS OF DYNAMICAL SYSTEMS*

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1. INTRODUCTION

Banach representations of dynamical systems is a relatively new theory already having several important applications. In the present course we study some hierarchies of topological dynamical systems and topological groups coming from Banach space theory. This allows to find new links between many different research lines. Among others: abstract topological dynamics, geometry of Banach spaces and the theory of Polish topological groups.

During this course I will expose some results and ideas from joint works with Eli Glasner (Tel Aviv University). You can download some related papers, from my homepage.

First let us ask some intuitive questions.

Question 1.1. *What is common between:*

- (1) *Grothendieck's Double Limit Property, (weak) almost periodicity and reflexive Banach spaces;*
- (2) *Lack of "butterfly effects", averages of functions on topological groups and Asplund Banach spaces;*
- (3) *Fibonacci binary sequence ("cutting sequences"), quasicrystals, monotonic maps and Rosenthal Banach spaces.*

By the Fibonacci binary sequence we mean the following particular case of a cutting binary sequence c_n of 0-s and 1-s (with the slope $\phi = \frac{1+\sqrt{5}}{2}$).

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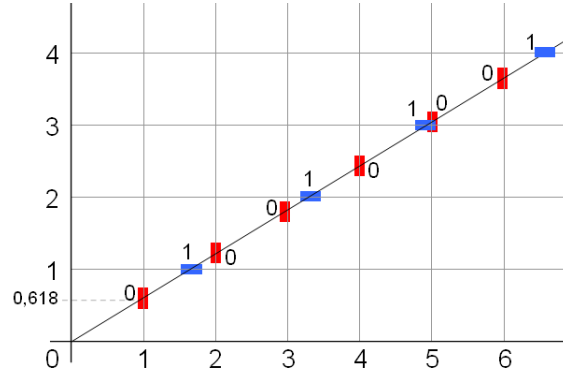


FIGURE 1. The Fibonacci binary sequence

It can be defined also by the finite blocks s_n using the Fibonacci substitution: $s_0 = 0, s_1 = 01, s_n = s_{n-1}s_{n-2}$. So, we have $0100101001001\dots$.

One may prove (Theorem 9.6) that the Fibonacci cutting sequence "lives in a Rosenthal Banach space" as a *generalized matrix coefficient*. That is, there exist: a Rosenthal Banach space V , a linear isometry $\sigma \in \text{Iso}(V)$ and two vectors $v \in V, f \in V^*$ such that

$$c_n = \langle \sigma^n(v), f \rangle = f(\sigma^n(v)) \quad \forall n \in \mathbb{N}.$$

It is impossible to choose V reflexive or even Asplund.

Our aim is to show that Questions like 1.1 can be studied by developing a relatively new tool: representations of dynamical systems on Banach spaces.

Like topological groups, compact dynamical systems, can be represented on (duals of) Banach spaces. We study dynamical analogs of Eberlein, Radon-Nikodým and weakly Radon-Nikodým compacta; that is the classes of dynamical systems which can be represented on reflexive, Asplund and Rosenthal Banach spaces. They correspond to important classes of compact metrizable dynamical systems: weakly almost periodic (WAP), hereditarily nonsensitive (HNS) and tame.

This approach naturally extends some classical research themes and at the same time opens new and sometimes quite unexpected directions. One of the examples is a connection between the lack of chaotic behavior (lack of "butterfly effects") of a dynamical system (HNS systems) and the existence of weak-star continuous representations on the dual of Asplund Banach spaces. The topological concept of the *fragmentability* (originally coming from Banach spaces) and the famous factorization theorem of Davis-Figiel-Johnson-Pelczyński are the main tools in the present theory.

We provide the necessary background. Besides some new results we give soft geometric proofs of several classical results (like: Teleman's regular representations of topological groups, Ellis and Ellis-Lawson theorems; Helmer's theorem about WAP functions; Ryll-Nardzewski's fixed point theorem, etc.). At the same time we discuss perspectives of the theory and pose several open questions.

1.1. Some concrete questions. To every Banach space V one may associate several important structures. For example: compact spaces $X \subset B^* := (B_{V^*}, w^*)$, topological groups $G \leq \text{Iso}(V)$ and continuous actions $G \times X \rightarrow X$, where $G \leq \text{Iso}(V)$ and X is a G -subset of (V^*, w^*) .

Let \mathcal{K} be a nice subclass of Banach spaces. For example: Hilbert, reflexive, Asplund, Rosenthal. There are several good reasons explaining our interest just to these classes.

Question 1.2.

- (1) Which compact dynamical G -systems X can be represented on some $V \in \mathcal{K}$?
- (2) Which topological groups can be embedded into $\text{Iso}(V)$ where $V \in \mathcal{K}$?

By classical results of Teleman every topological group G can be embedded into $\text{Iso}(V)$, where $V := \text{RUC}(G)$ (the algebra of all right uniformly continuous bounded functions on G). Moreover, every compact dynamical S -system X can be represented naturally on $V := C(X)$ via the canonical embedding into the dual $\delta : X \hookrightarrow C(X)^*$. However, the Banach spaces $\text{RUC}(G), C(X)$ very rarely are in a nice class.

For a given topological semigroup S , one way to measure the complexity of a compact dynamical S -system X is to investigate its representability on nice Banach spaces, [27, 26, 24]. Another way is to ask whether the points of X can be separated by a norm bounded S -invariant family $F \subset C(X)$ of continuous functions on X , such that F is “small” in some sense or another. More precisely, Questions 1.2 are closely related to the next question. For every $F \subset V$ and a weak-star compact subset $X \subset V^*$ one may consider the evaluation map

$$w : F \times X \rightarrow \mathbb{R}$$

induced by the canonical bilinear mapping $V \times V^* \rightarrow \mathbb{R}, (v, f) \mapsto f(v)$. For example, B_V separates the points of V^* (hence also of X). For every $F \subset C(X)$ the evaluation map $F \times X \rightarrow \mathbb{R}$ obviously is represented on $V := C(X)$.

Question 1.3. Which abstract evaluation maps can be realized as a part of the canonical bilinear map on a Banach space $V \in \mathcal{K}$?

Usually F is “small” means that the pointwise closure $\text{cl}_p(F)$ of F (the *envelope of F*) in \mathbb{R}^X is a “small” topological space. For example, (a) when $\text{cl}_p(F) \subset C(X)$, or (b) when $\text{cl}_p(F)$ consists of fragmented functions (Baire 1, when X is metrizable).

It turns out that the first case (a) characterizes the reflexively representable dynamical systems, (i.e., the dynamical analog of Eberlein compacta) or, for metric dynamical systems X , the class of Weakly Almost Periodic (in short: WAP) systems.

In the second case (b) we get the characterization of Rosenthal representable dynamical systems, or, for metric dynamical systems X , the class of tame systems (a Banach space V is said to be *Rosenthal* if it does not contain an isomorphic copy of the Banach space l_1 .)

We have the natural intermediate case of Asplund representable or hereditarily non sensitive (HNS) systems. Namely, an S -system X is HNS iff there exists a separating bounded family $F \subset C(X)$ which is a fragmented family.

2. PRELIMINARIES

2.1. Notation. The closure and the interior operators in topological spaces will be denoted by cl and int , respectively. ”Compact” will mean ”compact and Hausdorff”.

As usual *hereditarily Baire* means that every closed subspace is a Baire space. A function $f : X \rightarrow Y$ is *Baire class 1 function* if the inverse image $f^{-1}(O)$ of every open set is F_σ in X . A topological space X is said to be *Polish* if it admits a complete separable metrizable metric. For Polish spaces X a function $f : X \rightarrow \mathbb{R}$ is Baire 1 iff f is a pointwise limit of a sequence of continuous functions.

Banach spaces and locally convex vector spaces are over the field \mathbb{R} of real numbers.

When V is a Banach space we denote by B , or B_V , the closed unit ball of V . $B^* = B_{V^*}$ and $B^{**} := B_{V^{**}}$ will denote the weak* compact unit balls in the dual V^* and second dual V^{**} of V respectively.

2.2. From representations to compactifications. Let X be a topological space, Y be a compact Hausdorff space and let $f : X \rightarrow Y$ be a function such that $f(X)$ is dense in Y . If f is continuous, then Y (more precisely, the pair (Y, f)) is called a *compactification*

of X . If f is a homeomorphic embedding, then Y is called a *proper compactification* of X . Denote by $(\mathcal{C}(X), \leq)$ the partially ordered set of all compactifications of X up to the standard equivalence. For a topological space X denote by $C(X)$ the Banach algebra of real valued continuous and bounded functions equipped with the supremum norm. Recall that the unital closed subalgebras of $C(X)$ determine the compactifications of X .

Fact 2.1. (Gelfand-Kolmogoroff) *There exists a natural order preserving bijective correspondence between $\mathcal{C}(X)$ (different compactifications of X) and closed unital subalgebras of $C(X)$. In particular, $C(X)$ determines the greatest compactification $\beta : X \rightarrow \beta X$.*

Proof. (Sketch) Let \mathcal{A} be a Banach unital subalgebra \mathcal{A} of $C(X)$. Denote by \mathcal{A}^* the dual Banach space of \mathcal{A} . Consider the *canonical \mathcal{A} -compactification* $\alpha_{\mathcal{A}} : X \rightarrow X^{\mathcal{A}}$, where $X^{\mathcal{A}} \subset B^* \subset \mathcal{A}^*$ is the Gelfand space (or, the *spectrum*) of the algebra \mathcal{A} . The map

$$\alpha_{\mathcal{A}} : X \rightarrow X^{\mathcal{A}}, \quad x \mapsto \delta_x$$

is defined by the *Gelfand transform*, the evaluation at x multiplicative functional, that is $\alpha(x)(f) = \delta_x(f) = f(x)$ and $X^{\mathcal{A}}$ is the closure of $\alpha_{\mathcal{A}}(X)$ in \mathcal{A}^* with respect to the weak* topology ω^* . Therefore $X_{\mathcal{A}}$ is compact by Alaoglu Theorem.

Conversely, every compactification $\nu : X \rightarrow Y$ is equivalent to the *canonical \mathcal{A}_{ν} -compactification* $\alpha_{\mathcal{A}_{\nu}} : X \rightarrow X^{\mathcal{A}_{\nu}}$, where the algebra \mathcal{A}_{ν} (corresponding to ν) is defined as the image $j_{\nu}(C(Y))$ of the natural embedding of Banach algebras

$$j_{\nu} : C(Y) \rightarrow C(X), \quad \phi \mapsto \phi \circ \nu.$$

If $\mathcal{A}_1 \subset \mathcal{A}_2$ then the adjoint operator induces the weak-star continuous onto map $\mathcal{A}_2^* \rightarrow \mathcal{A}_1^*$. Its restriction on $c_2(X)$ gives the desired morphism of compactifications $c_2 \rightarrow c_1$. \square

Categorical view: the assignment $K \mapsto C(K)$ defines an important contravariant functor from the category **Comp** into the categories of Banach spaces **Ban** and Banach algebras.

For every (Tychonoff) space X and for the algebra $\mathcal{A} = C(X)$ we get the maximal (Chech-Stone) compactification.

$$\delta : X \rightarrow \beta(X) \subset B^* \subset C(X)^*$$

For every compact space K we have a topological embedding (Gelfand representation)

$$\delta : K \hookrightarrow C(K)^*, \quad x \mapsto \delta_x.$$

Its image $\delta(K)$ affinely generates $P(K)$ (i.e. $\overline{co}^{\omega^*}(\delta(K)) = P(K)$), where

$$P(K) := \{\mu \in C(K)^* : \|\mu\| = \mu(\mathbf{1}) = 1\}$$

the weak-star compact set of all *probability measures* on K . We have $K := \delta(K) \subset P(K) \subset B_{C(K)^*}$.

2.3. Topological prototypes. An important direction in the classical study of (large) compact spaces went via the following general principle: Given a compact space X find a nice class \mathcal{K} of Banach spaces such that there always is an element $V \in \mathcal{K}$ where X can be embedded into V^* equipped with its weak-star topology ?

Eberlein compacta in the sense of Amir and Lindenstrauss are exactly the weakly compact subsets in the class of all (equivalently, reflexive) Banach spaces. If X is a weak* compact subset in the dual V^* of an Asplund space V then, following Namioka [55], X is called a *Radon–Nikodým compactum* (in short: RN). In other words, reflexively representable compact spaces are the Eberlein compacta and Asplund representable compact spaces are the Radon–Nikodým compacta. Hilbert representable compacta are the so-called *uniformly Eberlein* compact spaces. Another interesting class of compact spaces, namely the *weakly Radon–Nikodým* (WRN) compacta, occurs by taking \mathcal{K} to be the class of Rosenthal Banach spaces (i.e. those Banach spaces which do not contain an isomorphic copy of l_1). Comparison

of the above mentioned classes of Banach spaces implies the inclusions of the corresponding classes of compact spaces:

$$(\mathbf{Comp} \cap \mathbf{Metr}) \subset uEb \subset Eb \subset RN \subset WRN \subset \mathbf{Comp}.$$

Note that this classification makes sense only for large compact spaces, where X is not metrizable. In fact, any compact metrizable space is even norm embeddable in a separable Hilbert space.

Example 2.2.

- (1) For every Hilbert space H the (weak compact) unit ball (B_H, w) is uniformly Eberlein.
- (2) 1-point compactification $A(\kappa)$ of a discrete space of cardinality κ is uniformly Eberlein.

A topological space (X, τ) is *scattered* if every (closed) subset $L \subset K$ has an isolated point in L . $A(\kappa)$ is Scattered. For every ordinal λ the linearly ordered compact space $[0, \lambda]$ is scattered.

Fact 2.3. (*Namioka-Phelps75*) *Let K be a compact space. The following are equivalent:*

- (1) K is scattered¹.
- (2) $C(K)$ is an Asplund space.

Example: $c = C(K)$ with $K = A(\omega)$.

Note that infinite dimensional space $C(K)$ never can be reflexive.

Corollary 2.4. *Every scattered compact space is RN.*

Remark 2.5.

- (1) $A(\kappa) \in Eberlein$.
- (2) $[0, \omega_1] \in RN \setminus Eberlein$.
- (3) Two arrows space $D \in WRN \setminus RN$.

Indeed, every compact linearly ordered space is WRN (a recent result [28]). D is not RN by a result of Namioka [55, Example 5.9].

- (4) $\beta\mathbb{N} \notin WRN$.

This was done by Todorčević (private communication).

One of the main directions taken in our research is the development of a dynamical analog, for compact S -dynamical systems (where S is a semigroup), of the above mentioned classification of large compact spaces (this is made precise in Definition 2.12 and Question 2.16 below).

Remark 2.6. Perhaps the first outstanding feature of this new theory is that, in contrast to the purely topological case (i.e., the case of trivial actions), for dynamical systems, the case of metrizable systems is "full of life". Moreover, the main interest of the dynamical theory is just within the class of metrizable dynamical systems. For example, even for $X := [0, 1]$, the unit interval, the action of the cyclic group \mathbb{Z} on X generated by the map $f(x) = x^2$ is RN and not Eberlein. There exists a compact metric \mathbb{Z} -system which is reflexively but not Hilbert representable, i.e., Eberlein but not uniformly Eberlein. There are compact metric \mathbb{Z} -systems which are WRN but not RN, etc. See Example 2.18.

It turns out that the corresponding classes of metric dynamical systems coincide with well known important classes whose study is well motivated by other independent reasons. For

¹*dispersed* in other terminology

example we have, Eberlein = WAP (weakly almost periodic systems), RN = HNS (hereditarily non-sensitive), WRN = tame systems. The investigation of Hilbert representable (i.e., “uniformly Eberlein”) systems is closely related to the study of unitary and reflexive representability of groups.

2.4. Some connections to Banach space theory. Another remarkable feature is the fact that the correspondence goes both ways. Thus, for example, to construct some non-trivial examples of Banach spaces. Every metric WRN but not RN \mathbb{Z} -system leads to an example of a separable Rosenthal Banach space which is not Asplund. One of the important questions in Banach space theory until the mid 70’s was to construct a separable Rosenthal space which is not Asplund. The first counterexamples were constructed independently by James and Lindenstrauss-Stegall.

In view of the representation Theorem below we now see that a fruitful way of producing such distinguishing examples comes from dynamical systems. Just consider a compact metric tame G -system which is not HNS and then represent it on a (separable) Rosenthal space V . Then V is not Asplund (otherwise, (G, X) is HNS). We have several examples of dynamical systems of this type; e.g. $(H_+[0, 1], [0, 1])$, the Sturmian cascades, or the projective actions of $GL_n(\mathbb{R})$ on the sphere or the projective space.

One may make this result sharper by using representation theorem. There exists a separable Rosenthal space V without the *adjoint continuity property*. Indeed, the Polish group $G := H_+[0, 1]$, which admits only trivial adjoint continuous representations (and, hence, trivial Asplund or reflexive representations), is however Rosenthal representable.

Finally, let us mention yet another potentially interesting direction, which may lead to a new classification inside Rosenthal Banach spaces induced by topological classification of Rosenthal compacta (Todorcevic trichotomy) applied to the Rosenthal compacta of the form $\mathcal{E}(V)$ (enveloping semigroups of Banach spaces V).

2.5. Some typical applications.

Theorem 2.7. [46, 49] *Let V be a reflexive space (remains true for PCP spaces). Then*

- (1) *norm topology = weak topology on every orbit Gv for every norm bounded $G \leq \text{Iso}(V)$.*
- (2) ***WOP=SOP.** The weak and the strong operator topologies coincide on $\text{Iso}(V)$.*
- (3) *Every weakly continuous (co)homomorphism $h : G \rightarrow \text{Iso}(V)$ is strongly continuous.*

Proof. It is enough to show (1). Let $z \in X := Gv$. Denote by τ the weak topology on $X \subset V$. We have to show that for every $\varepsilon > 0$ there exists a τ -neighborhood $O(z)$ of z in X such that O is ε -small. Since X is (τ, norm) -fragmented (*non-sensitivity* is enough), we can pick a *non-void* τ -open subset $W \subset X$ such that W is ε -small in V . Choose $g_0 \in G$ such that $g_0z \in W$. Denote by O the τ -open subset $g_0^{-1}W$ of (X, τ) . Then O is a τ -neighborhood of z and is ε -small. \square

Theorem 2.8. (*Shtern, Megrelishvili*) *If P is a compact semitopological monoid then there exists a reflexive Banach space V and an embedding $P \hookrightarrow \Theta(V)_w$ into the compact semitopological monoid*

$$\Theta(V)_w := \{\sigma \in L(V, V) : \|\sigma\| \leq 1\}.$$

Theorem 2.9.

- (1) *Let S be a compact semitopological monoid and G be its subgroup. Then G is a topological group.*
- (2) ***Ellis Theorem** Every compact semitopological **group** G is a topological group.*

Proof. It is enough to show (1). Combine previous two theorems. By Theorem 2.8, $G \leq \text{Iso}(V)_w \subset \Theta(V)_w$ for some reflexive V . By Theorem 2.7, WOP=SOP on $\text{Iso}(V)$ for every $V \in PCP$. So, $G \leq \text{Iso}(V)_w = \text{Iso}(V)_s$ is a topological group. \square

Theorem. ([26], 2012) $\mathbf{Ros}_r \neq \mathbf{Asp}_r$

The Polish topological group $H_+[0, 1]$ is representable on a separable Rosenthal Banach space (and not representable on any Asplund space, [23], 2007).

\Downarrow

(Well known) There exists a separable Rosenthal Banach space which is not Asplund.

Remark 2.10. Well known but once it was a famous problem, resolving by James (JT space) and Lindenstrauss-Stegall (JF space).

Another corollary: There exists a separable Rosenthal space V without the *adjoint continuity property*.

2.6. The hierarchy of Banach representations. With every Banach space V one may naturally associate several structures which are related to the theories of topological dynamics, topological groups and compact right topological semigroups:

Definition 2.11.

- (1) $\text{Iso}(V)$ is the group of linear onto self-isometries of V . It is a topological (semitopological) group with respect to the strong (respectively, weak) operator topology. It is naturally included in the semigroup $\Theta(V) := \{\sigma \in L(V, V) : \|\sigma\| \leq 1\}$ of non-expanding linear operators. The latter is a topological (semitopological) monoid with respect to the strong (respectively, weak) operator topology. Notation: $\Theta(V)_s$, $\text{Iso}(V)_s$ (respectively, $\Theta(V)_w$, $\text{Iso}(V)_w$) or simply $\Theta(V)$ and $\text{Iso}(V)$, where the topology is understood.
- (2) For every subsemigroup $S \leq \Theta(V)^{op}$ the pair (S, B^*) is a dynamical system, where B^* is the weak star compact unit ball in the dual space V^* , and $\Theta(V)^{op}$ is the opposite semigroup (which can be identified with the adjoint) to $\Theta(V)$. The action is jointly (separately) continuous where S carries the strong (weak) operator topology.
- (3) The enveloping semigroup $E(S, B^*)$ of the system (S, B^*) is a compact right topological semigroup (it can be identified with the pointwise closure of S in B^{*B^*}). In particular, $\mathcal{E}(V) := E(\Theta(V)^{op}, B^*)$ will be called the *enveloping semigroup of V* . Its topological center is just $\Theta(V)_w^{op}$ which is densely embedded into $\mathcal{E}(V)$. Note that $\mathcal{E}(V) = \Theta(V)^{op}$ iff V is reflexive.

A *representation* of a semigroup S (with identity element e) on a Banach space V is a co-homomorphism $h : S \rightarrow \Theta(V)$, where $\Theta(V) := \{T \in L(V) : \|T\| \leq 1\}$ and $h(e) = id_V$. Here $L(V)$ is the space of continuous linear operators $V \rightarrow V$ and id_V is the identity operator. This is equivalent to the requirement that $h : S \rightarrow \Theta(V)^{op}$ be a monoid homomorphism, where $\Theta(V)^{op}$ is the opposite semigroup of $\Theta(V)$. If $S = G$, is a group then $h(G) \subset \text{Iso}(V)$, where $\text{Iso}(V)$ is the group of all linear isometries from V onto V .

Definition 2.12. [49, 22, 23, 24] Let X be a dynamical S -system.

- (1) A *representation* of (S, X) on a Banach space V is a pair

$$h : S \rightarrow \Theta(V), \quad \alpha : X \rightarrow V^*$$

where $h : S \rightarrow \Theta(V)$ is a weakly continuous representation (co-homomorphism) of semigroups and $\alpha : X \rightarrow V^*$ is a weak* continuous bounded S -mapping with respect to the dual action

$$S \times V^* \rightarrow V^*, \quad (s\varphi)(v) := \varphi(h(s)(v)).$$

We say that the representation is *strongly continuous* if h is strongly continuous. A representation (h, α) is said to be *faithful* if α is a topological embedding.

- (2) If $S := G$ is a group then a representation of (G, X) on V is a pair (h, α) , where α is as above and $h : G \rightarrow \text{Iso}(V)$ is a group co-homomorphism.
- (3) If \mathcal{K} is a subclass of the class of Banach spaces, we say that a dynamical system (S, X) is (*strongly*) \mathcal{K} -*representable* if there exists a weakly (respectively, strongly) continuous faithful representation of (S, X) on a Banach space $V \in \mathcal{K}$.
- (4) A dynamical system (S, X) is said to be (*strongly*) \mathcal{K} -*approximable* if it can be embedded in a product of (strongly) \mathcal{K} -representable S -spaces.

Remark 2.13. The notion of a reflexively (Asplund) representable compact dynamical system is a dynamical version of the purely topological notion of an *Eberlein* (respectively, a *Radon-Nikodym* (RN, in short)) compactum, in the sense of Amir and Lindenstrauss (respectively, in the sense of Namioka). As in [24], we call Rosenthal representable systems *Weakly Radon-Nikodym* (WRN) systems.

Remark 2.14.

- (1) Of course not every \mathcal{K} -approximable is \mathcal{K} -representable. Take for example, (S, X) with $S := \{e\}$ and $X := [0, 1]^{\mathbb{R}}$. Then (S, X) is clearly reflexively-approximable but not reflexively-representable (because X , as a compactum, is not Eberlein).
- (2) In some particular cases \mathcal{K} -approximability and \mathcal{K} -representability are equivalent. This happens for example if X is metrizable and \mathcal{K} is closed under countable l_2 -sums.

For a compact space X we denote by $H(X)$ the topological group of all self-homeomorphisms of X endowed with the compact open topology.

Lemma 2.15. *Let X be a compact G -space, where G is a topological subgroup of $H(X)$. Assume that (h, α) is a faithful representation (that is, $\alpha : X \rightarrow (V^*, w^*)$ is an embedding) of (G, X) on a Banach space V . Then h is a topological group embedding.*

Proof. Recall that the strong operator topology on $\text{Iso}(V)^{op}$ is identical with the compact open topology inherited from the action of this group on the weak-star compact unit ball (B^*, w^*) . \square

We repeat here some general questions asked in the Introduction.

Query 2.16. *Let \mathcal{K} be a “nice” class of Banach spaces.*

- (1) *Which dynamical S -systems X admit a faithful representation on some Banach space $V \in \mathcal{K}$?*
- (2) *Which topological groups can be embedded into $\text{Iso}(V)$ for some $V \in \mathcal{K}$?*
- (3) *Which compact right topological semigroups (in particular, which enveloping semigroups of dynamical systems) can be embedded into $\mathcal{E}(V)$ for some $V \in \mathcal{K}$?*

Remark 2.17. An old result of Teleman [69] (see also the survey of Pestov [59] for a detailed discussion) is that every (Hausdorff) topological group can be embedded into $\text{Iso}(V)$ for some Banach space V (namely, one can take $V := \text{RUC}(G)$). Furthermore, every continuous dynamical system (G, X) has a faithful representation on $V := C(X)$, where one can identify $x \in X$ with the point mass δ_x viewed as an element of $C(X)^*$. This is true also for semigroup actions. So, any compact dynamical S -system X is Banach representable (on $C(X)$).

Example 2.18.

- (1) Let $X = [0, 1]$ be the unit interval. Consider the cascade (\mathbb{Z}, X) generated by the homeomorphism $\sigma(x) = x^2$. Then (\mathbb{Z}, X) , as a dynamical system, is RN and not Eberlein.

Indeed, observe that the pair of sequences $x_n = 1 - \frac{1}{n}$ in $X = [0, 1]$ and $g_m \in G$ with $g_m(x) = x^{2^m}$ does not satisfy DLP. The corresponding limits are 0 and 1. This means that (\mathbb{Z}, X) is not Eberlein. The enveloping semigroup $E(\mathbb{Z}, X)$ is metrizable being homeomorphic to the two-point compactification of \mathbb{Z} . Hence, by [29], (\mathbb{Z}, X) is RN.

- (2) The Sturmian symbolic dynamical system (in particular, the symbolic dynamical system generated by the Fibonacci binary sequence) is WRN but not RN.
- (3) $(H_+[0, 1], [0, 1])$ is WRN but not RN.
- (4) The Bernoulli shift system $(\mathbb{Z}, \{0, 1\}^{\mathbb{Z}})$ is not WRN. Indeed, it is well known that the enveloping semigroup in this case can be identified with the compact right topological semigroup $\beta\mathbb{Z}$. Now use the dynamical version of BFT dichotomy (Fact 8.9).

In the following table we encapsulate some features of the trinity: dynamical systems, enveloping semigroups, and Banach representations. Here X is a compact metrizable G -space and $E(X)$ denotes the corresponding enveloping semigroup. The symbol f stands for an arbitrary function in $C(X)$ and $fG = \{f \circ g : g \in G\}$ denotes its orbit. Finally, $\text{cl}(fG)$ is the pointwise closure of fG in \mathbb{R}^X .

DS	Dynamical characterization	Enveloping semigroup	Banach representation
WAP	$\text{cl}(fG)$ is a subset of $C(X)$	Every element is continuous	Reflexive
HNS	$\text{cl}(fG)$ is metrizable	$E(X)$ is metrizable	Asplund
Tame	$\text{cl}(fG)$ is Fréchet	Every element is Baire 1	Rosenthal

TABLE 1. The hierarchy of Banach representations

2.7. Actions and semigroups.

Definition 2.19. Let S be a semitopological semigroup with a neutral element e . Let $\pi : S \times X \rightarrow X$ be a *left action* of S on a topological space X . This means that $ex = x$ and $s_1(s_2x) = (s_1s_2)x$ for all $s_1, s_2 \in S$ and $x \in X$, where as usual, we write sx instead of $\pi(s, x) = \lambda_s(x) = \rho_x(s)$. Let $S \times X \rightarrow X$ and $S \times Y \rightarrow Y$ be two actions. A map $f : X \rightarrow Y$ between S -spaces is an *S-map* if $f(sx) = sf(x)$ for every $(s, x) \in S \times X$.

We say that X is a *dynamical S-system* (or an *S-space* or an *S-flow*) if the action π is separately continuous (that is, if all orbit maps $\rho_x : S \rightarrow X$ and all translations $\lambda_s : X \rightarrow X$ are continuous). We sometimes write it as a pair (S, X) .

A *right system* (X, S) can be defined analogously. If S^{op} is the *opposite semigroup* of S with the same topology then (X, S) can be treated as a left system (S^{op}, X) (and vice versa).

Fact 2.20. [41] *Let G be a Čech-complete (e.g., locally compact or completely metrizable) semitopological group. Then every separately continuous action of G on a compact space X is continuous.*

Notation: *All semigroups S are assumed to be monoids, i.e., semigroups with a neutral element which will be denoted by e . Also actions are monoidal (meaning $ex = x, \forall x \in X$) and separately continuous. We reserve the symbol G for the case when S is a group.*

Let $h : S_1 \rightarrow S_2$ be a semigroup homomorphism, S_1 act on X_1 and S_2 on X_2 . A map $f : X_1 \rightarrow X_2$ is said to be *h-equivariant* if $f(sx) = h(s)f(x)$ for every $(s, x) \in S_1 \times X_1$. For $S_1 = S_2$ with $h = 1_S$, we say *S-map*. The map $h : S_1 \rightarrow S_2$ is an *antihomomorphism* iff $h : S_1 \rightarrow S_2^{op}$ (the same assignment) is a homomorphism.

Given $x \in X$, its *orbit* is the set $Sx = \{sx : s \in S\}$. A point x with $\text{cl}(Sx) = X$ is called a *transitive point*, and the set of transitive points is denoted by X_{tr} . We say that the system is *point-transitive* when $X_{tr} \neq \emptyset$. The system is called *minimal* if $X_{tr} = X$.

By an (invertible) *cascade* on X we mean a continuous action $S \times X \rightarrow X$, where $S := \mathbb{N} \cup \{0\}$ is the additive semigroup of all nonnegative integers (respectively, $S = (\mathbb{Z}, +)$).

For a pair of topological spaces X and Y we let $C(X, Y)$ denote the space of continuous functions from X into Y . We will take $C(X)$ to be the Banach space of *bounded* continuous functions (when X is not necessarily compact). Let $G \times X \rightarrow X$ be an action. It induces the right action $C(X) \times S \rightarrow C(X)$. A function $f \in C(X)$ is said to be:

(a) *Almost periodic* if the norm closure $\text{cl}(fS)$ of the orbit fS is norm compact in $C(X)$; Notation: $f \in \text{AP}(X)$.

(b) *Weakly almost periodic* if the weak closure $\text{cl}_w(fS)$ of the orbit fS is weak compact in $C(X)$; Notation: $f \in \text{WAP}(X)$.

(c) *Right uniformly continuous* if the orbit map $\tilde{f} : S \rightarrow C(X)$ is continuous. That is, if

$$\forall \varepsilon > 0 \forall s_0 \in S \exists U \in N(s_0) \quad |f(sx) - f(s_0x)| < \varepsilon \quad \forall x \in X, s \in U.$$

Notation: $f \in \text{RUC}(X)$.

In particular we have $\text{WAP}(S)$ for usual left action of S on itself. Note that for the right action of S on itself the corresponding (right) version of $\text{WAP}(S)$ gives the equivalent definition [10, 6].

A compact S -space X is said to be (*weakly*) *almost periodic* if (resp., $C(X) = \text{WAP}(X)$) $C(X) = \text{AP}(X)$. For any S -space X the collections $\text{WAP}(X)$ and $\text{AP}(X)$ are S -invariant subalgebras of $C(X)$. The corresponding Gelfand spaces and compactifications define S -equivariant compactifications $u_{ap} : X \rightarrow X^{AP}$ and $u_{wap} : X \rightarrow X^{WAP}$. The compactification $S \rightarrow S^{WAP}$ (for $X := S$) is the universal semitopological semigroup compactification of S . Furthermore, $\text{WAP}(X)$ and $\text{AP}(X)$ is the set of all functions which come from WAP and AP S -compactifications of X , respectively.

For every topological group G , treated as a G -space, the corresponding universal AP compactification is the classical *Bohr compactification* $b : G \rightarrow bG$, where bG is a compact topological group.

2.8. Semigroups. Let S be a semigroup which is also a topological space. By $\lambda_a : S \rightarrow S, x \mapsto ax$ and $\rho_a : S \rightarrow S, x \mapsto xa$ we denote the left and right a -translations. The subset $\Lambda(S) := \{a \in S : \lambda_a \text{ is continuous}\}$ is called the *topological center* of S .

Definition 2.21. A semigroup S with a given topology is called:

- (1) *right topological semigroup* if every ρ_a is continuous;
- (2) *admissible* if S is right topological and $\Lambda(S)$ is dense in S ;
- (3) *semitopological* if the multiplication $m : S \times S \rightarrow S$ is separately continuous (i.e., if S is left and right topological);
- (4) *topological* if the multiplication $S \times S \rightarrow S$ is continuous;
- (5) *topological group* if S , algebraically, is a group, topological semigroup and if the inversion $s \mapsto s^{-1}$ is continuous.

2.9. Semigroup compactifications.

Definition 2.22. Let S be a semitopological semigroup. [6, p. 105] A *right topological semigroup compactification* of S is a pair (γ, T) such that T is a compact right topological semigroup, and γ is a continuous semigroup homomorphism from S into T , where $\gamma(S)$ is dense in T and the left translation $\lambda_s : T \rightarrow T, x \mapsto \gamma(s)x$ is continuous for every $s \in S$, that is, $\gamma(S) \subset \Lambda(T)$.

It follows that the *associated action*

$$\pi_\gamma : S \times T \rightarrow T, \quad (s, x) \mapsto \gamma(s)x = \lambda_s(x)$$

is separately continuous.

Example 2.23.

- (1) Maximal (jointly continuous) G -compactification $G \hookrightarrow \beta_G G := G^{\text{RUC}}$ (the greatest ambit). The corresponding algebra is $\text{RUC}(G)$.
- (2) Universal semitopological compactification: $G \rightarrow wG := G^{\text{WAP}}$. The corresponding algebra is $\text{WAP}(G)$.
- (3) universal topological compact group compactification (Bohr compactification): $G \rightarrow bG = G^{\text{AP}}$.
- (4) 1-point compactification for LC groups: $G \hookrightarrow G \cup \{\infty\}$ with the algebra $C_0(G) \oplus \mathbb{R}$.

Note that by [52] the projection $q : G^{\text{RUC}} \rightarrow G^{\text{WAP}}$ is a homeomorphism iff G is pre-compact.

For every (separately continuous) compact S -system X we have a (pointwise continuous) monoid homomorphism $j : S \rightarrow C(X, X)$, $j(s) = \tilde{s}$, where $\tilde{s} : X \rightarrow X, x \mapsto sx = \pi(s, x)$ is the s -translation ($s \in S$).

Definition 2.24. The *enveloping semigroup* $E(S, X)$ (or just $E(X)$) of the compact dynamical S -system X is defined as the pointwise closure $E(S, X) = \text{cl}_p(j(S))$ of $\tilde{S} = j(S)$ in X^X .

The associated homomorphism $j : S \rightarrow E(X)$ is a right topological semigroup compactification (say, *Ellis compactification*) of S , $j(e) = \text{id}_X$ and the associated action $\pi_j : S \times E(X) \rightarrow E(X)$ is separately continuous. Furthermore, if the S -action on X is continuous then π_j is continuous. $E(X)$ is always a right topological compact monoid. Algebraic and topological properties of the families $j(S)$ and $E(X)$ reflect the asymptotic dynamical behavior of (S, X) .

Remark 2.25. Every enveloping semigroup $E(S, X)$ is an example of a compact right topological admissible semigroup. Conversely, every compact right topological *admissible* semigroup P is an enveloping semigroup (of $(\Lambda(P), P)$).

Example 2.26. A reach source of semigroup compactifications are operator compactifications. For every weakly continuous representation $h : S \rightarrow \Theta(V)^{\text{op}}$ one may consider the *operator compactification* $h_0 : S \rightarrow S_0 := E(S, B^*) \subset \mathcal{E}(V)$.

Example 2.27. For example: $\mathbb{Z} \cup \{\infty\}$ and $\beta(\mathbb{Z})$ are operator compactifications of \mathbb{Z} . First for the natural representation of \mathbb{Z} on $l_2(\mathbb{Z})$ and the second for the regular representation of \mathbb{Z} on $\text{RUC}(\mathbb{Z}) = l_\infty(\mathbb{Z})$. The semigroup compactification $\mathbb{Z} \rightarrow \mathbb{Z}^{\text{Distal}}$ is not equivalent to an operator compactification, [26].

Question 2.28. *When G can be embedded into a good right topological compact semigroup P ? For example: when P can be semitopological, metrizable, Frechet, ... ?*

Remark 2.29.

- (1) There exists a nontrivial Polish group G whose universal semitopological compactification G^{WAP} is trivial. This is shown in [47] for the Polish group $G := H_+[0, 1]$ of orientation preserving homeomorphisms of the unit interval. Equivalently: every (weakly) continuous representation $G \rightarrow \text{Iso}(V)$ of G on a reflexive Banach space V is trivial.

- (2) A stronger result is shown in [23]: every continuous representation $G \rightarrow \text{Iso}(V)$ of G on an Asplund space V is trivial (and every Asplund function on G is constant). Every nontrivial right topological semigroup compactification of the Polish topological group $G := H_+[0, 1]$ is not metrizable [29]. In contrast we show that G is Rosenthal representable.

Exercise 2.30. Every locally compact topological group is closed in every Hausdorff topological group.

For example, \mathbb{Z} and \mathbb{R} cannot be embedded into compact groups. In particular, such groups do not admit finite dimensional orthogonal representations $h : G \rightarrow O_n(\mathbb{R})$ where h is an embedding. More precisely, we have

Fact 2.31. *Let G be a topological group. The following are equivalent:*

- (1) G can be embedded into a compact topological semigroup.
- (2) G can be embedded into a compact group (i.e., the Bohr compactification $b : G \rightarrow bG$ is an embedding).
- (3) G is embedded into a product of finite dimensional orthogonal group $O_n(\mathbb{R})$.

The equivalence of (1) and (3) is a consequence of Peter-Weyl theorem. The equivalence of (1) and (2) easily follows from the following

Exercise 2.32. If S is a compact ² topological semigroup and if G is a subgroup of S then $cl(G)$ is a (compact) topological group.

One of the standard examples of a proper compactification is the so-called *1-point compactification* $\nu : X \hookrightarrow X_\infty := X \cup \{\infty\}$ defined for every locally compact non-compact Hausdorff space (X, τ) . Recall the topology

$$\tau_\infty := \tau \cup \{X_\infty \setminus K : K \text{ is compact in } X\}.$$

Exercise 2.33. Let (G, \cdot, τ) be a locally compact non-compact Hausdorff topological group. Denote by $S := G \cup \{\infty\}$ the 1-point compactification of G . Show that (S, \cdot, τ_∞) is a semitopological but not topological semigroup.

Corollary 2.34. *Let G be a locally compact group. Then*

- (1) G is embedded into a topological semigroup iff G is compact.
- (2) G is embedded into a compact semitopological semigroup.

Below we show:

Theorem 2.35. (*Ellis thm*) *Every (locally) compact semitopological group is a topological group.*

Remark 2.36. Let A be a subsemigroup of a right topological semigroup S . If $A \subset \Lambda(S)$ then the closure $cl(A)$ is a right topological semigroup. In general, $cl(A)$ is not necessarily a subsemigroup of S (even if S is compact right topological and A is a left ideal). Also $\Lambda(S)$ may be empty for general compact right topological semigroup S .

Exercise 2.37. Let S be the interval $[0, 1]$ with the multiplication

$$st = \begin{cases} t, & \text{if } 0 \leq t < \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Show that: S is a compact right topological semigroup with $\Lambda(S) = \emptyset$. The subset $T := [0, \frac{1}{2})$ is a subsemigroup (even, a left ideal) of S and $cl(T) = [0, \frac{1}{2}]$ is not a subsemigroup of S .

²Hausdorff, by our standing assumptions

Exercise 2.38. Let $S := \mathbb{Z} \cup \{-\infty, \infty\}$ be the two-point compactification of \mathbb{Z} . Extend the usual addition by:

$$n + t = t + n = s + t = t \quad n \in \mathbb{Z}, \quad s, t \in \{-\infty, \infty\}$$

Show:

- (1) $(S, +)$ is a noncommutative compact right topological semigroup having dense topological centre $\Lambda(S) = \mathbb{Z}$.
- (2) S is topologically isomorphic to the *enveloping semigroup* of the invertible cascade $(\mathbb{Z}, [0, 1])$ generated by the homeomorphism $\sigma : [0, 1] \rightarrow [0, 1], \sigma(x) = x^2$.

2.10. Examples of (semi)groups.

Exercise 2.39. Prove that:

- (1) for every metric space (M, d) the semigroup $S := \Theta(M, d)$ of all *contractive*³ $f : X \rightarrow X$ (that is, $d(f(x), f(y)) \leq d(x, y)$) is a topological monoid with respect to the topology of pointwise convergence;
- (2) the group $\text{Iso}(M)$ of all onto isometries is a topological group;
- (3) the evaluation map $S \times M \rightarrow M$ is a continuous monoidal action.

An action $S \times X \rightarrow X$ on a metric space (X, d) is *contractive* if every s -translation $\tilde{s} : X \rightarrow X$ lies in $\Theta(X, d)$. It defines a natural homomorphism $h : S \rightarrow \Theta(X, d)$.

Exercise 2.40. Let $S \times X \rightarrow X$ be contractive action of S on (X, d) . Show that the following conditions are equivalent:

- (i) The action is continuous.
- (ii) The action is separately continuous.
- (iii) The natural homomorphism $h : S \rightarrow \Theta(X, d)$ of monoids is continuous.

Exercise 2.41. Let $\text{Unif}(Y, Y)$ be the set of all uniform self-maps of a uniform space (Y, μ) . Denote by μ_{sup} the uniformity of uniform convergence on $\text{Unif}(Y, Y)$. Show that

- (1) under the corresponding topology $\text{top}(\mu_{\text{sup}})$ on $\text{Unif}(Y, Y)$ and the usual composition we get a topological monoid;
- (2) If G is any subgroup of the monoid $\text{Unif}(Y, Y)$ then G is a topological group;
- (3) For every subsemigroup $S \subset \text{Unif}(Y, Y)$ the induced action $S \times Y \rightarrow Y$ is continuous;

Exercise 2.42. Let Y be a compact space. Show that:

- (1) The semigroup $C(Y, Y)$ endowed with the compact open topology is a topological monoid;
- (2) The subset $H(Y)$ in $C(Y, Y)$ of all homeomorphisms $Y \rightarrow Y$ is a topological group;
- (3) For every subsemigroup $S \subset C(Y, Y)$ the induced action $S \times Y \rightarrow Y$ is continuous;
- (4) Furthermore, it satisfies the following remarkable minimality property. If τ is an arbitrary topology on S such that $(S, \tau) \times Y \rightarrow Y$ is continuous then $\tau_{\text{co}} \subset \tau$.

2.11. Operator topologies.

Definition 2.43. Let V be a Banach space. The *strong operator topology* (SOT) on $L(V, V)$ is the pointwise topology inherited from $(V, \|\cdot\|)^V$. That is, a net s_i converges to s iff $s_i(v)$ converges to $s(v)$ in the norm topology for every $v \in V$. Replacing the norm topology of V by its weak topology we obtain the *weak operator topology* (WOT).

A net s_i in $(L(V, V), \text{WOT})$ converges to s iff $f(s_i(v))$ converges to $f(s(v))$ in \mathbb{R} for every given pair of vectors $(v, f) \in V \times V^*$.

³in another terminology: *Lipschitz 1 maps*

Exercise 2.44. Prove that $\Theta(V)$ and $L(V, V)$ are semitopological monoids with respect to WOT.

Exercise 2.45. Show that:

- (1) The semigroup $\Theta(V)$ endowed with the SOT is a topological monoid;
- (2) The subspace $\text{Iso}(V)$ of all linear onto isometries is a topological group.

Below in Theorem 10.3 we show that the weak and strong operator topologies on $\text{Iso}(V)$ are the same for a large class of Banach spaces (PCP). SOT and WOT are different on $\text{Iso}(C(I^2))$, [46].

It is well known (see de Leeuw-Glicksberg [12]) that $\Theta(V)_w$ is a compact (semitopological) semigroup for every reflexive V . One may show that this property characterizes reflexive spaces.

Lemma 2.46. *For any Banach space V and any given norm 1 vector $v \in S_V$ the map*

$$\Theta(V)_w \rightarrow (B, w), \quad s \mapsto sv$$

is onto (and continuous).

Proof. Take $f \in S_{V^*}$ such that $f(v) = 1$. For every $z \in B$ define the rank 1 operator

$$A(f, z) : V \rightarrow V, \quad x \mapsto f(x)z.$$

Then $A(f, z)(v) = z$ and $A(f, z) \in \Theta$ since $\|A(f, z)\| = \|f\| \cdot \|z\| = \|z\| \leq 1$. □

Proposition 2.47. *The following are equivalent:*

- (1) V is reflexive;
- (2) $B := B_V$ is weakly compact;
- (3) $\Theta(V)_w$ is compact.

Proof. (1) \Leftrightarrow (2) is a well known criterion.

(2) \Rightarrow (3) $\Theta(V) \subset (B, w)^B$ is a closed subset.

(3) \Rightarrow (2) Is a corollary of Lemma 2.46. □

Every left action $\pi : S \times X \rightarrow X$ induces the co-homomorphism $h_\pi : S \rightarrow C(X)$ and the right action $C(X) \times S \rightarrow C(X)$ where $(fs)(x) = f(sx)$. While the translations $\tilde{s} : V \rightarrow V$ are continuous, the orbit maps $\tilde{f} : S \rightarrow C(X)$ are not necessarily norm (even weakly) continuous and require additional assumptions for their continuity. Denote by $\text{RUC}_S(X)$, or simply by $\text{RUC}(X)$, the set of all functions $f \in C(X)$ such that the orbit map \tilde{f} is norm continuous (see [6]).

For every normed space V the usual adjoint map

$$\text{adj} : L(V) \rightarrow L(V^*), \quad s \mapsto s^* \quad (\langle s(v), f \rangle = \langle v, s^*(f) \rangle)$$

is an injective co-homomorphism of monoids. Sometimes we write simply s instead of s^* .

Lemma 2.48. *For every normed space V the injective map*

$$\gamma : \Theta(V)_s^{op} \hookrightarrow C(B^*, B^*)$$

induced by the adjoint map $\text{adj} : L(V) \rightarrow L(V^)$, is a topological (even uniform) monoid embedding. In particular,*

$$\Theta(V)_s^{op} \times B^* \rightarrow B^*$$

is a jointly continuous monoidal action of $\Theta(V)_s^{op}$ on the compact space B^ .*

Proof. The strong uniformity on $\Theta(V)$ is generated by the family of pseudometrics $\{p_v : v \in V\}$, where $p_v(s, t) = \|sv - tv\|$. On the other hand the family of pseudometrics $\{q_v : v \in V\}$, where $q_v(s, t) = \sup\{|(fs)(v) - (ft)(v)| : f \in B^*\}$ generates the natural uniformity inherited from $C(B^*, B^*)$. Now observe that $p_v(s, t) = q_v(s, t)$ by the Hahn-Banach theorem. This proves that γ is a uniform (and hence, also, topological) embedding. \square

Corollary 2.49. *Let V be a Banach space. Suppose that $\pi : V \times S \rightarrow V$ is a right action of a topologized semigroup S by linear contractive operators. The following are equivalent:*

- (i) *The co-homomorphism $h : S \rightarrow \Theta(V)$, $h(s)(v) := vs$ is strongly continuous.*
- (ii) *The induced affine action $S \times B^* \rightarrow B^*$, $(s\psi)(v) := \psi(vs)$ is jointly continuous.*

2.12. Theorems of Teleman and Uspenskij. When a topological group G can be represented on a Banach space by linear isometries. That is, when $G \hookrightarrow \text{Iso}(V)$? In fact Teleman's theorem (rediscovered by many authors) below 2.52 shows that always.

For example, every finite group G can be represented on \mathbb{R}^n with $n = |G|$. Indeed, take

$$G \hookrightarrow S_n \hookrightarrow \text{Iso}_{lin}(\mathbb{R}^n) = O_n(\mathbb{R}).$$

Exercise 2.50. Every discrete group G admits an effective isometric representation on the Banach space $l_\infty(G)$ and also on the Hilbert space $H := l_2(G)$.

So, on $l_\infty := l_\infty(\mathbb{N})$ and on $l_2 := l_2(\mathbb{N})$ if G is countable.

For every compact space K we have a Gelfand representation $\delta : K \hookrightarrow C(K)^*$. Now Let $S \times K \rightarrow K$ be a continuous action. The induced action $C(K) \times S \rightarrow C(K)$ is: linear, norm-preserving and continuous. We obtain a natural representation (h, δ) , where $h : S \rightarrow \Theta(V)$ is a strongly continuous co-homomorphism of monoids. Call it the *Teleman's representation*. This is a dynamical version of Gelfand's representation.

Theorem 2.51. [69] *Let $S \times K \rightarrow K$ be a continuous action on a compact space K . Then*

- (1) *The Teleman's representation (h, δ) of the dynamical system (S, K) on the Banach space $V = C(K)$ is faithful and strongly continuous. That is, $h : S \rightarrow \Theta(V)_s$ is continuous. If $S = G$ is a group then h is a co-homomorphism of groups $h : G \rightarrow \text{Iso}(V)_s$;*
- (2) *Moreover, if S carries the compact open topology inherited from $C(K, K)$ then the homomorphisms $S \rightarrow C(B^*, B^*)$ and $h : S \rightarrow \Theta(V)_s^{op}$ are topological embeddings.*

Proof. (1) The induced right linear action $C(K) \times S \rightarrow C(K)$ is continuous (because the orbit maps are norm continuous). This action is contractive. It follows that $h : S \rightarrow \Theta(V)^{op}$ is a well defined strongly continuous homomorphism. Moreover, B^* is an S -subset under the dual action. By Lemma 2.48 we obtain that the action $S \times B^* \rightarrow B^*$ is continuous.

(2) Let $S \subset C(K, K)$. Denote by τ_0 the induced topology on S . The action $S \times K \rightarrow K$ can be treated as a restriction of the bigger action $S \times B^* \rightarrow B^*$, where K naturally is embedded into B^* via Gelfand's map. Then the topology τ on S inherited from $C(B^*, B^*)$ majors the original topology τ_0 . Hence, $\tau_0 \subset \tau$.

On the other hand, the continuity of $S \times B^* \rightarrow B^*$ easily implies that $\tau \subset \tau_0$ on S . Summing up we conclude that $\tau = \tau_0$ on S . \square

Theorem 2.52. (*Teleman's theorems*) *Let G be a topological group. Then*

- (1) *G is embedded into $\text{Iso}(V)$ for some Banach space V .*
- (2) *G is embedded into $\text{Iso}(M, d)$ for some metric space (M, d) .*
- (3) *G is embedded into $\text{Homeo}(K)$ for some compact space K .*

Proof. Clearly, (1) \Rightarrow (2). By Theorem 2.51, (3) \Rightarrow (1). So it is enough to show (3).

The left action $G \times G \rightarrow G$ is G -compactifiable. The algebra $\text{RUC}(G)$ separates the points and closed subsets. Then the maximal G -compactification $G \rightarrow \beta_G G$ is an embedding and $G \hookrightarrow H(K)$, where $K := \beta_G G$. \square

Theorem 2.53. [50] (*universal small actions*) Let $K := [0, 1]^{\mathbb{N}}$ be the Hilbert cube. Denote by $U := C(K, K)$ the topological monoid and the natural action $U \times K \rightarrow K$. Then this action is universal for all monoidal actions on compact metrizable spaces. The pair $(H(K), K)$ is universal for continuous group actions on compact metrizable spaces.

[That is, for every compact metrizable X and a topological submonoid of U there exists an equivariant representation $(h, \alpha) : (S, X) \rightrightarrows (U, I^\omega)$ such that $h : S \hookrightarrow U$ is an embedding of topological semigroups and $\alpha : X \hookrightarrow I^\omega$ is a topological embedding.]

Proof. Use Equivariant Teleman's representation on $V := C(K)$ and the fact that by Keller's theorem $B^* = (B_{V^*}, w^8)$ is homeomorphic to the Hilbert cube K for every separable V . \square

Corollary 2.54. (*Uspenskij* [72]) $H([0, 1]^{\mathbb{N}})$ is a universal Polish topological group.

2.13. Representations of topological groups. By a (*t-faithful*) **representation** of a topological group G on a Banach space V we mean a continuous homomorphism (resp., *topological group embedding*) $h : G \rightarrow \text{Iso}(V)$ of G into the top. group $\text{Iso}(V)$ of all linear onto isometries of V with SOT. Every (topological) group is topologically isomorphic to its opposite group by the assignment $g \mapsto g^{-1}$. Hence, G is representable on V iff it is co-representable on V .

Problem 2.55. Which (Polish) groups can be represented on nice Banach spaces ?

Turn to some closely related concepts and questions.

Problem 2.56. Which Polish groups admit:

- (a) small *t-faithful* semigroup compactifications ?
(semitopological, metrizable, Fréchet, first-countable ... ?)
- (b) *t-effective* actions on compact metrizable spaces X such that (G, X) is a nice DS ?

By [52], $\text{WAP}(G) = \text{RUC}(G)$ iff G is precompact ("large room").

$$\text{AP}(G) \subset \text{WAP}(G) \subset \dots \subset \text{RUC}(G)$$

$$b(G) = G^{\text{AP}} \leftarrow G^{\text{WAP}} \leftarrow \dots \leftarrow \beta_G(G) = G^{\text{RUC}}$$

Hilb \subset **Ref** \subset **Asp** \subset **Ros** \subset **Ban**

uEb \subset **Eb** \subset **RN** \subset **WRN** \subset **Comp** "Dynamical analogs".

Let $\mathcal{K} \subset \mathbf{Ban}$ be a subclass of Banach spaces. We write: $G \in \mathcal{K}_r$ if \exists a *t-faithful* representation of G on $V \in \mathcal{K}$.

$$\mathbf{TGr} = \mathbf{Ban}_r \supseteq \mathbf{Ros}_r \supseteq \mathbf{Asp}_r \supseteq \mathbf{Ref}_r \supseteq \mathbf{Hilb}_r \supseteq \{LC \text{ top. gr.}\}$$

Remark 2.57.

- (1) (Teleman's theorem) Any topological group G is Banach representable.
- (2) (Gelfand-Raikov) Every locally compact group is Hilbert representable. If G is a locally compact topological group then the regular representation of $H = L_2(G, \mu)$ (where μ is the Haar measure) defines an embedding $G \hookrightarrow \text{Iso}(H)$.
- (3) (Me [48]) $G := L_4[0, 1] \in \mathbf{Ref}_r \setminus \mathbf{Hilb}_r$ (For proofs see Section 11).
(Glasner-Weiss 2012) $\exists G \in \mathbf{Ref}_r \setminus \mathbf{Hilb}_r$ s.t. $G = l_2/D$ is Polish monothetic.
- (4) **Unknown if: $\mathbf{Asp}_r = \mathbf{Ref}_r$, $\mathbf{Ban}_r = \mathbf{Ros}_r$.**

It is an open question if every Polish group is Rosenthal representable (enough to examine $G := \text{Homeo}([0, 1]^{\mathbb{N}})$).

Remark 2.58.

- (1) (Me 2001) $\mathbf{TGr} \neq \mathbf{Ref}_r$ $H_+[0, 1] \notin \mathbf{Ref}_r$
 \forall representation $h : H_+ \rightarrow \text{Iso}(V)$ is trivial $\forall V \in \mathbf{Ref}$.
 \Downarrow
 Every semitopological compactification of H_+ is trivial.
- (2) (Gl-Me 2007) \forall representation $h : H_+ \rightarrow \text{Iso}(V)$ is trivial $\forall V \in \mathbf{Asp}$.
 \Downarrow
- (3) \forall metrizable semigroup compactification of H_+ is trivial.
- (4) Assertion 2 proved independently also by Uspenskij.
- (5) \exists a semigroup compactification of H_+ which is Fréchet.
 \Uparrow

We will sketch

Theorem. (Gl-Me 2012) $\mathbf{Ros}_r \neq \mathbf{Asp}_r$

The Polish topological group $H_+[0, 1]$ is representable on a separable Rosenthal Banach space (and not representable on any Asplund space – 2007).

\Downarrow

(Well known) There exists a separable Rosenthal Banach space which is not Asplund.

Remark 2.59. Well known but once it was a famous problem, resolving by James (JT space) and Lindenstrauss-Stegall (JF space).

3. MATRIX COEFFICIENTS

For every $h : S \rightarrow L(V, V)$ and any pair of vectors $v \in V$ and $\psi \in V^*$, we have a canonically associated (*generalized*) *matrix coefficient* $m_{v, \psi} : S \rightarrow \mathbb{R}$, $s \mapsto \langle vs, \psi \rangle = \langle v, s\psi \rangle$

$$\begin{array}{ccc} S & \xrightarrow{m_{v, \psi}} & \mathbb{R} \\ h \downarrow & & \uparrow \psi \\ L(V, V) & \xrightarrow{\tilde{v}} & V \end{array}$$

Easy to adopt this definition for any bilinear mappings $V \times W \rightarrow \mathbb{R}$ and a pair of compatible (co)-homomorphisms from S to $L(V, V)$ and $L(W, W)$.

Remark 3.1. In order to justify the name "matrix coefficient" note the following. For $V = \mathbb{R}^n$ of finite dimension (rows $1 \times n$) consider V^* (columns $n \times 1$) and $v = e_i \in V$ and $w = e_j^t \in V^*$ taken from the standard basis. Then for a matrix $A = (a_{ij})_{n \times n}$ we have

$$a_{ij} = \langle e_i A, e_j^t \rangle .$$

Exercise 3.2. Let $h : S \rightarrow \Theta(V)$ be a co-homomorphism. The following are equivalent:

- (1) h is weakly continuous;
- (2) The action $S \times B^* \rightarrow B^*$ is separately continuous (where $B^* := (B^*, w^*)$);
- (3) Every (*matrix coefficient*) $m_{v, \psi} : S \rightarrow \mathbb{R}$, $s \mapsto \langle vs, \psi \rangle$ is continuous for any $(v, \psi) \in V \times V^*$.

It is natural to expect that matrix coefficients reflect good properties of flow representations. We recall two well-known facts. The first example is the case of Hilbert representations. Let $h : G \rightarrow \text{Iso}(H)$ be a continuous homomorphism, where H is Hilbert with its scalar product $H \times H \rightarrow \mathbb{R}$. Then the corresponding matrix coefficient $m_{u, v}$ is the so-called *Fourier-Stieltjes functions* on G . If $u = v$, then we get *positive definite functions* on G . The

converse is also true: every continuous positive definite function comes from some continuous Hilbert representation. Recall that a continuous bounded function $f : G \rightarrow \mathbb{R}$ is said to be *positive definite* if $\sum \alpha_i \alpha_j f(g_i^{-1} g_j) \geq 0$ for all $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, $g_1, \dots, g_n \in G$. Every positive definite function is WAP.

The second example comes from Eberlein (see [6, Examples 1.2.f]). If V is reflexive, then every bounded V -representation (h, α) and arbitrary pair (v, ψ) lead to a weakly almost periodic function $m_{v, \psi}$ on S . This follows easily by the (weak) continuity of the natural operators defined by the following rule. For every fixed $\psi \in V^*$ ($v \in V$) define *introversion type operators* by

$$L_\psi : V \rightarrow C(S) \text{ and } R_v : V^* \rightarrow C(S), \text{ where } L_\psi(v) = R_v(\psi) = m_{v, \psi}.$$

We say that a vector $v \in V$ is strong (weak) continuous if the corresponding orbit map $\tilde{v} : S \rightarrow V$, $\tilde{v}(s) = vs$, defined through $h : S \rightarrow \Theta(V)$, is strongly (weakly) continuous.

Fact 3.3. *Let $h : S \rightarrow \Theta(V)$ be a weakly continuous cohomomorphism (homomorphism). Then*

- (1) $L_\psi : V \rightarrow C(S)$ is a linear bounded S -operator between right (left) S -actions.
- (2) If $v \in V$ ψ is strong continuous, then $m_{v, \psi}$ is right uniformly continuous on S . If ψ is norm continuous then $m_{v, \psi}$ is left uniformly continuous on S .

Proof. (1) Is straightforward.

- (2) In order to establish that $m_{v, \psi} \in \text{RUC}(S)$, observe that

$$\begin{aligned} |m_{v, \psi}(st) - m_{v, \psi}(s_0t)| &= | \langle vst, \psi \rangle - \langle vs_0t, \psi \rangle | = \\ &| \langle vs, t\psi \rangle - \langle vs_0, t\psi \rangle | \leq \|vs - vs_0\| \cdot \|t\psi\| = \|vs - vs_0\|. \end{aligned}$$

□

Fact 3.4. *Let $(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{opp}, B^*)$ be an equivariant pair with weak* continuous α .*

- (i) *The map $T : V \rightarrow C(X)$, $v \mapsto T(v)$, where $T(v) : X \rightarrow \mathbb{R}$ is defined by*

$$T(v)(x) = \langle v, \alpha(x) \rangle$$

is a linear S -operator (between right S -actions) with $\|T\| \leq 1$.

- (ii) $T(v_0) \in \text{RUC}(X)$ for every strongly continuous vector v_0 in V . Hence, if h is strongly continuous then $T(V) \subset \text{RUC}(X)$.
- (iii) If V is reflexive, then $T(V) \subset \text{WAP}(X)$.

Proof. (i) Is straightforward.

- (ii) Observe that $\|\alpha(x)\| \leq 1$ for every $x \in X$. We get

$$\begin{aligned} \|T(v_0)s - T(v_0)s_0\| &= \sup\{ | \langle v_0s - v_0s_0, \alpha(x) \rangle | : x \in X \} \leq \\ &\leq \|v_0s - v_0s_0\| \cdot \|\alpha(x)\| \leq \|v_0s - v_0s_0\|. \end{aligned}$$

This implies that $T(v_0) \in \text{RUC}(X)$.

(iii) If V is reflexive, the orbit vS is relatively weakly compact for each $v \in V$. By the (weak) continuity of the S -operator T , the same is true for the orbit of $T(v)$ in $C(X)$. Therefore we get $T(v) \in \text{WAP}(X)$. □

Proposition 3.5. *For every S -flow X the following are equivalent:*

- (1) $f \in \text{RUC}(X)$.
- (2) *There exist: a Banach space V , a strongly continuous antihomomorphism $h : S \rightarrow \Theta(V)$, a weak* continuous equivariant map $\alpha : X \rightarrow B^*$, and a vector $v \in V$ such that*

$$f(x) = \langle v, \alpha(x) \rangle = T(v)(x).$$

Proof. (1) \implies (2) The function f belongs to an S -invariant Banach subalgebra \mathcal{A} of $RUC_S(X)$. The right action of S on $V := \mathcal{A}$ is jointly continuous. Then by Corollary 2.49, corresponding left action of S on the dual ball (B^*, w^*) is jointly continuous. Then the naturally associated map $\alpha : X \rightarrow B^*$ and the vector $v := f$ satisfy the desired property.

(1) \impliedby (2) Immediate by Fact 3.4 (ii). \square

Proposition 3.6. *For every semitopological monoid S the following are equivalent:*

- (1) $f \in RUC(S)$.
- (2) *There exist: a Banach space V , a strongly continuous antihomomorphism $h : S \rightarrow \Theta(V)$, and a pair of vectors $v \in V$ and $\psi \in V^*$ such that $f = m_{v,\psi}$.
If G is a topological group then $h(G) \subset Is(V)$.*

Proof. (1) \implies (2) Consider the Gelfand compactification $u_R : S \rightarrow S^R$ defined by $RUC(S) = C(S^R)$. Then the action $S \times S^R \rightarrow S^R$ is jointly continuous. Now define: $V := C(S^R)$, corresponding strongly continuous $h : S \rightarrow \Theta(V)$ (induced by the right action of S on $C(S^R)$), $v := f \in V$ and $\psi = u_R(e) \in V^*$.

(1) \impliedby (2) Immediate by Fact 3.3.2. \square

So we see that every right uniformly continuous function on a (semi)group can be represented as a matrix coefficient $m_{v,\psi}$ of some strongly continuous *Banach representation*. We mentioned also that a positive definite function on a topological group G is a matrix coefficient of some Hilbert representation. One of our aims is to understand the role of matrix coefficients for intermediate cases of reflexive and Asplund representations. We show that wap functions are exactly the *reflexive matrix coefficients*. In the ‘‘Asplund case’’ this approach leads to a definition of *Asplund functions*.

3.1. Enveloping semigroups of Banach spaces.

Definition 3.7. Given a Banach space V we denote by $\mathcal{E}(V)$ the enveloping semigroup of the dynamical system $(\Theta(V)^{op}, B^*)$. We say that $\mathcal{E}(V)$ is the *enveloping semigroup of V* .

In the sequel whenever V is understood we use the following simple notations $\mathcal{E} := \mathcal{E}(V)$, $\Theta := \Theta(V)$, $\Theta^{op} := \Theta(V)^{op}$. By S_V we denote the unit sphere of V .

Lemma 3.8. *For every Banach space V , every $v \in S_V$ and $\psi \in S_{V^*}$ we have*

- (1) $\Theta v = B$.
- (2) $v\mathcal{E} = B^{**}$.
- (3) $cl_{w^*}(\Theta^{op}\psi) = B^*$.
- (4) $\mathcal{E}\psi = B^*$.
- (5) $\Lambda(\mathcal{E}) = \Theta^{op}$.

Proof. (1) Take $f \in S_{V^*}$ such that $f(v) = 1$. For every $z \in B$ define the rank 1 operator

$$A(f, z) : V \rightarrow V, \quad x \mapsto f(x)z.$$

Then $A(f, z)(v) = z$ and $A(f, z) \in \Theta$ since $\|A(f, z)\| = \|f\| \cdot \|z\| = \|z\| \leq 1$.

(2) By (1), $v\Theta^{op} = \Theta v = B$ which is pointwise dense in B^{**} by Goldstine theorem. So, $v\mathcal{E} = B^{**}$ because $\mathcal{E} \rightarrow (V^{**}, w^*), p \mapsto vp$ is continuous and $\mathcal{E} = \overline{\Theta^{op}}$.

(3) We can suppose that V is infinite-dimensional (use (1) for the finite-dimensional case). Then the unit sphere S_{V^*} is norm (hence, weak*) dense in B^* . So it is enough to prove that the weak* closure of $\Theta^{op}\psi$ contains S_{V^*} . Let $\phi \in S_{V^*}$. We have to show that for every $\varepsilon > 0$ and $v_1, v_2, \dots, v_n \in V$ there exists $s \in \Theta$ such that $\|s^*\psi(v_i) - \phi(v_i)\| < \varepsilon$ for every $i = 1, 2, \dots, n$, where $s^* \in \Theta^{op}$ is the adjoint of the operator s . Since $\psi \in V^*$ and $\|\psi\| = 1$ one may choose $z \in B_V$ such that

$$|\phi(v_i)(\psi(z) - 1)| < \varepsilon$$

for every $i = 1, 2, \dots, n$. Define $s := A(\phi, z)$. Then

$$|(s^*\psi)(v_i) - \phi(v_i)| = |\psi(sv_i) - \phi(v_i)| = |\psi(\phi(v_i)z) - \phi(v_i)| = |\phi(v_i)(\psi(z) - 1)| < \varepsilon$$

for every i .

(4) Follows from (3) because \mathcal{E} is the weak* operator closure of Θ^{op} .

(5) Trivially, $\Lambda(\mathcal{E}) \supseteq \Theta^{op}$. Conversely, let $\sigma \in \Lambda(\mathcal{E})$. Then $\sigma \in L(V^*)$ with $\|\sigma\| \leq 1$. Consider the adjoint operator $\sigma^* : V^{**} \rightarrow V^{**}$. We have to show that $\sigma^*(v) \in V \subset V^{**}$, for every $v \in \Lambda$, where we treat V as a Banach subspace of V^{**} . By Banach-Grothendieck theorem (Fact 3.9) it is enough to show that $\sigma^*(v)|_{B^*} : B^* \rightarrow \mathbb{R}$ is w^* -continuous. By our assumption, $\sigma \in \Lambda(\mathcal{E})$. That is, the left translation $l_\sigma : \mathcal{E} \rightarrow \mathcal{E}$ is continuous. Choose a point $z \in S_{V^*}$ and consider the orbit map $\tilde{z} : \mathcal{E} \rightarrow B^*, p \mapsto pz$. Then, $\tilde{z} \circ l_\sigma = \sigma|_{B^*} \circ \tilde{z}$. By (4) we have $\mathcal{E}z = B^*$, hence, $\tilde{z} : \mathcal{E} \rightarrow B^*$ is onto. Since \mathcal{E} is compact, it follows that the map $\sigma|_{B^*} : B^* \rightarrow \mathbb{R}$ is continuous. This implies that $\sigma^*(v)|_{B^*} : B^* \rightarrow \mathbb{R}$ is w^* -continuous (for any $v \in V$), as desired. \square

Fact 3.9. (Banach-Grothendieck theorem) *If V is a Banach space then for every continuous linear functional $u : V^* \rightarrow \mathbb{R}$ on the dual space V^* the following are equivalent:*

- (1) u is w^* -continuous.
- (2) The restriction $u|_{B^*}$ is w^* -continuous.
- (3) u is the evaluation at some point of V . That is, $u \in i(V)$, where $i : V \hookrightarrow V^{**}$ is the canonical embedding.

4. REFLEXIVE SPACES AND WAP SYSTEMS

4.1. Double Limit Property. Let F, X, Y be topological spaces and $w : F \times X \rightarrow Y, w(f, x) := f(x)$ be a function. We say that F has the Double Limit Property (DLP) on X if for every sequence $\{f_n\} \subset F$ and every sequence $\{x_m\} \subset X$ the limits

$$\lim_n \lim_m f_n(x_m) \quad \text{and} \quad \lim_m \lim_n f_n(x_m)$$

are equal whenever they both exist.

Example 4.1. Let V be a reflexive space. Then B has DLP on B^* .

Proof. By Eberlein-Shmulian theorem B and B^* in their weak topologies are sequentially compact. \square

Theorem 4.2. (Raynaud [61, Prop. 1.1], Krivine-Maurey [39, Theorem II.3] for metrizable X, F) *Let $w : F \times X \rightarrow \mathbb{R}$ be a separately continuous bounded function with compact spaces F and X . Then it can be represented on a reflexive space. That is, there exists a reflexive space V and weak continuous maps $\nu : F \rightarrow V, \alpha : X \rightarrow V^*$ such that $\langle \nu(f), \alpha(x) \rangle = w(f, x)$.*

Remark 4.3. One may refine these results (even keeping the general action setting) as follows. The fundamental DFJP-factorization construction from [11] has an ‘‘isometric modification’’ [42]. Taking into account this modification note that we can prove a little bit more. Namely, if the given family $F \subset C(X)$ is *bounded by constant 1*, then we can assume that $\nu(F) \subset B$ and $\alpha(X) \subset B^*$. Hence the following sharper diagram commutes:

$$\begin{array}{ccc} F \times X & \longrightarrow & [-1, 1] \\ \nu \downarrow & & \downarrow \alpha \\ B \times B^* & \longrightarrow & [-1, 1] \end{array}$$

For more details see [26].

Corollary 4.4. *Let F and X be compact spaces and $w : F \times X \rightarrow \mathbb{R}$ be a separately continuous bounded function. Then:*

- (1) F has DLP on X ;
- (2) The induced (bounded) images $j_1(F) \subset C_p(X)$ and $j_2(X) \subset C_p(F)$ are Eberlein compacta (hence, Frechet and sequentially compact).

Note that (1) admits also a direct proof easily reducing the proof to the case of metrizable F, X .

Lemma 4.5. (Grothendieck) *Let X be a compact space. Then a bounded subset F of $C(X)$ is weakly compact iff F is pointwise compact.*

Proof. By Lebesgue dominated convergence theorem it follows that any pointwise converging bounded sequence in $C(X)$ is weakly converging. So, $id : (F, p) \rightarrow (F, w)$ is sequentially continuous. The evaluation map $(F, p) \times X \rightarrow \mathbb{R}$ is separately continuous. Therefore, (F, p) is a Frechet compactum (Corollary 4.4). So we obtain that the pointwise and weak topologies on $F \subset C(X)$ are the same. \square

Lemma 4.6. (Grothendieck; see for example [6, Appendix A]) *Let F be a bounded subset in a Banach space V . The following are equivalent:*

- (1) The weak closure of F in V is weakly compact;
- (2) F has DLP on B^* .

Theorem 4.7. *Let V be a Banach space. The following conditions are equivalent:*

- (1) V is reflexive.
- (2) B has DLP on B^* .
- (3) every bounded subset $F \subset V$ has DLP on every bounded $X \subset V^*$.
- (4) $B \subset V$ is weakly compact.

Proof. (1) \Rightarrow (2) As in Example 4.1 use Eberlein-Shmulian theorem.

(2) \Rightarrow (3) Is trivial.

(3) \Rightarrow (4) Apply Lemma 4.6.

(4) \Rightarrow (1) B_V is w^* -closed in V^{**} . By Goldstine's theorem the w^* -closure of B_V in V^{**} is $B^{**} := B_{V^{**}}$. Hence, $B = B^{**}$. This implies directly that $V = V^{**}$. \square

Exercise 4.8. Let X be a compact space and $F \subset C(X)$ be a bounded subset. Show that F has DLP on X iff F has DLP on B^* , where $B^* = B_{C(X)^*}$.

4.2. WAP dynamical systems. Given a function $f \in C(X)$ we consider its orbit $fS := \{f \circ \tilde{s} : s \in S\} \subset C(X)$. For every $f \in C(X)$ the function $E \rightarrow \mathbb{R}^X, s \mapsto fs$ is pointwise continuous. So we have $fE = cl_p(fs)$.

One may estimate the dynamical complexity of f is by considering the pointwise compact subset $cl_p(fs)$ in \mathbb{R}^X . Various kinds of "smallness" of this compactum leads to a natural hierarchy. The classical example is (weakly) almost periodic functions.

Definition 4.9. Let X be a compact S -system.

- (1) $f \in C(X)$ is said to be WAP if one of the following equivalent conditions is satisfied:
 - (a) fS is weakly precompact in $C(X)$;
 - (b) $cl_p(fs) \subset C(X)$;
 - (c) fS has DLP on X .
- (2) (S, X) is said to be WAP if one of the following equivalent conditions is satisfied:
 - (a) every member $p \in E(S, X)$ is a continuous function $X \rightarrow X$;
 - (b) $WAP(X) = C(X)$.

The equivalences can be verified using Grothendieck's classical results. See for example, [6, Theorem A4] and [6, Theorem A5]. If X is metrizable (or, sequentially compact) then (a) and (b) in (2) are equivalent to the condition: (c) $S \times X \rightarrow X$ has DLP.

The following characterization of WAP dynamical systems is due to Ellis and Nerurkar [15].

Theorem 4.10. *Let X be a compact S -dynamical system. The following conditions are equivalent.*

- (1) (S, X) is WAP.
- (2) The enveloping semigroup $E(S, X)$ consists of continuous maps. That is, $E(S, X) \subset C(X, X)$.

Proof. (1) \Rightarrow (2) By Definition 4.9, $\text{cl}_p(fS) \subset C(X)$ for every $f \in C(X) = \text{WAP}(X)$. Therefore, $fp : X \rightarrow \mathbb{R}$ is continuous for every $f \in C(X)$ and $p \in E(X)$. Since X is compact this guarantees that every $p : X \rightarrow X$ is continuous.

(2) \Rightarrow (1) If $E \subset C(X, X)$ then $fE = \text{cl}_p(fS) \subset C(X)$. By Grothendieck's Lemma 4.5, $\text{cl}_p(fS)$ is weakly compact. Hence, $f \in \text{WAP}(X)$. \square

Corollary 4.11. *When (S, X) is WAP the enveloping semigroup $E(X)$ is a semitopological semigroup. The converse holds if in addition we assume that (S, X) is point transitive.*

Example 4.12.

- (1) (Eberlein, see for example [6, Examples 1.2.f]) If V is reflexive, then every weakly continuous representation (h, α) of an S -system X on V and every pair $(v, \psi) \in V \times V^*$ lead to a weakly almost periodic function $m_{v, \psi}$ on S . This follows easily by the (weak) continuity of the bounded operator $L_\psi : V \rightarrow C(S)$, where $L_\psi(v) = m_{v, \psi}$.
- (2) Analogously, every $v \in V$ (with reflexive V) defines a wap function $T_v : X \rightarrow \mathbb{R}$ on the G -system X which naturally comes from the given dynamical system representation (h, α) . Precisely, define

$$T(V) = T_v : X \rightarrow \mathbb{R}, \quad x \mapsto \langle v, \alpha(x) \rangle.$$

Then the set of functions $\{T_v\}_{v \in V}$ is a subset of $\text{WAP}(X)$.

Proof. (1) If the orbit of vS is relatively weakly compact in V . Then $L_\psi(vS) = m_{v, \psi}S$ is relatively weakly compact in $C(S)$. Thus, $m_{v, \psi} \in \text{WAP}(S)$.

For the case when h is a "homomorphism" recall (see [10] or [6]) that fS is weakly precompact iff Sf is weakly precompact in $C(S)$.

- (2) Is similar. \square

If in (2) α is an embedding (which implies that X is reflexively representable) then it follows that the collection $\{T_v\}_{v \in V}$ (and hence also $\text{WAP}(X)$) separates the points of X . If, in addition, X is compact it follows that $\text{WAP}(X) = C(X)$ (because $\text{WAP}(X)$ is always a closed subalgebra of $C(X)$). That is, in this case (S, X) is WAP in the sense of Ellis and Nerurkar.

The converses of Example 4.12 is also true as we show below.

Theorem 4.13. [49] *Let $S \times X \rightarrow X$ be a separately continuous action of a semitopological semigroup S on a compact space X . For every $f \in \text{WAP}(X)$ there exist: a reflexive space V , a functional $\phi \in V^*$ and an equivariant pair*

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V), B_V)$$

such that $h : S \rightarrow \Theta(V)$ is a weakly continuous homomorphism, $\alpha : X \rightarrow B_V$ is a weakly continuous S -map, and $f(x) = \langle \phi, \alpha(x) \rangle = \phi(\alpha(x))$ for every $x \in X$.

If $S = G$ is a semitopological group then one can assume in addition that $h(G) \subset \text{Iso}(V)$ and $h : G \rightarrow \text{Iso}(V)$ is strongly continuous.

Theorem 4.14. [49, section 4] *Let S be a semitopological semigroup.*

- (1) *A compact (continuous) S -space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively approximable.*
- (2) *A compact (continuous) metric S -space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively representable.*
- (3) *Every $f \in \text{WAP}(S)$ is a matrix coefficient of a reflexive representation.*

It is important to take into account the following characterization of reflexive spaces.

Lemma 4.15. *Let V be a Banach space. The following conditions are equivalent:*

- (1) *V is reflexive.*
- (2) *The semitopological semigroup $\Theta(V)_w$ is compact.*
- (3) *The compact right topological semigroup $\mathcal{E}(V)$ is semitopological.*
- (4) *(Θ^{op}, B^*) is a WAP system.*

Proof. (1) \Rightarrow (2) Always, $\Theta(V)$ is a closed subset of the product $(B, w)^B$. So, if V is reflexive then by Theorem 4.7 $(B, w)^B$ is compact. Hence, we obtain by Tychonoff theorem that $\Theta(V)$ is compact.

(2) \Rightarrow (3) Use the fact that $\Theta(V)_w^{op}$ is dense in $\mathcal{E}(V)$ (Lemma 3.8).

(3) \Rightarrow (4) One may apply Corollary 4.11 because $(\Theta(V)_w^{op}, B^*)$ is transitive.

(4) \Rightarrow (1) Choose any $v \in S_V$ and treat it as a (continuous) function on the dynamical Θ^{op} -system B^* . Then $v \in \text{WAP}(B^*)$. Then its orbit has DLP on B^* . So, $v\Theta^{op} = \Theta v = B$ has DLP on B^* . \square

Another consequence of Theorem 4.14 (taking into account Lemma 4.15) is

Theorem 4.16. ([67] and [46]) *Every compact semitopological semigroup S can be embedded into $\Theta(V)$ for some reflexive V .*

Thus, compact semitopological semigroups S can be characterized as closed subsemigroups of $\mathcal{E}(V)$ for reflexive Banach spaces V .

4.3. DFJP factorization for WAP dynamical systems.

Theorem 4.17. *Let X be a compact S -space and $F \subset C(X)$ a norm bounded S -invariant subset of $C(X)$. The following are equivalent:*

- (1) *(F, S, X) admits a reflexive representation.*
- (2) *$\text{cl}_p(F) \subset C(X)$.*
- (3) *F has DLP on X .*

Proof. For the “only if” part observe that if V is a reflexive space then every bounded subset F of the dual V^* has DLP on every bounded subset $X \subset V$. This follows from the Eberlein-Šmulian theorem. See Theorem 4.7.

For the “if part” we use [49, Theorem 4.11] which we reformulate here in the present terms of Definition 9.1. Let $w : F \times X \rightarrow \mathbb{R}$ be a separately continuous bounded map with compact spaces F and X such that S acts on X (from left) and on F (from right) via separately continuous actions such that $w(f, sx) = w(fs, x)$. Assume that F (thinking it as a (bounded) family of maps $X \rightarrow \mathbb{R}$) separates the points of X . Then according to [49, Theorem 4.11] there exists a reflexive space V and a faithful representation (ν, h, α) of the triple (F, S, X) .

Another important part (which is close to Grothendieck’s double limit theorem) of the proof is [6, Theorem A.4]. It asserts that for every compact space X a bounded family $F \subset C(X)$ has the Double Limit Property on X iff $\text{cl}_p(F) \subset C(X)$. \square

Theorem 4.18. *(S, X) is a WAP (continuous) system if and only if (S, X) is weakly (respectively, strongly) reflexively-approximable. If X is metrizable then “approximable” can be replaced by “representable”. Moreover, the corresponding Banach space can be assumed to be separable.*

Proof. The “only if” part: Use the fact that (Θ^{op}, B^*) is a WAP system (Theorem 8.4) for every reflexive space V .

The “if” part: (1) For every $f \in C(X) = \text{Tame}(X)$ the orbit fS has DLP (being weakly precompact) family for X . Applying Theorem 9.4 below we conclude that every $f \in C(X) = \text{Tame}(X)$ on a compact S -space X comes from a reflexive representation. Since continuous functions separate points of X , this implies that reflexive representations of (S, X) separate points of X . So, it is enough to prove the following result which gives a proof of Theorem 4.17.

Theorem 4.19. *Let X be a compact S -space and let $F \subset C(X)$ be a bounded S -invariant pointwise compact family. Then there exist: a reflexive Banach space V , an injective mapping $\nu : F \rightarrow B_V$ and a representation*

$$h : S \rightarrow \Theta(V), \quad \alpha : X \rightarrow V^*$$

of (S, X) on V such that h is weakly continuous, α is a weak (in fact, weakly) continuous map and*

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall f \in F \quad \forall x \in X.$$

Thus the following diagram commutes

$$(4.1) \quad \begin{array}{ccc} F \times X & \longrightarrow & \mathbb{R} \\ \nu \downarrow & \downarrow \alpha & \downarrow id_{\mathbb{R}} \\ V \times V^* & \longrightarrow & \mathbb{R} \end{array}$$

If X is metrizable then in addition we can suppose that V is separable.

If the action $S \times X \rightarrow X$ is continuous we may assume that h is strongly continuous.

If $S = G$ is a group then $h(G) \subset \text{Iso}(V)$.

If F separates points of X then $\alpha : X \rightarrow (V^, w^*)$ is a topological embedding.*

Proof. Step 1: The construction of V .

For brevity of notation let $\mathcal{A} := C(X)$ denote the Banach space $C(X)$, B will denote its unit ball, and B^* will denote the weak* compact unit ball of the dual space $\mathcal{A}^* = C(X)^*$. Let W be the symmetrized convex hull of F ; that is, $W := \text{co}(F \cup -F)$. Consider the sequence of sets

$$(4.2) \quad M_n := 2^n W + 2^{-n} B.$$

Then W is convex and symmetric. We apply the construction of Davis-Figiel-Johnson-Pelczyński [11] as follows. Let $\|\cdot\|_n$ be the Minkowski functional of the set M_n , that is,

$$\|v\|_n = \inf \{ \lambda > 0 \mid v \in \lambda M_n \}.$$

Then $\|\cdot\|_n$ is a norm on \mathcal{A} equivalent to the given norm of \mathcal{A} . For $v \in \mathcal{A}$, set

$$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \quad \text{and let } V := \{v \in \mathcal{A} \mid N(v) < \infty\}.$$

Denote by $j : V \hookrightarrow \mathcal{A}$ the inclusion map. Then (V, N) is a Banach space, $j : V \rightarrow \mathcal{A}$ is a continuous linear injection and

$$(4.3) \quad W \subset j(B_V) = B_V \subset \bigcap_{n \in \mathbb{N}} M_n = \bigcap_{n \in \mathbb{N}} (2^n W + 2^{-n} B)$$

Indeed, if $v \in W$ then $2^n v \in M_n$, hence $\|v\|_n \leq 2^{-n}$ and $N(v)^2 \leq \sum_{n \in \mathbb{N}} 2^{-2n} < 1$. This proves $W \subset j(B_V)$. In order to prove the second inclusion recall that the norms $\|\cdot\|_n$ on \mathcal{A} are equivalent to each other. It follows that if $v \in B_V$ then $\|v\|_n < 1$ for all $n \in \mathbb{N}$. That is, for every $n \in \mathbb{N}$, $v \in \lambda_n M_n$ for some $0 < \lambda_n < 1$. By the construction M_n is a convex subset containing the origin. This implies that $\lambda_n M_n \subset M_n$. Hence $j(v) = v \in M_n$ for every $n \in \mathbb{N}$.

Step 2: The construction of the representation (h, α) of (S, X) on V .

The given action $S \times X \rightarrow X$ induces a natural linear norm preserving continuous right action $C(X) \times S \rightarrow C(X)$ on the Banach space $\mathcal{A} = C(X)$. It follows by the construction that W and B are S -invariant subsets in \mathcal{A} . This implies that V is an S -invariant subset of \mathcal{A} and the restricted natural linear action $V \times S \rightarrow V$, $(v, g) \mapsto vg$ satisfies $N(vs) \leq N(v)$. Therefore, the co-homomorphism $h : S \rightarrow \Theta(V)$, $h(s)(v) := vs$ is well defined.

Let $j^* : \mathcal{A}^* \rightarrow V^*$ be the adjoint map of $j : V \rightarrow \mathcal{A}$. Define $\alpha : X \rightarrow V^*$ as follows. For every $x \in X \subset C(X)^*$ set $\alpha(x) = j^*(x)$. Then (h, α) is a representation of (S, X) on the Banach space V .

By the construction, $F \subset W \subset B_V$. Define $\nu : F \hookrightarrow B_V$ as the natural inclusion. Then

$$(4.4) \quad f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall f \in F \quad \forall x \in X.$$

Step 3: Weak continuity of $h : S \rightarrow \Theta(V)$.

By our construction $j^* : C(X)^* \rightarrow V^*$, being the adjoint of the bounded linear operator $j : V \rightarrow C(X)$, is a norm and weak* continuous linear operator. By [16, Lemma 1.2.2] we obtain that $j^*(C(X)^*)$ is norm dense in V^* . Since V is Rosenthal, Haydon's theorem (Fact 5.9.4) gives $Q := cl_{w^*}(co(Y)) = cl_{norm}(co(Y))$, where $Y := j^*(X)$. Now observe that $j^*(P(X)) = Q$. Since $S \times X \rightarrow X$ is separately continuous, every orbit map $\tilde{x} : S \rightarrow X$ is continuous, and each orbit map $\widetilde{j^*(x)} : S \rightarrow j^*(X)$ is weak* continuous. Then also $\widetilde{j^*(z)} : S \rightarrow V^*$ is weak* continuous for each $z \in cl_{norm}(co(j^*(X))) = Q$. Since $sp(Q)$ is norm dense in V^* (and $\|h(s)\| \leq 1$ for each $s \in S$) it easily follows that $j^*(z) : S \rightarrow V^*$ is weak* continuous for every $z \in V^*$. This is equivalent to the weak continuity of h .

If the action $S \times X \rightarrow X$ is continuous we may assume that h is strongly continuous. Indeed, by the definition of the norm N , we can show that the action of S on V is norm continuous (use the fact that, for each $n \in \mathbb{N}$, the norm $\|\cdot\|_n$ on \mathcal{A} is equivalent to the given norm on \mathcal{A}).

Step 4: V is a reflexive space.

Proof. See [11]. Sketch: $j^{**}(B_{V^{**}}) \subset \mathcal{A}$. Hence, j^{**} is 1-1 and $(j^{**})^{-1}(\mathcal{A}) = V$. It follows that $V^{**} \subset V$ (reflexivity). □

If the compact space X is metrizable then $C(X)$ is separable and it is also easy to see that (V, N) is separable. □

This proves Theorem 9.4 and hence also Theorem 9.3.1.

Finally note that if X is metrizable then “approximable” can be replaced by “representable” using an l_2 -sum of a sequence of separable reflexive spaces (see Lemma 5.11.3). \square

4.4. Applications.

Theorem 4.20. (WAP Representation Theorem) *Let X be a compact semitopological S -system and $f \in C(X)$. The following conditions are equivalent:*

- (i) $f : X \rightarrow \mathbb{R}$ is weakly almost periodic.
- (ii) There exist: a representation (h, α) of (S, X) into reflexive V with a weak continuous antihomomorphism $h : S \rightarrow \Theta(V)$, weak (eq., weak-star) continuous $\alpha : X \rightarrow B^*$, and a vector $v \in V$ such that $f(x) = \langle v, \alpha(x) \rangle$.

If either: a) $S = G$ is a semitopological group; or b) X is compact and the action $S \times X \rightarrow X$ is jointly continuous, then in (ii) we can suppose that h is strongly continuous.

Proof. Recall that a continuous function $f \in C(X)$ is WAP iff the orbit fS is relatively weakly compact in $C(X)$. That is, $cl_w(fS)$ is weakly compact. Apply Theorem for $F := cl_w(fS)$. \square

Theorem 4.21. *A compact S -system X is WAP iff X is reflexively approximable.*

Proof. If X is REFL-approximable then X is wap by Fact 3.4 (iii) (in fact, WAP separates points but since $WAP(X)$ is an algebra and X is compact it is enough). Another proof by enveloping semigroup.

The nontrivial part follows from Theorem 4.20 because if X has sufficiently many wap functions, then (S, X) has sufficiently many reflexive representations. \square

Corollary 4.22. *Every wap flow (S, X) is RN-approximable.*

It is well known that a countable product of Eberlein (RN) compacta is again Eberlein (resp.: RN). We show that the same is true for dynamical systems.

Lemma 4.23. *The classes of Eberlein and RN S -systems are closed under countable products.*

Proof. Let X_n be a sequence of Eberlein (or, RN) S -flows. By the definition there exists a sequence of reflexive (Asplund) representations

$$(h_n, \alpha_n) : (S, X_n) \rightrightarrows (\Theta(V_n)^{opp}, B(V_n^*)).$$

We can suppose that each X_n is compact and $\alpha_n(X_n) \subset 2^{-n}B(V_n^*)$. Turn to the l_2 -sum of representations. That is, consider

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{opp}, B(V^*))$$

where $V := (\sum_n V_n)_{l_2}$, $h(s)(v) = \sum_n h(s)(v_n)$ for every $v = \sum_n v_n$, and $\alpha(x) = \sum_n \alpha_n(x_n)$ for every $x = (x_1, x_2, \dots) \in \prod_n X_n$. It is easy to show that $\alpha(x) \in B(V^*)$, α is weak* continuous and injective (hence, a topological embedding). Now use the fact that the l_2 -sum of reflexive (Asplund) spaces is again reflexive (Asplund) [16]. \square

Corollary 4.24. (i) *Every second countable compact wap system is Eberlein.*
(ii) *Every second countable compact RN-approximable system is an RN system.*

Proof. Assertion (ii) is immediate by Lemma 4.23. Indeed, use the following observation: if a family $F := \{f_i : X \rightarrow Y_i\}$ of continuous functions separates the points of X then there exists a countable subfamily which also separates the points.

For (i), we need also Theorem 4.21. \square

The following theorem provides, in particular, a flow generalization of a result by Amir-Lindenstrauss which states that if X is an Eberlein compact then $B^* = (B(C(X)^*), w^*)$ is Eberlein, too.

Theorem 4.25. *Let X be a compact semitopological S -flow. The following are equivalent:*

- (i) X is S -Eberlein.
- (ii) There exists a Banach space E , a homomorphism $h : S \rightarrow \Theta(V)$ (no continuity assumptions on h), and an S -embedding $\alpha : X \rightarrow (V, w)$.
- (iii) There exists a compact space Y and a (right) strict S -duality $Y \times X \rightarrow \mathbb{R}$.
- (iv) There exists a sequence of S -invariant weakly compact subsets $K_n \subset C(X)$ such that $\bigcup_{n \in \mathbb{N}} K_n$ separates the points of X .
- (v) There exists an S -invariant weakly compact subset M in $C(X)$ such that $cl(sp(M)) = C(X)$.
- (vi) B^* is S -Eberlein.

Proof. (i) \implies (ii) By the definition there exists a faithful reflexive V -representation. That is, we can choose a weakly continuous homomorphism $h : S \rightarrow \Theta(V)^{opp} = \Theta(V^*)$ and an equivariant embedding $\alpha : X \rightarrow B(V^*)$. It suffices to choose $E := V^*$.

(ii) \implies (iii) By our assumption X is S -embedded into (E, w) . Define the right strict S -duality $Y \times X \rightarrow \mathbb{R}$ as a restriction of the canonical duality where $Y := (B(E^*), w^*)$.

(iii) \implies (iv) Our right strict $Y \times X \rightarrow \mathbb{R}$ duality induces the strict S -duality $Y_q \times X \rightarrow \mathbb{R}$. We can suppose in addition that this duality is bounded. Now define simply $K_n := Y_q \subset C(X)$ for each n and use Fact 4.5.

(iv) \implies (v) Look at K_n as an S -subflow of $(C(X), w)$. We can suppose that $K_n \subset B(C(X))$. Following a method of Rosenthal, consider S -invariant set M_n consisting of the constant function equal to 1 on X and of all products of functions $f_1 \cdot f_2 \cdots f_n$ where $f_i \in (\bigcup_{m=1}^n K_m) \cup \{1\}$. By Fact 4.5 and the Eberlein-Smulian theorem, it is easy to see that each M_n is weakly compact. Then $M := \bigcup_{n \in \mathbb{N}} 2^{-n} M_n$ is also S -invariant and weakly compact in $C(X)$. By the Stone-Weierstrass theorem, $sp(M)$ is dense in $C(X)$.

(v) \implies (vi) We can suppose that $M \subset B(C(X))$. The corresponding left strict S -duality $M \times B^* \rightarrow [-1, 1]$ is also right strict because $cl(sp(M)) = C(X)$. Now we can apply Theorem 4.19.

(vi) \implies (i) is trivial because X can be treated as an S -subflow of B^* . □

Theorem 4.26. *For every semitopological monoid S the function $f : S \rightarrow \mathbb{R}$ is wap iff f is a matrix coefficient of a weak continuous antihomomorphism $S \rightarrow \Theta(V)$ for a reflexive V . That is, there exist $v \in V$ and $\psi \in V^*$ such that $f(s) = \langle vs, \psi \rangle$.*

If $S = G$ is a group then $h(G) \subset Is(V)$ (and h is strongly continuous).

Proof. Apply Theorem 4.20 to the flow (S, S) . Then for $f \in WAP(S)$ there exists a reflexive V and a representation $h : S \rightarrow \Theta(V)$, $\alpha : S \rightarrow B(V^*)$ such that $f(s) = \langle v, \alpha(s) \rangle$ for a suitable $v \in V$. Denote by e the identity of S . Then $f = m_{v, \psi}$ where $\psi = \alpha(e)$. □

If we wish to get a homomorphism, just consider $h : S \rightarrow \Theta(V)^{opp} = \Theta(V^*)$.

Fact 4.27. ([67] and [46]) *Let S be a semitopological semigroup. The following are equivalent:*

- (i) S is embedded into a compact semitopological monoid.
- (ii) There exists a reflexive space E such that S is embedded (as a semitopological subsemigroup) into $\Theta(E)_w$.

Therefore, compact semitopological semigroups are exactly the class of all closed subsemigroups of $\Theta(E)_w$ for some reflexive V .

Proof. (i) \implies (ii) We can suppose that S is a monoid. Consider $X := S^W$ the universal semitopological compactification of S . Then the corresponding universal map $u_W : S \rightarrow S^W$ is a topological embedding by (i) and hence, the action (S, S^W) is *left strict*. That is, there is no strictly coarser topology on S under which S is a semitopological semigroup and S^W is still a semitopological S -flow. By Theorem 4.20 there exists a separating family (h_i, α_i) of reflexive V_i -representations ($i \in I$) of (S, S^W) . Then the l_2 -sum of these representations defined on the Banach space $V := (\sum_i V_{i \in I})_{l_2}$ will induce a weakly continuous antihomomorphism $h : S \rightarrow \Theta(V)$. Since the original action is left strict, it is easy to show that h must be a topological embedding. Define $E := V^*$. It is clear that the antihomomorphism h defines the desired homomorphism $h : S \rightarrow \Theta(V)^{opp} = \Theta(V^*) = \Theta(E)$.
(ii) \implies (i) Use Lemma 4.15. □

By Theorem 2.7, $Is(V)_s = Is(V)_w$ for every reflexive V . Therefore we obtain

Fact 4.28. *Let G be a topological group. The following are equivalent:*

- (i) $G \rightarrow G^w$ is an embedding.
- (ii) G is a topological subgroup of the group $Is(V)_s$ (endowed with the strong operator topology) of all linear isometries for a suitable reflexive V .

It is well known (as noted for example by Arhangel'skij or Namioka-Wheeler that a compact space is Eberlein iff it can be included into some right strict duality. Theorem 4.19 provides “a flow version”.

We next recall a version of Lawson's theorem and a soft geometric proof using representations of dynamical systems on reflexive spaces.

Theorem 4.29. (Ellis-Lawson Joint Continuity Theorem) *Let G be a subgroup of a compact semitopological monoid S . Suppose that $S \times X \rightarrow X$ is a separately continuous action with compact X . Then the action $G \times X \rightarrow X$ is jointly continuous (and G is a topological group).*

Proof. A sketch of the proof from [49]: We show the joint continuity of $G \times X \rightarrow X$ (for the last part take $X := S$ and the natural action $G \times S \rightarrow S$). It is easy to see by Grothendieck's Lemma (Theorem 4.5) that $C(X) = WAP(X)$. Hence (S, X) is a weakly almost periodic system. By Theorem 4.14 the proof can be reduced to the particular case where $(S, X) = (\Theta(V)^{op}, B_{V^*})$ for some reflexive Banach space V with $G := Iso(V)$, where $\Theta(V)^{op}$ is endowed with the weak operator topology. By Theorem 2.7, the weak and strong operator topologies coincide on $Iso(V)$ for reflexive V . In particular, G acts continuously on B_{V^*} . (For the last part use apply our assertion to the following action $G \times X \rightarrow X$ where $X = S$) □

As a corollary one gets the classical result of Ellis.

Theorem 4.30. (Ellis' Theorem) *Every compact semitopological group is a topological group.*

5. FRAGMENTABILITY AND BANACH SPACES

The concept of fragmentability originally comes from Banach space theory and has several applications in Topology, and more recently also in Topological Dynamics.

Definition 5.1. Let (X, τ) be a topological space and (Y, μ) a uniform space.

- (1) [35] X is (τ, μ) -fragmented by a (typically, not continuous) function $f : X \rightarrow Y$ if for every nonempty subset A of X and every $\varepsilon \in \mu$ there exists an open subset O of X such that $O \cap A$ is nonempty and the set $f(O \cap A)$ is ε -small in Y . We also say in that case that the function f is *fragmented*. Notation: $f \in \mathcal{F}(X, Y)$, whenever the uniformity μ is understood. If $Y = \mathbb{R}$ then we write simply $\mathcal{F}(X)$.

- (2) [22] We say that a *family of functions* $F = \{f : (X, \tau) \rightarrow (Y, \mu)\}$ is *fragmented* if condition (1) holds simultaneously for all $f \in F$. That is, $f(O \cap A)$ is ε -small for every $f \in F$.
- (3) [28] We say that F is an *eventually fragmented family* if every infinite subfamily $C \subset F$ contains an infinite fragmented subfamily $K \subset C$.
- (4) We say that F is an *eventually weakly fragmented family* if for every infinite subfamily $L \subset F$ and every $\varepsilon \in \xi$ there exists an infinite ε -Fr subfamily $K \subset L$.

In Definition 5.1.1 when $Y = X$, $f = id_X$ and μ is a metric uniform structure, we get the usual definition of fragmentability (more precisely, (τ, μ) -fragmentability) in the sense of Jayne and Rogers [36]. Implicitly it already appears in a paper of Namioka and Phelps [56].

Remark 5.2. [22, 24]

- (1) It is enough to check the condition of Definition 5.1 for closed subsets $A \subset X$ and for $\epsilon \in \mu$ from a *subbase* γ of μ (that is, the finite intersections of the elements of γ form a base of the uniform structure μ).
- (2) When X and Y are Polish spaces, $f : X \rightarrow Y$ is fragmented iff f is a Baire class 1 function.
- (3) When X is compact and (Y, ρ) metrizable uniform space then $f : X \rightarrow Y$ is fragmented iff f has a *point of continuity property* (i.e., for every closed nonempty $A \subset X$ the restriction $f|_A : A \rightarrow Y$ has a continuity point).
- (4) When Y is compact with its unique compatible uniformity μ then $p : X \rightarrow Y$ is fragmented if and only if $f \circ p : X \rightarrow \mathbb{R}$ has a point of continuity property for every $f \in C(Y)$.
- (5) A topological space (X, τ) is scattered iff X is (τ, μ) -fragmented, for every uniform structure μ on the set X .
- (6) Let (X, τ) be a separable metrizable space and (Y, ρ) a pseudometric space. Suppose that $f : X \rightarrow Y$ is a fragmented onto map. Then Y is separable.
- (7) $F = \{f_i : X \rightarrow (Y, \xi)\}_{i \in I}$ is a fragmented family iff the induced map $X \rightarrow (Y^F, \xi_U)$ is fragmented, where ξ_U is the uniformity of uniform convergence.
- (8) Let $\alpha : X \rightarrow X'$ be a continuous onto map between compact spaces. Assume that (Y, ξ) is a uniform space, $F := \{f_i : X \rightarrow Y\}_{i \in I}$ and $F' := \{f'_i : X' \rightarrow Y\}$ are families such that $f'_i \circ \alpha = f_i$ for every $i \in I$. Then F is a fragmented family iff F' is a fragmented family.

The first assertion in the following lemma can be proved using Namioka's joint continuity theorem.

- Lemma 5.3.** (1) *Suppose F is a compact space, X is Čech-complete, Y is a uniform space and we are given a separately continuous map $w : F \times X \rightarrow Y$. Then the naturally associated family $\tilde{F} := \{\tilde{f} : X \rightarrow Y\}_{f \in F}$ is fragmented, where $\tilde{f}(x) = w(f, x)$.*
- (2) *Suppose F is a compact metrizable space, X is hereditarily Baire and M is separable and metrizable. Assume we are given a map $w : F \times X \rightarrow M$ such that every $\tilde{x} : F \rightarrow M$, $f \mapsto w(f, x)$ is continuous and $y : X \rightarrow M$ is continuous at every $\tilde{y} \in Y$ for some dense subset Y of F . Then the family \tilde{F} is fragmented.*
- (3) (version of Osgood's theorem) *Let $f_n : X \rightarrow \mathbb{R}$ be a pointwise convergent sequence of continuous functions on a hereditarily Baire space X . Then $F := \{f_n\}_{n \in \mathbb{N}}$ is a fragmented family.*

Proof. (1): There exists a collection of uniform maps $\{\varphi_i : Y \rightarrow M_i\}_{i \in I}$ into metrizable uniform spaces M_i which generates the uniformity on Y . Now for every closed subset

$A \subset X$ apply Namioka's joint continuity theorem to the separately continuous map $\varphi_i \circ w : F \times A \rightarrow M_i$ and take into account Remark 5.2.1.

(2): Since every $\tilde{x} : F \rightarrow M$ is continuous, the natural map $j : X \rightarrow C(F, M)$, $j(x) = \tilde{x}$ is well defined. Every closed nonempty subset $A \subset X$ is Baire. By [29, Proposition 2.4], $j|_A : A \rightarrow C(F, M)$ has a point of continuity, where $C(F, M)$ carries the sup-metric. Hence, $\tilde{F}_A = \{\tilde{f}|_A : A \rightarrow M\}_{f \in F}$ is equicontinuous at some point $a \in A$. This implies that the family \tilde{F} is fragmented.

(3) follows from (2) applied to the evaluation map $w : F \times X \rightarrow \mathbb{R}$, where $F := \{f\} \cup \{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ with $f := \lim f_n$, the pointwise limit. □

Remark 5.4. Let us briefly describe one of the ideas linking fragmentability and dynamical systems. Suppose that X is a weak* compact dual ball of some Banach space V . One of the major themes in Banach space theory is the study of the relationship between the norm and the weak* topologies on $X \subset V^*$. When these two coincide, we say that X is a *Kadec subset* of V^* . If moreover X is a subsystem (under some action by linear isometries) then X , as a dynamical system, is equicontinuous. In general, as an attempt to measure “the level of equicontinuity”, we can ask how close are the two natural topologies on X inherited from V^* . A more concrete, but sufficiently flexible, question is: for which dynamical system representations is the natural mapping $1_X : (X, \text{weak}^*) \rightarrow (X, \text{norm})$ fragmented? The latter means that every nonempty subset of X admits relatively weak* open nonempty subsets with arbitrarily small diameters. Every point of continuity of 1_X is a point of equicontinuity of the dynamical system X .

5.1. Banach spaces defined by fragmentability. We recall the definitions of three important classes of Banach spaces: Asplund, Rosenthal and PCP. Each of them can be characterized in terms of fragmentability.

5.1.1. *Asplund Banach spaces.* Recall that a Banach space V is an *Asplund space* if the dual of every separable linear subspace is separable.

In the following result the equivalence of (1), (2) and (3) is well known and (4) is a reformulation of (3) in terms of fragmented families.

Theorem 5.5. [56, 55] *Let V be a Banach space. The following conditions are equivalent:*

- (1) V is an Asplund space.
- (2) V^* has the Radon-Nikodým property.
- (3) Every bounded subset A of the dual V^* is (weak*, norm)-fragmented.
- (4) B is a fragmented family of real valued maps on the compactum B^* .

Clearly, (4) is equivalent to saying that every bounded subset $F \subset V$ is a fragmented family of functions on every weak-star compact subset $X \subset V^*$.

Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund. By [56] the Banach space $C(K)$ for compact K is Asplund iff K is a scattered compactum (see also Lemma 5.2.4). Namioka's Joint Continuity Theorem implies that every weakly compact set in a Banach space is norm fragmented, [55]. This explains why every reflexive space is Asplund.

5.1.2. *Banach spaces not containing l_1 .* Let $f_n : X \rightarrow \mathbb{R}$ be a uniformly bounded sequence of functions on a set X . Following to Rosenthal we say that this sequence is an l_1 -sequence on X if there exists a real constant $a > 0$ such that for all $n \in \mathbb{N}$ and choices of real scalars c_1, \dots, c_n we have

$$a \cdot \sum_{i=1}^n |c_i| \leq \left\| \sum_{i=1}^n c_i f_i \right\|.$$

This is the same as requiring that the closed linear span in $l_\infty(X)$ of the sequence f_n be linearly homeomorphic to the Banach space l_1 . In fact, in this case the map

$$l_1 \rightarrow l_\infty(X), \quad (c_n) \rightarrow \sum_{n \in \mathbb{N}} c_n f_n$$

is a linear homeomorphic embedding.

A sequence f_n of real valued functions on a set X is said to be *independent* if there exist $a < b$ such that

$$\bigcap_{n \in P} f_n^{-1}(-\infty, a) \cap \bigcap_{m \in M} f_m^{-1}(b, \infty) \neq \emptyset$$

for all finite disjoint subsets P, M of \mathbb{N} . Every subsequence of an independent sequence is again independent.

Definition 5.6. A Banach space V is said to be *Rosenthal* if it does not contain an isomorphic copy of l_1 .

Every Asplund space is Rosenthal (because l_1^* is the nonseparable space l_∞).

Definition 5.7. Let X be a topological space. We say that a subset $F \subset C(X)$ is a *Rosenthal family* (for X) if F is norm bounded and the pointwise closure $\text{cl}_p(F)$ of F in \mathbb{R}^X consists of fragmented maps, that is, $\text{cl}_p(F) \subset \mathcal{F}(X)$.

The following useful result synthesizes some known results.

It is based on results of Rosenthal [63], Talagrand [68, Theorem 14.1.7] and van Dulst [?]. In [24, Prop. 4.6] we show why eventual fragmentability of F can be included in the following list.

Theorem 5.8. *Let X be a compact space and $F \subset C(X)$ a bounded subset. The following conditions are equivalent:*

- (1) F does not contain an independent subsequence.
- (2) F does not contain a subsequence equivalent to the unit basis of l_1 .
- (3) Each sequence in F has a pointwise convergent subsequence in \mathbb{R}^X .
- (4) F is a Rosenthal family for X .
- (5) F is an eventually fragmented family.

We need also some characterizations of Rosenthal spaces.

Theorem 5.9. *Let V be a Banach space. The following conditions are equivalent:*

- (1) V is a Rosenthal Banach space.
- (2) (E. Saab and P. Saab [65]) Each $x^{**} \in V^{**}$ is a fragmented map when restricted to the weak* compact ball B^* . Equivalently, $B^{**} \subset \mathcal{F}(B^*)$.
- (3) B is a Rosenthal family for the weak* compact unit ball B^* .
- (4) B is an eventually fragmented family of maps on B^* .
- (5) (Haydon [33, Theorem 3.3]) For every weak* compact subset $Y \subset V^*$ the weak* and norm closures of the convex hull $\text{co}(Y)$ in V^* coincide: $\text{cl}_{w^*}(\text{co}(Y)) = \text{cl}_{\text{norm}}(\text{co}(Y))$.

Condition (2) is a reformulation (in terms of fragmented maps) of a criterion from [65] which was originally stated in terms of the point of continuity property. The equivalence of (1), (3) and (4) follows from Theorem 5.8.

5.1.3. Banach spaces with PCP. A Banach space V is said to have the *point of continuity property* (PCP for short) if every bounded weakly closed subset $C \subset V$ admits a point of continuity of the identity map $(C, \text{weak}) \rightarrow (C, \text{norm})$ (see for example Edgar-Wheeler [13] and [36]). Every Banach space with RNP has PCP. In particular, this is true for the duals of Asplund spaces and for reflexive spaces. This concept was studied, among others, by

Bourgain and Rosenthal. They show, for instance, that there are separable Banach spaces with PCP which do not satisfy RNP.

Theorem 5.10. (*Jayne and Rogers [36]*) *Let V be a Banach space. The following conditions are equivalent:*

- (1) V has PCP.
- (2) Every bounded subset $A \subset V$ is (weak, norm)-fragmented.

Let $\{V_i\}_{i \in I}$ be a family of Banach spaces. The l_2 -sum of this family, denoted by $V := (\Sigma_{i \in I} V_i)_{l_2}$, is defined as the space of all functions $(x_i)_{i \in I}$ on I such that $x_i \in V_i$ and

$$\|x\| := \left(\sum_{i \in I} \|x_i\|^2 \right)^{\frac{1}{2}} < \infty.$$

- Lemma 5.11.**
- (1) $V^* = (\Sigma_{i \in I} V_i)^* = (\Sigma_{i \in I} V_i^*)_{l_2}$ and the pairing $V \times V^* \rightarrow \mathbb{R}$ is defined by $\langle v, f \rangle = \sum_{i \in I} f_i(v_i)$.
 - (2) If every V_i is reflexive (Asplund, Rosenthal) then V is reflexive (respectively: Asplund, Rosenthal).
 - (3) For every semitopological semigroup S the classes of reflexively (Asplund, Rosenthal) representable compact S -spaces are closed under countable products.

Proof. (1) This is well known (see, for example, [56]).

(2) The reflexive case follows easily from (1). For the Asplund case see [56] (or [16] for a simpler proof). Now suppose that each V_i is Rosenthal and $l_1 \subset V = (\Sigma_{i \in I} V_i)_{l_2}$. Since l_1 is separable one may easily reduce the question to the case of countably many Rosenthal spaces V_i . So we can suppose that $V := (\Sigma_{n \in \mathbb{N}} V_n)_{l_2}$. In view of Fact 5.9 it suffices to show that every element $u = (u_n)_{n \in \mathbb{N}}$ is a fragmented map on the weak* compact unit ball B_{V^*} . That is, we need to check that $u \in \mathcal{F}(B_{V^*})$. The set $\mathcal{F}(X) \cap l_\infty(X)$ is a Banach subspace of $l_\infty(X)$ for every topological space X . So the proof can be reduced to the case of coordinate functionals u_{n_0} . Also, $\langle u_{n_0}, (f_n)_{n \in \mathbb{N}} \rangle = f_{n_0}(u_{n_0})$. Now use the fact that u_{n_0} is a fragmented map on $B_{V_{n_0}^*}$ because V_{n_0} is Rosenthal (Fact 5.9).

(3) Similar to [51, Lemma 3.3] (or [49, Lemma 4.9]) using (2) and the l_2 -sum of representations (h_n, α_n) of (S, X_n) on V_n where $\|\alpha_n(x)\| \leq 2^{-n}$ for every $x \in X_n$ and $n \in \mathbb{N}$. \square

5.2. More properties of fragmented families. Here we demonstrate a general principle: the fragmentability of a family of continuous maps on a compact space is “countably-determined”. Formally the following theorem is new, though its proof (the part (3) \Rightarrow (1)) is inspired by a result of Namioka [55, Theorem 3.4].

Theorem 5.12. *Let $F = \{f_i : X \rightarrow Y\}_{i \in I}$ be a bounded family of **continuous** maps from a compact (not necessarily metrizable) space (X, τ) into a pseudometric space (Y, d) . The following conditions are equivalent:*

- (1) F is a fragmented family of functions on X .
- (2) Every countable subfamily K of F is fragmented.
- (3) For every countable subfamily K of F the pseudometric space $(X, \rho_{K,d})$ is separable, where

$$\rho_{K,d}(x_1, x_2) := \sup_{f \in K} d(f(x_1), f(x_2)).$$

Proof. (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (3): Let K be a countable subfamily of F . Consider the natural map

$$\pi : X \rightarrow Y^K, \pi(x)(f) := f(x).$$

By (2), K is a fragmented family. Thus by Lemma 5.2.6 the map π is (τ, μ_K) -fragmented, where μ_K is the uniformity of d -uniform convergence on $Y^K := \{f : K \rightarrow (Y, d)\}$. Then the map π is also (τ, d_K) -fragmented, where d_K is the pseudometric on Y^K defined by

$$d_K(z_1, z_2) := \sup_{f \in K} d(z_1(f), z_2(f)).$$

Since d is bounded, $d_K(z_1, z_2)$ is finite and d_K is well-defined. Denote by (X_K, τ_p) the subspace $\pi(X) \subset Y^K$ in pointwise topology. Since $K \subset C(X)$, the induced map $\pi_0 : X \rightarrow X_K$ is a continuous map onto the compact space (X_K, τ_p) . Denote by $i : (X_K, \tau_p) \rightarrow (Y^K, d_K)$ the inclusion map. So, $\pi = i \circ \pi_0$, where the map π is (τ, d_K) -fragmented. Then by Lemma 5.2.7 we obtain that i is (τ_p, d_K) -fragmented. It immediately follows that the identity map $id : (X_K, \tau_p) \rightarrow (X_K, d_K)$ is (τ_p, d_K) -fragmented.

Since K is countable, $(X_K, \tau_p) \subset Y^K$ is metrizable. Therefore, (X_K, τ_p) is second countable (being a metrizable compactum). Now, since d_K is a pseudometric on Y^K , and $id : (X_K, \tau_p) \rightarrow (X_K, d_K)$ is (τ_p, d_K) -fragmented, we can apply Lemma 5.2.5. It directly implies that the set X_K is a separable subset of (Y^K, d_K) . This means that $(X, \rho_{K,d})$ is separable.

(3) \Rightarrow (1) : Suppose that F is not fragmented. Thus, there exists a non-empty closed subset $A \subset X$ and an $\varepsilon > 0$ such that for each non-empty open subset $O \subset X$ with $O \cap A \neq \emptyset$ there is some $f \in O$ such that $f(O \cap A)$ is not ε -small in (Y, d) . Let V_1 be an arbitrary non-empty relatively open subset in A . There are $a, b \in V_1$ and $f_1 \in F$ such that $d(f_1(a), f_1(b)) > \varepsilon$. Since f_1 is continuous we can choose relatively open subsets V_2, V_3 with $\text{cl}(V_2 \cup V_3) \subset V_1$ such that $d(f(x), f(y)) > \varepsilon$ for every $(x, y) \in V_2 \times V_3$.

By induction we can construct a sequence $\{V_n\}_{n \in \mathbb{N}}$ of non-empty relatively open subsets in A and a sequence $K := \{f_n\}_{n \in \mathbb{N}}$ in F such that:

- (i) $\text{cl}(V_{2n} \cup V_{2n+1}) \subset V_n$ for each $n \in \mathbb{N}$;
- (ii) $d(f_n(x), f_n(y)) > \varepsilon$ for every $(x, y) \in V_{2n} \times V_{2n+1}$.

We claim that $(X, \rho_{K,d})$ is not separable, where

$$\rho_{K,d}(x_1, x_2) := \sup_{f \in K} d(f(x_1), f(x_2)).$$

In fact, for each *branch*

$$\alpha := V_1 \supset V_{n_1} \supset V_{n_2} \supset \dots$$

where for each $i, n_{i+1} = 2n_i$ or $2n_i + 1$, by compactness of X one may choose an element

$$x_\alpha \in \bigcap_{i \in \mathbb{N}} \text{cl}(V_{n_i}).$$

If $x = x_\alpha$ and $y = x_\beta$ come from different branches, then there is an $n \in \mathbb{N}$ such that $x \in \text{cl}(V_{2n})$ and $y \in \text{cl}(V_{2n+1})$ or (vice versa). In any case it follows from (ii) and the continuity of f_n that $d(f_n(x), f_n(y)) \geq \varepsilon$, hence $\rho_{K,d}(x, y) \geq \varepsilon$. Since there are uncountably many branches we conclude that A and hence also X are not $\rho_{K,d}$ -separable. \square

Definition 5.13. [16, 49] Let X be a compact space and $F \subset C(X)$ a norm bounded family of continuous real valued functions on X . Then F is said to be an *Asplund family for X* if for every countable subfamily K of F the pseudometric space $(X, \rho_{K,d})$ is separable, where

$$\rho_{K,d}(x_1, x_2) := \sup_{f \in K} |f(x_1) - f(x_2)|.$$

Any Asplund family for a compact space X can be viewed, by [16, Lemma 1.5.3], as a particular case of the more general concept of an *Asplund set* in the Banach space $C(X)$.

Corollary 5.14. *Let X be a compact space and $F \subset C(X)$ a norm bounded family of continuous real valued functions on X . Then F is fragmented if and only if F is an Asplund family for X .*

Theorem 5.15. *Let $F = \{f_i : X \rightarrow Y\}_{i \in I}$ be a family of continuous maps from a compact (not necessarily metrizable) space (X, τ) into a pseudouniform space (Y, μ) . Then F is fragmented if and only if every countable subfamily $A \subset F$ is fragmented.*

Proof. The proof can be reduced to Theorem 5.12. Every pseudouniform space can be uniformly approximated by pseudometric spaces. Using Lemma 5.2.1 we can suppose that (Y, μ) is pseudometrizable; i.e. there exists a pseudometric d such that $\text{unif}(d) = \mu$. Moreover, replacing d by the uniformly equivalent metric $\frac{d}{1+d}$ we can suppose that $d \leq 1$. \square

5.3. The natural affine extension map $T : b\mathcal{B}_1(X) \rightarrow b\mathcal{B}_1(B^*)$.

Definition 5.16. Let X be a topological space.

- (1) A function $f : X \rightarrow \mathbb{R}$ is said to be Baire 1 if $f^{-1}(O)$ is an F_σ in X for every open $O \subset \mathbb{R}$. Notation: $f \in \mathcal{B}_1(X)$.
- (2) Denote by $\mathcal{B}_1^l(X)$ the set of all pointwise limits of sequences of continuous functions on X .
- (3) Bounded functions in $\mathcal{B}_1(X)$ and $\mathcal{B}_1^l(X)$ are denoted by $b\mathcal{B}_1(X)$ and $b\mathcal{B}_1^l(X)$.

Always, $\mathcal{B}_1^l(X) \subset \mathcal{B}_1(X)$ (van Dulst p. 137 for every X) and $\mathcal{B}_1(X) \subset \mathcal{F}(X)$, for every hereditarily Baire space [8, Lemma 1C(c)].

Below X be a compact space. It naturally is embedded into $(C(X)^*, w^*)$. This embedding induces a natural injective map

$$T : b\mathcal{B}_1^l(X) \rightarrow b\mathcal{B}_1^l(B^*),$$

where B^* , as before, is the weak* compact unit ball of $C(X)^*$. In the definition of T below we will use Riesz representation theorem and Lebesgue's Dominated Convergence Theorem.

For compact X we have $\mathcal{F}(X) = B_r^\S$ in terms of [8]. Each $f \in \mathcal{F}(X)$ is universally measurable for every compact space X (see for example [8, Prop. 1F]). Therefore, for every measure $\mu \in B^*$ we can define

$$(Tf)(\mu) := \int f d\mu.$$

This map is well defined. Indeed, first note that when $f \in C(X)$, $T(f) = i(f)$, where

$$i : C(X) \hookrightarrow C(B^*), \quad i(f)(\mu) := \langle f, \mu \rangle = \int f d\mu$$

is the canonical isometric inclusion of the corresponding Banach spaces and

$$\langle \cdot, \cdot \rangle : C(X) \times C(X)^* \rightarrow \mathbb{R}$$

is the canonical bilinear mapping. Now if $f \in b\mathcal{B}_1^l(X)$ then by definition f is a pointwise limit of a sequence of continuous functions $h_n \in C(X)$. Since $f : X \rightarrow \mathbb{R}$ is a bounded function we can assume in addition that the sequence h_n is uniformly bounded. By Lebesgue's Convergence Theorem it follows that $T(f)$ is a pointwise limit of the sequence $T(h_n) = i(h_n)$, $n \in \mathbb{N}$. Since every $i(h_n) \in C(B^*)$ we conclude (by Definition 5.16) that $T(f) \in b\mathcal{B}_1^l(B^*)$. The sequence $i(h_n)$ is uniformly bounded in $C(B^*)$ hence $T(f)$ is a bounded function. This means that $T(f) \in b\mathcal{B}_1^l(B^*)$.

The map T is injective because $T(f)(\delta_x) = f(x)$ for every point mass $\delta_x \in B^*$ ($x \in X$).

Remark 5.17. Each $T(f)$ for $f \in b\mathcal{B}_1^l(X)$ can be treated as an element of the second dual $C(X)^{**}$ of $C(X)$. Moreover the pointwise topology of $\mathcal{B}_1^l(B^*)$ and the weak*-topology on $C(X)^{**}$ agree on $T(b\mathcal{B}_1^l(X))$.

Lemma 5.18. *Let X be a compact space. For every uniformly bounded subset $A \subset b\mathcal{B}_1^l(X)$ the restriction $T|_A$ of the natural injective map*

$$T : b\mathcal{B}_1^l(X) \rightarrow b\mathcal{B}_1^l(B^*) \cap C(X)^{**}$$

on A is sequentially continuous. Furthermore, $T(A)$ is also uniformly bounded.

Proof. Lebesgue's Convergence Theorem implies that T is sequentially continuous. The boundedness of $T(A)$ is easy. \square

Namioka [55] gave a kind of duality between uniform separability and pointwise metrizability.

Theorem 5.19. (Namioka [55, Theorem 4.1]) *Let $F \subset \mathbb{R}^X$ be a bounded set of maps. The following are equivalent:*

- (1) *The pseudometric space (X, ρ_F) is separable, where*

$$\rho_F(x_1, x_2) := \sup\{|f(x_1) - f(x_2)| : f \in F\}.$$

- (2) *The pointwise closure $\text{cl}_p(F)$ is a metrizable subspace of \mathbb{R}^X .*

Proof. (1) \Rightarrow (2) F is an equicontinuous family on the pseudometric space (X, ρ_F) . Then its pointwise closure $\text{cl}_p(F)$ is also equicontinuous. So, it follows that the topology of pointwise convergence on X for $\text{cl}_p(F)$ is the same as the topology of pointwise convergence on a countable ρ_F -dense subset of X . Hence, $\text{cl}_p(F)$ is metrizable.

(2) \Rightarrow (1) Let $K := \text{cl}_p(F)$ be pointwise metrizable. Then $C(K)$ is norm separable. Denote by $\varphi : X \rightarrow C(K)$ the induced map defined by $\varphi(x)(f) := f(x)$ for every $f \in C(K)$. Then $\varphi(X) \subset C(K)$ is norm separable. In particular (since $F \subset K$) we obtain that (X, ρ_F) is separable. \square

Lemma 5.20. (Namioka [55, Theorem 4.4]) *Let $F \subset C(K)$ be a bounded family of continuous functions on a compact space K . The following are equivalent:*

- (1) *F is a fragmented family of functions on K .*
(2) *The pseudometric space (K, ρ_A) is separable for every countable $A \subset F$.*
(3) *The pointwise closure $\text{cl}(A)$ is a metrizable subspace of \mathbb{R}^K for every countable $A \subset F$.*

Proof. Combine Theorem 5.20 and Corollary 5.14. \square

Proposition 5.21. *If $F \subset C(X)$ is a countable bounded fragmented family on a compact space X then*

- (1) *$\text{cl}_p(F)$ is a metrizable subset of $b\mathcal{B}_1^l(X)$.*
(2) *the restriction of T on $\text{cl}_p(F)$ induces a homeomorphism*

$$\text{cl}_p(F) \rightarrow \text{cl}_p(T(F)) \subset b\mathcal{B}_1^l(B^*) \cap C(X)^{**}.$$

Proof. (1) $\text{cl}_p(F)$ is a uniformly bounded subset of \mathbb{R}^X because F is bounded. Since F is fragmented we obtain that (X, ρ_F) is separable. So, by Lemma 5.20, $\text{cl}_p(F)$ is metrizable in \mathbb{R}^X . Therefore, every $\phi \in \text{cl}_p(F)$ is a pointwise closure of a subsequence in F . Hence, $\phi \in b\mathcal{B}_1^l(X)$.

(2) In view of Lemma 5.18 the restricted (injective!) map $T : \text{cl}_p(F) \rightarrow b\mathcal{B}_1^l(B^*)$ is sequentially continuous. This restriction is even continuous because $\text{cl}_p(F)$ is metrizable by (1). We conclude that the map $T : \text{cl}_p(F) \rightarrow b\mathcal{B}_1^l(B^*)$ is a continuous injection, and therefore a homeomorphism, of $\text{cl}_p(F)$ onto its image in $b\mathcal{B}_1^l(B^*)$. \square

Proposition 5.22. *Let X be a compact space and $F \subset C(X)$ be bounded family. The following conditions are equivalent:*

- (1) F is a (eventually) fragmented family for X iff F_{B^*} is a (resp., eventually) fragmented family for B^* .
- (2) F is a Rosenthal family for X iff F_{B^*} is a Rosenthal family for B^* .

Proof. (1) Let A be a countable subfamily of F . Then since A is fragmented, Lemma 5.20 implies that $\text{cl}(A_X) \subset b\mathcal{B}_1^l(X)$ is metrizable. By Proposition 5.21.2 $\text{cl}(A_{B^*})$ is homeomorphic to $\text{cl}(A_X)$. Therefore, $\text{cl}(A_{B^*}) \subset b\mathcal{B}_1^l(B^*)$ is metrizable, too. Now again by Lemma 5.20 we obtain that A_{B^*} is fragmented. It is true for every countable subfamily of F_{B^*} . Thus, F_{B^*} is fragmented. This proves the "fragmented case". The "eventually fragmented case" is verbatim the same.

(2) is a reformulation of the eventually fragmented case of (1). \square

Of course if F_{B^*} is fragmented then $F_{P(X)}$ is fragmented, too.

Corollary 5.23. $C(K)$ is Asplund iff the compact space K is scattered.

Proof. Let K is scattered. Then it is fragmented by any uniformity, in particular with respect to the norm of $C(K)^*$. Then Proposition 5.22 guarantees that B^* is also fragmented by the norm. Therefore, $C(K)$ is Asplund.

Second direction comes from the following Exercise. \square

Exercise 5.24. Let K be a compact space which is norm-fragmented in $C(K)^*$. Show that K is scattered.

Hint: the norm uniformity on $X \subset C(K)^*$ is discrete.

Theorem 5.25. Let $F \subset V$ be a norm bounded subset and $K \subset V^*$ be a weak-star compact subset. Then F is a fragmented family on K iff F is a fragmented family on $Q := \overline{\text{co}}(K)$.

Proof. Consider the restriction operator

$$r_K : V \rightarrow C(K), \quad r_K(v)(x) = \langle v, x \rangle \quad \forall x \in K$$

$F_K := r_K(F)$ is a fragmented family on K . Then by Proposition 5.22, F_Q is also fragmented on $P(K) \subset B^*$. Now consider the adjoint $r_K^* : C(K)^* \rightarrow V^*$. Then $r_K^*(P(K)) = Q$. \square

Corollary 5.26. (Namioka [55, Thm 2.5]) Let $K \subset V^*$ be a weak-star compact subset which is norm fragmented. Then $\overline{\text{co}}(K)$ is also norm fragmented.

Proof. Take $F := B_V$. \square

Lemma 5.27. (Fabian's Lemma [16, Lemma 1.5.3]) Let K be a compact space, let $F \subset B_{C(K)}$ be a nonempty set, and consider on $C(K)^*$ the pseudometric ρ_F defined as

$$\rho_F(\lambda, \mu) = \sup_{f \in F} \langle \lambda - \mu, f \rangle \quad \lambda, \mu \in C(K)^*.$$

Assume that (K, ρ_F) (as a subset of $C(K)^*$) is separable. Then $(C(K)^*, \rho_F)$ is also separable.

Proof. By Theorem 5.19, $cl_p(F) \subset b\mathcal{B}_1^l(X)$ is metrizable. Since $T : cl_p(F) \subset b\mathcal{B}_1^l(X) \rightarrow b\mathcal{B}^l(B^*)$ is sequentially continuous we obtain that this map is a homeomorphic embedding. Hence, $T(cl_p(F)) \subset b\mathcal{B}^l(B^*)$ is also metrizable. Again by Theorem 5.19 we obtain that (B^*, ρ_F) is separable. This implies that $(C(K)^*, \rho_F)$ is also separable. \square

Lemma 5.28. Let F be a fragmented family of real valued functions on (X, μ) . Then $\text{co}(F)$ is also fragmented.

Proof. If $f_i(D)$ is ε -small for every $i = 1, \dots, n$ and $\sum_{i=1}^n c_i = 1$, $c_i > 0$ then $\sum_{i=1}^n c_i f_i(D)$ is ε -small. \square

6. SOME CLASSES OF RIGHT TOPOLOGICAL SEMIGROUPS AND DYNAMICAL SYSTEMS

To the basic classes of right topological semigroups listed in 2.21 above, we add the following two which have naturally arisen in the study of tame and HNS dynamical systems.

Definition 6.1. [22, 24] A compact admissible right topological semigroup P is said to be:

- (1) [24] *tame* if the left translation $\lambda_a : P \rightarrow P$ is a fragmented map for every $a \in P$.
- (2) *HNS-semigroup* if $\{\lambda_a : P \rightarrow P\}_{a \in P}$ is a fragmented family of maps.

These classes are closed under factors. We have the inclusions:

$$\{\text{compact semitopological semigroups}\} \subset \{\text{HNS-semigroups}\} \subset \{\text{Tame semigroups}\}$$

Lemma 6.2.

- (1) *Every compact semitopological semigroup P is a HNS-semigroup.*
- (2) *Every HNS-semigroup is tame.*
- (3) *If P is a metrizable compact right topological admissible semigroup then P is a HNS-semigroup.*

Proof. (1) Apply Lemma 5.3.1 to $P \times P \rightarrow P$.

(2) is trivial.

(3) Apply Lemma 5.3.2 to $P \times P \rightarrow P$. □

If P is Fréchet, as a topological space, then P is a tame semigroup by Corollary 8.7 below.

7. HNS-SEMIGROUPS, DYNAMICAL SYSTEMS AND ASPLUND BANACH SPACES

Definition 7.1. We say that a compact S -system X is *hereditarily non-sensitive* (HNS, in short) if one of the following equivalent conditions are satisfied:

- (1) For every closed nonempty subset $A \subset X$ and for every entourage ε from the unique compatible uniformity on X there exists an open subset O of X such that $A \cap O$ is nonempty and $s(A \cap O)$ is ε -small for every $s \in S$.
- (2) The family of translations $\tilde{S} := \{\tilde{s} : X \rightarrow X\}_{s \in S}$ is a fragmented family of maps.
- (3) $E(S, X)$ is a fragmented family of maps from X into itself.

The equivalence of (1) and (2) is evident from the definitions. Clearly, (3) implies (2). As to the implication (2) \Rightarrow (3), observe that the pointwise closure of a fragmented family is again a fragmented family, [24, Lemma 2.8].

Note that if $S = G$ is a group then in Definition 7.1.1 one may consider only closed subsets A which are G -invariant (see the proof of [22, Lemma 9.4]).

Lemma 7.2.

- (1) *For every S the class of HNS compact S -systems is closed under subsystems, arbitrary products and factors.*
- (2) *For every HNS compact S -system X the corresponding enveloping semigroup $E(X)$ is HNS both as an S -system and as a semigroup.*
- (3) *Let P be a HNS-semigroup. Assume that $j : S \rightarrow P$ be a continuous homomorphism from a semitopological semigroup S into P such that $j(S) \subset \Lambda(P)$. Then the S -system P is HNS.*
- (4) $\{\text{HNS-semigroups}\} = \{\text{enveloping semigroups of HNS systems}\}$.

Theorem 7.3. *Let V be a Banach space. The following are equivalent:*

- (1) *V is an Asplund Banach space.*
- (2) (Θ^{op}, B^*) *is a HNS system.*
- (3) \mathcal{E} *is a HNS-semigroup.*

Proof. (1) \Rightarrow (2): Use Definition 7.1.2 and the following well known characterization of Asplund spaces: V is Asplund iff B^* is (w^*, norm) -fragmented (Fact 5.5).

(2) \Rightarrow (1) By Fact 5.5 we have to show that B is a fragmented family for B^* . Choose a vector $v \in S_V$. Since Θ^{op} is a fragmented family of self-maps on B^* and as $v : B^* \rightarrow \mathbb{R}$ is uniformly continuous we get that the system $v\Theta^{op} = \Theta v$ of maps from B^* to \mathbb{R} is also fragmented. Now recall that $\Theta v = B$ by Lemma 3.8.1.

(2) \Rightarrow (3): Follows from Lemma 7.2.2 and the fact that \mathcal{E} is the enveloping semigroup $E(\Theta^{op}, B^*)$.

(3) \Rightarrow (2): $\Lambda(\mathcal{E}) = \Theta^{op}$ (Lemma 3.8.5) and \mathcal{E} is a HNS-semigroup. So, (S, \mathcal{E}) is HNS by Lemma 7.2.3 with $S = \Theta^{op}$. Take $\psi \in B^*$ with $\|\psi\| = 1$. The map $q : \mathcal{E} \rightarrow B^*$, $p \mapsto p\psi$ defines a continuous homomorphism of Θ^{op} -systems. By Lemma 3.8.4, we have $\mathcal{E}\psi = B^*$. So q is onto. Now observe that the HNS property is preserved by factors of S -systems (Lemma 7.2.1). \square

Our next two theorems are based on ideas from Glasner-Megrelishvili-Uspenskij [29].

Theorem 7.4. *Let V be a Banach space. The following are equivalent:*

- (1) V is a separable Asplund space.
- (2) \mathcal{E} is homeomorphic to the Hilbert cube $[-1, 1]^{\mathbb{N}}$ (for infinite-dimensional V).
- (3) \mathcal{E} is metrizable.

Proof. (1) \Rightarrow (2) Since \mathcal{E} is a compact affine subset in the Fréchet space $\mathbb{R}^{\mathbb{N}}$ we can use Keller's Theorem [9, p. 100].

(2) \Rightarrow (3) Is trivial.

(3) \Rightarrow (1) \mathcal{E} is a HNS-semigroup by Lemma 6.2.3. Now Theorem 7.3 implies that V is Asplund. It is also separable; indeed, by Lemma 3.8.4, B^* is a continuous image of \mathcal{E} , so that B^* is also w^* -metrizable, which in turn yields the separability of V . \square

Theorem 7.5. *Let X be a compact S -system. Consider the following assertions:*

- (a) $E(X)$ is metrizable.
- (b) (S, X) is HNS.

Then:

- (1) (a) \Rightarrow (b).
- (2) If X , in addition, is metrizable then (a) \Leftrightarrow (b).

Proof. (1) By Definition 7.1 we have to show that $E(X)$ is a fragmented family of maps from X into itself. The unique compatible uniformity on the compactum X is the weakest uniformity on X generated by $C(X)$. Using Remark 5.2.1 one may reduce the proof to the verification of the following claim: $E^f := \{f \circ p : p \in E(X)\}$ is a fragmented family for every $f \in C(X)$. In order to prove this claim apply Lemma 5.3.2 to the induced mapping $E(X) \times X \rightarrow \mathbb{R}$, $(p, x) \mapsto f(px)$ (using our assumption that $E(X)$ is metrizable).

(2) If X is a metrizable HNS S -system then by Theorem 9.3 below, (S, X) is representable on a separable Asplund space V . We can assume that X is S -embedded into B^* . The enveloping semigroup $E(S, B^*)$ is embedded into \mathcal{E} . The latter is metrizable by virtue of Theorem 7.4. Hence $E(S, X)$ is also metrizable, being a continuous image of $E(S, B^*)$. \square

Theorem 7.6. *For a compact metric S -space X the following conditions are equivalent:*

- (1) the dynamical system (S, X) is RN (that is, Asplund representable);
- (2) (S, X) is HNS;
- (3) the enveloping semigroup $E(S, X)$ is metrizable.

Theorem 7.7. *For a discrete monoid S and a finite alphabet A let $X \subset A^S$ be a subshift. The following conditions are equivalent:*

- (1) (S, X) is Asplund representable (that is, RN).
- (2) (S, X) is HNS.
- (3) X is scattered.

If, in addition, X is metrizable (e.g., if S is countable) then each of the conditions above is equivalent also to:

- (4) X is countable.

Proof. (1) \Rightarrow (2): Follows directly from Theorem 9.3.2.

(2) \Rightarrow (3): Let μ be the natural uniformity on X and μ_S the (finer) uniformity of uniform convergence on $X \subset X^S$ (we can treat X as a subset of X^S under the assignment $x \mapsto \hat{x}$, where $\hat{x}(s) = sx$). If X is HNS then the family \tilde{S} is fragmented. This means that X is μ_S -fragmented. As we already mentioned, every subshift X is uniformly S -expansive. Therefore, μ_S coincides with the discrete uniformity μ_Δ on X (the largest possible uniformity on the set X). Hence, X is also μ_Δ -fragmented. This means, by Lemma 5.2.4, that X is a scattered compactum.

(3) \Rightarrow (1): If X is a scattered compactum then the Banach space $C(X)$ is Asplund by [56]. We have the canonical faithful representation of (S, X) on $C(X)$. Hence, (S, X) is RN.

If X is metrizable then

- (4) \Leftrightarrow (3): A scattered compactum is metrizable iff it is countable. □

Theorem 7.8. *Every scattered (e.g., countable) compact S -space X is HNS*

Proof. Recall that $C(X)$ is Asplund if (and only if) the compactum X is scattered. □

Theorem 7.9. *For a discrete monoid S and a finite alphabet A let $X \subset A^S$ be a subshift. The following conditions are equivalent:*

- (1) (S, X) is Asplund representable (that is, RN).
- (2) (S, X) is HNS.
- (3) X is scattered.

If, in addition, X is metrizable (e.g., if S is countable) then each of the conditions above is equivalent also to:

- (4) X is countable.

Proof. (1) \Rightarrow (2): Follows directly from Theorem 9.3.2.

(2) \Rightarrow (3): Let μ be the natural uniformity on X and μ_S the (finer) uniformity of uniform convergence on $X \subset X^S$ (we can treat X as a subset of X^S under the assignment $x \mapsto \hat{x}$, where $\hat{x}(s) = sx$). If X is HNS then the family \tilde{S} is fragmented. This means that X is μ_S -fragmented. As we already mentioned, every subshift X is uniformly S -expansive. Therefore, μ_S coincides with the discrete uniformity μ_Δ on X (the largest possible uniformity on the set X). Hence, X is also μ_Δ -fragmented. This means, by Lemma 5.2.4, that X is a scattered compactum.

(3) \Rightarrow (1): If X is a scattered compactum then the Banach space $C(X)$ is Asplund by [56]. We have the canonical faithful representation of (S, X) on $C(X)$. Hence, (S, X) is RN.

If X is metrizable then

- (4) \Leftrightarrow (3): A scattered compactum is metrizable iff it is countable. □

8. TAME SEMIGROUPS, TAME SYSTEMS AND ROSENTHAL SPACES

Definition 8.1. A compact separately continuous S -system X is said to be *tame* if the translation $\lambda_a : X \rightarrow X$, $x \mapsto ax$ is a fragmented map for every element $a \in E(X)$ of the enveloping semigroup.

Lemma 8.2. *Every WAP system is HNS and every HNS is tame.*

Proof. If (S, X) is WAP then $E(X) \times X \rightarrow X$ is separately continuous. By Lemma 5.3.1 we obtain that E is a fragmented family of maps from X to X . In particular, its subfamily $\{\tilde{s} : X \rightarrow X\}_{s \in S}$ of all translations is fragmented. Hence, (S, X) is HNS.

Directly from the definitions we conclude that every HNS is tame. \square

Another proof of Lemma 8.2 comes also from Banach representations theory for dynamical systems because every reflexive space is Asplund and every Asplund is Rosenthal.

By [28], a compact metrizable S -system X is tame iff S is *eventually fragmented* on X , that is, for every infinite (countable) subset $C \subset G$ there exists an infinite subset $K \subset C$ such that K is a fragmented family of maps $X \rightarrow X$.

Lemma 8.3.

- (1) *For every S the class of tame S -systems is closed under closed subsystems, arbitrary products and factors.*
- (2) *For every tame compact S -system X the corresponding enveloping semigroup $E(X)$ is tame both as an S -system and as a semigroup.*
- (3) *Let P be a tame right topological compact semigroup and let $\nu : S \rightarrow P$ be a continuous homomorphism from a semitopological semigroup S into P such that $\nu(S) \subset \Lambda(P)$. Then the S -system P is tame.*
- (4) $\{\text{tame semigroups}\} = \{\text{enveloping semigroups of tame systems}\}.$

Theorem 8.4. *Let V be a Banach space. The following are equivalent:*

- (1) *V is a Rosenthal Banach space.*
- (2) *(Θ^{op}, B^*) is a tame system.*
- (3) *$p : B^* \rightarrow B^*$ is a fragmented map for each $p \in \mathcal{E}$.*
- (4) *\mathcal{E} is a tame semigroup.*

Proof. (2) \Leftrightarrow (3): Follows from the definition of tame flows because $\mathcal{E} = E(\Theta^{op}, B^*)$.

(2) \Rightarrow (4): Since $\mathcal{E} = E(\Theta^{op}, B^*)$, Lemma 8.3.2 applies.

(4) \Rightarrow (2): By our assumption, \mathcal{E} is a tame semigroup. Then by Lemma 8.3.3 the system $(\Theta^{op}, \mathcal{E})$ is tame. Its factor (Lemma 3.8.4) (Θ^{op}, B^*) is tame, too.

(2) \Rightarrow (1): By a characterization of Rosenthal spaces [24, Prop. 4.12] (see also Fact 5.9) it suffices to show that $B^{**} \subset \mathcal{F}(B^*)$. Since (Θ^{op}, B^*) is tame, $p : B^* \rightarrow B^*$ is fragmented for every $p \in E(\Theta^{op}, B^*) = \mathcal{E}$. Pick an arbitrary $v \in B_V$ with $\|v\| = 1$. Then $v\mathcal{E}$ is exactly B^{**} by Lemma 3.8.2. So every $\phi \in B^{**}$ is a composition $v \circ p$, where p is a fragmented map. Since $v : B^* \rightarrow \mathbb{R}$ is weak* continuous we conclude that $\phi : B^* \rightarrow B^*$ is fragmented.

(1) \Rightarrow (3): We have to show that $\mathcal{E} \subset \mathcal{F}(B^*, B^*)$ for every Rosenthal space V . Let $p \in \mathcal{E}$. Then $p \in \Theta(V^*)$. That is, p is a linear map $p : V^* \rightarrow V^*$ with norm ≤ 1 . Then, for every vector $f \in V$, the composition $f \circ p : V^* \rightarrow \mathbb{R}$ is a linear bounded functional on V^* . That is, $f \circ p \in V^{**}$ belongs to the second dual. Again, by the above mentioned characterization of Rosenthal spaces, the corresponding restriction $f \circ p|_{B^*} : B^* \rightarrow \mathbb{R}$ is a fragmented function for every $f \in V$. Since V separates points of B^* we can apply [24, Lemma 2.3.3]. It follows that $p : B^* \rightarrow B^*$ is fragmented for every $p \in \mathcal{E}$. \square

8.1. Some concrete examples of systems and families.

Example 8.5.

- (1) Let $X = [0, 1]$ be the unit interval. Consider the cascade (\mathbb{Z}, X) generated by the homeomorphism $\sigma(x) = x^2$. Then (\mathbb{Z}, X) , as a dynamical system, is RN and not Eberlein. To see this observe that the pair of sequences $x_n = 1 - \frac{1}{n}$ in $X = [0, 1]$ and $\sigma^m \in G$ with $\sigma^m(x) = x^{2^m}$ does not satisfy DLP. The corresponding limits are 0 and

1. This means that $(\mathbb{Z}, [0, 1])$ is not Eberlein. The enveloping semigroup $E(\mathbb{Z}, [0, 1])$ is metrizable being homeomorphic to the two-point compactification of \mathbb{Z} . Hence, by [29], $(\mathbb{Z}, [0, 1])$ is RN. The sequence $\{\sigma^m : [0, 1] \rightarrow [0, 1]\}_{m \in \mathbb{N}}$ is a fragmented family which does not satisfy DLP.
- (2) The Sturmian symbolic dynamical system (X, σ) is WRN but not RN. The sequence $\{\sigma^n : X \rightarrow X\}_{n \in \mathbb{N}}$ is an eventually fragmented but not fragmented family.
- (3) The natural action of the Polish group $H_+[0, 1]$ of all increasing homeomorphisms of $[0, 1]$ on $[0, 1]$ is tame but not HNS. The family $H_+[0, 1]$ of functions (or its dense subsequence) is eventually fragmented but not fragmented.
- (4) The Bernoulli shift system $(\mathbb{Z}, \{0, 1\}^{\mathbb{Z}})$ is not WRN (equivalently, nontame). In fact, it is well known that the enveloping semigroup of this system can be identified with $\beta\mathbb{Z}$. Now use the dynamical version of BFT dichotomy (Fact 8.9).

Another way to see that the shift system is not tame is the well known fact that the sequence of projections

$$\{\pi_m : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}\}_{m \in \mathbb{N}}$$

is independent. Hence by Theorem 5.8 this family fails to be eventually fragmented.

- (5) Another example of an independent sequence is the sequence of Rademacher functions

$$r_n : [0, 1] \rightarrow \mathbb{R}, \quad r_n(x) := \text{sgn}(\sin(2^n \pi x)).$$

8.2. A dynamical BFT dichotomy. Recall that a topological space K is a *Rosenthal compactum* [32] if it is homeomorphic to a pointwise compact subset of the space $\mathcal{B}_1(X)$ of functions of the first Baire class on a Polish space X . All metric compact spaces are Rosenthal. An example of a separable non-metrizable Rosenthal compactum is the *Helly compact* of all nondecreasing selfmaps of $[0, 1]$ in the pointwise topology. Recall that a topological space K is *Fréchet* (or, *Fréchet-Urysohn*) if for every $A \subset K$ and every $x \in \text{cl}(A)$ there exists a sequence of elements of A which converges to x . Every Rosenthal compact space K is Fréchet by a result of Bourgain-Fremlin-Talagrand [8, Theorem 3F], generalizing a result of Rosenthal.

Theorem 8.6. *If the enveloping semigroup $E(X)$ is a Fréchet (e.g., Rosenthal) space, as a topological space, then (S, X) is a tame system (and $E(X)$ is a tame semigroup).*

Proof. Let $p \in E(X)$. We have to show that $p : X \rightarrow X$ is fragmented. By properties of fragmented maps [24, Lemma 2.3.3] it is enough to show that $f \circ p : X \rightarrow \mathbb{R}$ is fragmented for every $f \in C(X)$. By the Fréchet property of $E(X)$ we may choose a sequence s_n in S such that the sequence $j(s_n)$ converges to p in $E(X)$. Hence the sequence of continuous functions $f \circ s_n = f \circ j(s_n)$ converges pointwise to $f \circ p$ in \mathbb{R}^X . Apply Lemma 5.3.2 to the evaluation map $F \times X \rightarrow \mathbb{R}$, where $F := \{f \circ p\} \cup \{f \circ j(s_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^X$ carries the pointwise topology. We conclude that F is a fragmented family. In particular, $f \circ p$ is a fragmented map. ($E(X)$ is a tame semigroup by Lemma 8.3.2.) \square

Corollary 8.7. *Let P be a compact right topological admissible semigroup. If P is Fréchet (e.g., when it is Rosenthal), as a topological space, then P is a tame semigroup.*

Proof. Applying Theorem 8.6 to the system (S, P) , with $S := \Lambda(P)$ we obtain that $E(S, P) = P$ is a tame semigroup. \square

The following theorem is due to Bourgain-Fremlin-Talagrand [8, Theorem 3F], generalizing a result of Rosenthal. The second assertion (BFT dichotomy) is presented as in the book of Todorčević [70] (see Proposition 1 of Section 13).

Theorem 8.8. (1) *Every Rosenthal compact space K is Fréchet.*

- (2) (BFT dichotomy) Let X be a Polish space and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of continuous real valued functions on X which is bounded. Then, either the sequence $\{f_n\}_{n \in \mathbb{N}}$ contains a pointwise convergent subsequence, or it contains a subsequence whose closure in \mathbb{R}^X is homeomorphic to $\beta\mathbb{N}$, the Stone-Ćech compactification of \mathbb{N} .

The following result was proved in [22, Theorem 3.2] using the Bourgain-Fremlin-Talagrand (BFT) dichotomy in the setting of continuous group actions. The same arguments work also for separately continuous semigroup actions. For the sake of completeness we include a simplified proof.

Fact 8.9 (A dynamical BFT dichotomy). [22, Theorem 3.2] *Let X be a compact metric dynamical S -system and let $E = E(X)$ be its enveloping semigroup. We have the following alternative. Either*

- (1) E is a separable Rosenthal compact, hence $\text{card } E \leq 2^{\aleph_0}$; or
- (2) the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence $\text{card } E = 2^{2^{\aleph_0}}$.

The first possibility holds iff X is a tame S -system.

Proof. For every $f \in C(X)$ define $E^f := \{f \circ p : p \in E\}$. Then E^f is a pointwise compact subset of \mathbb{R}^X , being a continuous image of E under the map $q_f : E \rightarrow E^f$, $p \mapsto f \circ p$. Since X is metrizable by the separability of E there exists a sequence $\{s_m\}_{m=1}^\infty$ in S such that $\{j(s_m)\}_{m=1}^\infty$ is dense in $E(X)$. In particular, the sequence of real valued functions $\{f \circ s_m\}_{m=1}^\infty$ is pointwise dense in E^f .

Choose a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $C(X)$ which separates the points of X . For every pair s, t of distinct elements of E there exist a point $x_0 \in X$ and a function f_{n_0} such that $f_{n_0}(sx_0) \neq f_{n_0}(tx_0)$. It follows that the continuous diagonal map

$$\Phi : E \rightarrow \prod_{n \in \mathbb{N}} E^{f_n}, \quad p \mapsto (f_1 \circ p, f_2 \circ p, \dots)$$

separates the points of E and hence is a topological embedding. Now if for each n the space E^{f_n} is a Rosenthal compactum then so is $E \cong \Phi(E) \subset \prod_{n=1}^\infty E^{f_n}$, because the class of Rosenthal compacta is closed under countable products and closed subspaces. On the other hand if at least one $E^{f_n} = \text{cl}_p(\{f_n \circ s_m\}_{m=1}^\infty)$ is not Rosenthal then, by BFT-dichotomy it contains a homeomorphic copy of $\beta\mathbb{N}$ and it is easy to see that so does its preimage E . In fact if $\beta\mathbb{N} \cong Z \subset E^{f_n}$ then any closed subset Y of E which projects onto Z and is minimal with respect to these properties is also homeomorphic to $\beta\mathbb{N}$.

Now we show the last assertion. If X is tame then every $p \in E(X)$ is a fragmented self-map of X . Hence every $f \circ p \in E^f$ is fragmented. By Remark 5.2.2 this is equivalent to saying that every $f \circ p$ is Baire 1. So $E^f \subset \mathcal{B}_1(X)$ is a Rosenthal compactum. Therefore, $E \cong \Phi(E) \subset \prod_{n \in \mathbb{N}} E^{f_n}$ is also Rosenthal. Conversely, if E is a Rosenthal compactum then (S, X) is tame by Theorem 8.6. \square

Theorem 8.10 (BFT dichotomy for Banach spaces). *Let V be a separable Banach space and let $\mathcal{E} = \mathcal{E}(V)$ be its (separable) enveloping semigroup. We have the following alternative. Either*

- (1) \mathcal{E} is a Rosenthal compactum, hence $\text{card } \mathcal{E} \leq 2^{\aleph_0}$; or
- (2) the compact space \mathcal{E} contains a homeomorphic copy of $\beta\mathbb{N}$, hence $\text{card } \mathcal{E} = 2^{2^{\aleph_0}}$.

The first possibility holds iff V is a Rosenthal Banach space.

Proof. Recall that $\mathcal{E} = E(\Theta^{op}, B^*)$. By Theorem 8.4, V is Rosenthal iff (Θ^{op}, B^*) is tame. Since V is separable, B^* is metrizable. So we can apply Fact 8.9. \square

8.3. Some topological corollaries. Answering a question of Talagrand [68, Problem14-2-41], R. Pol [58] gave an example of a separable compact Rosenthal space K which cannot be embedded in $\mathcal{B}_1(X)$ for any compact metrizable X . In [24] we say that a compact space K is *strongly Rosenthal* if it is homeomorphic to a subspace of $\mathcal{B}_1(X)$ for a compact metrizable X ; and that it is *admissible* if there exists a metrizable compact space X and a bounded subset $Z \subset C(X)$ with $K \subset \text{cl}_p(Z)$, such that the pointwise closure $\text{cl}_p(Z)$ of Z in \mathbb{R}^X consists of Baire 1 functions. Clearly every admissible compactum is strongly Rosenthal.

Theorem 8.11. [24] *Let X be a compact metrizable S -system. Then (S, X) is tame iff the compactum $K := E(X)$ is Rosenthal iff $E(X)$ is admissible.*

Thus, Pol's separable compactum mentioned above cannot be of the form $E(X)$. We do not know if every separable strongly Rosenthal space is admissible. If the answer to this question is in the negative, then this will yield another topological obstruction on being an enveloping semigroup.

Finally, as a consequence of the representation theorem 9.3.1 below we obtain the following result: *A compact space K is an admissible Rosenthal compactum iff it is homeomorphic to a weak* closed bounded subset in the second dual of a separable Rosenthal Banach space V .*

Essentially the same result (using different terminology and setting) was obtained earlier by Marciszewski (see [43, Section 6.2] and [44, Theorem 8.2]).

8.4. Some classes of functions.

Definition 8.12. Let $f \in C(X)$ on a (not necessarily, compact) S -system X .

- (1) We say that f comes from the S -compactification $q : X \rightarrow Y$ (where the action of S on Y is at least separately continuous) if there exists a continuous function $f' : Y \rightarrow \mathbb{R}$ such that $f = f' \circ q$.
- (2) We say that $f \in C(X)$ is *RMC* (right multiplicatively continuous) if f comes from some S -compactification $q : X \rightarrow Y$. For every compact S -system X we have $\text{RMC}(X) = C(X)$.
- (3) If we consider only jointly continuous S -actions on Y then the functions $f : X \rightarrow \mathbb{R}$ which come from such G -compactifications $q : X \rightarrow Y$ are *right uniformly continuous*. Notation: $f \in \text{RUC}(X)$.
- (4) f is said to be: a) *WAP*; b) *Asplund*; c) *tame* if f comes from an S -compactification $q : X \rightarrow Y$ such that (S, Y) is: WAP, HNS or tame respectively. For the corresponding classes of functions we use the notation: $\text{WAP}(X)$, $\text{Asp}(X)$, $\text{Tame}(X)$, respectively. Each of these is a norm closed S -invariant subalgebra of the S -algebra $\text{RMC}(X) \subset C(X)$ and

$$\text{WAP}(X) \subset \text{Asp}(X) \subset \text{Tame}(X).$$

For more details see [26, 27].

- (5) Note that as a particular case of (3) we have defined the algebras $\text{WAP}(S)$, $\text{Asp}(S)$, $\text{Tame}(S)$ corresponding to the left action of S on $X := S$.

Definition 8.13. [22, 26] We say that a compact dynamical S -system X is *cyclic* if there exists $f \in C(X)$ such that (S, X) is topologically S -isomorphic to the Gelfand space X_f of the S -invariant unital subalgebra $\mathcal{A}_f \subset C(X)$ (generated by the orbit fS).

Remark 8.14. Let X be a (not necessarily compact) S -system and $f \in \text{RMC}(X)$. Then, as was shown in [26], there exist: a cyclic S -system X_f , a continuous S -compactification $\pi_f : X \rightarrow X_f$, and a continuous function $\tilde{f} : X_f \rightarrow \mathbb{R}$ such that $f = \tilde{f} \circ \pi_f$; that is, f comes from the S -compactification $\pi_f : X \rightarrow X_f$. The collection of functions $\tilde{f}S$ separates points of X_f . Finally, $f \in \text{RUC}(X)$ iff the action of S on X_f is jointly continuous.

As expected the cyclic S -systems X_f provide “building blocks” for compact S -systems. That is, every compact S -space can be embedded into the S -product of S -spaces X_f , where $f \in C(X)$.

Proposition 8.15. *Let X be a compact S -space and $f \in C(X)$.*

- (1) $f \in WAP(X)$ iff fS has DLP on X .
- (2) $f \in Asp(X)$ iff fS is a fragmented family.
- (3) $f \in Tame(X)$ iff fS is eventually fragmented iff fS does not contain an l_1 -sequence.

9. REPRESENTATIONS OF DYNAMICAL SYSTEMS AND OF FUNCTIONS

Next we deal with the representability of families of real-valued functions on compact systems. This topic is closely related to the “smallness” of the family F in terms of its pointwise closure in the spirit of Theorem 5.8.

Definition 9.1. Let $\mathcal{K} \subset \mathbf{Ban}$ be a subclass of Banach spaces.

- (1) Let X be a compact S -system and (h, α) a representation of (S, X) on a Banach space V . Let $F \subset C(X)$ be a bounded S -invariant family of continuous functions on X and $\nu : F \rightarrow V$ a bounded mapping. We say that (ν, h, α) is an F -representation of the triple (F, S, X) if ν is an S -mapping (i.e., $\nu(fs) = \nu(f)s$ for every $(f, s) \in F \times S$) and

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall f \in F, \quad \forall x \in X.$$

In other words, the following diagram commutes

$$(9.1) \quad \begin{array}{ccc} F \times X & \longrightarrow & \mathbb{R} \\ \nu \downarrow & & \downarrow \alpha \\ V \times V^* & \longrightarrow & \mathbb{R} \end{array} \quad \begin{array}{c} \downarrow id_{\mathbb{R}} \\ \downarrow \end{array}$$

- (2) We say that a family $F \subset C(X)$ is \mathcal{K} -representable if there exists a Banach space $V \in \mathcal{K}$ and a representation (ν, h, α) of the triple (F, S, X) . A function $f \in C(X)$ is said to be \mathcal{K} -representable if the orbit fS is \mathcal{K} -representable.

Note that we do not assume in (1) or (2) that α is injective. However, when the family F separates points on X it follows that the map α is necessarily an injection.

- (3) In particular, we obtain the definitions of reflexively, Asplund and Rosenthal representable families of functions on dynamical systems.

Clearly, every bounded S -invariant $F \subset C(X)$ on an S -system X is Banach representable via the canonical representation on $V = C(X)$.

9.1. Representation theorems. Let S be a semitopological semigroup and X a compact S -system with a separately continuous action.

Theorem 9.2. *(Small families of functions)*

Let $F \subset C(X)$ be a norm bounded S -invariant subset of $C(X)$.

- (1) (F, S, X) admits a Rosenthal representation iff F is an eventually fragmented family iff $\text{cl}_p(F) \subset \mathcal{F}(X)$.
- (2) (F, S, X) admits an Asplund representation iff F is a fragmented family iff the envelope $\text{cl}_p(F)$ of F is a fragmented family.
- (3) (F, S, X) admits a reflexive representation iff $\text{cl}_p(F) \subset C(X)$ iff F has DLP on X .

Proof. (3) Already was proved in Section 4.3.

(1) and (2): The “only if part” is a consequence of the characterizations of Asplund and Rosenthal spaces in terms of fragmented and eventually fragmented families, Theorems 5.5.4 and 5.9.4. \square

Theorem 9.3.

- (1) (S, X) is a tame (continuous) system if and only if (S, X) is weakly (respectively, strongly) Rosenthal-approximable.
- (2) (S, X) is a HNS (continuous) system if and only if (S, X) is weakly (respectively, strongly) Asplund-approximable.

If X is metrizable then in (1) and (2) “approximable” can be replaced by “representable”. Moreover, the corresponding Banach space can be assumed to be separable.

Proof. “only if” part: For (1) use the fact that (Θ^{op}, B^*) is a tame system (Theorem 8.4) for every Rosenthal V and for (2), the fact that (Θ^{op}, B^*) is HNS (Theorem 7.3) for Asplund V .

“if” part: (1) For every $f \in C(X) = \text{Tame}(X)$ the orbit fS is a Rosenthal family for X (Proposition 8.15). Applying Theorem 9.4 below we conclude that every $f \in C(X) = \text{Tame}(X)$ on a compact G -space X comes from a Rosenthal representation. Since continuous functions separate points of X , this implies that Rosenthal representations of (S, X) separate points of X . So, for (1) it is enough to prove the following result which gives a proof of Theorem 9.2.1. The proof of (2) is similar.

Theorem 9.4. *Let $F \subset C(X)$ be a Rosenthal family (Asplund family) for X such that F is S -invariant; that is, $fS \subset F \ \forall f \in F$. Then there exist: a Rosenthal (respectively, Asplund) Banach space V , an injective mapping $\nu : F \rightarrow B_V$ and a representation*

$$h : S \rightarrow \Theta(V), \quad \alpha : X \rightarrow V^*$$

of (S, X) on V such that h is weakly continuous, α is a weak continuous map and*

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall f \in F \quad \forall x \in X.$$

Thus the following diagram commutes

$$(9.2) \quad \begin{array}{ccc} F \times X & \longrightarrow & \mathbb{R} \\ \nu \downarrow & \downarrow \alpha & \downarrow id_{\mathbb{R}} \\ V \times V^* & \longrightarrow & \mathbb{R} \end{array}$$

If X is metrizable then in addition we can suppose that V is separable.

If the action $S \times X \rightarrow X$ is continuous we may assume that h is strongly continuous.

Proof. Step 1: The construction of V .

For brevity of notation let $\mathcal{A} := C(X)$ denote the Banach space $C(X)$, B will denote its unit ball, and B^* will denote the weak* compact unit ball of the dual space $\mathcal{A}^* = C(X)^*$. Let W be the symmetrized convex hull of F ; that is, $W := \text{co}(F \cup -F)$. Consider the sequence of sets

$$(9.3) \quad M_n := 2^n W + 2^{-n} B.$$

Then W is convex and symmetric. We apply the construction of Davis-Figiel-Johnson-Pelczyński [11] as follows. Let $\| \cdot \|_n$ be the Minkowski functional of the set M_n , that is,

$$\|v\|_n = \inf \{ \lambda > 0 \mid v \in \lambda M_n \}.$$

Then $\|\cdot\|_n$ is a norm on \mathcal{A} equivalent to the given norm of \mathcal{A} . For $v \in \mathcal{A}$, set

$$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \quad \text{and let } V := \{v \in \mathcal{A} \mid N(v) < \infty\}.$$

Denote by $j : V \hookrightarrow \mathcal{A}$ the inclusion map. Then (V, N) is a Banach space, $j : V \rightarrow \mathcal{A}$ is a continuous linear injection and

$$(9.4) \quad W \subset j(B_V) = B_V \subset \bigcap_{n \in \mathbb{N}} M_n = \bigcap_{n \in \mathbb{N}} (2^n W + 2^{-n} B)$$

Indeed, if $v \in W$ then $2^n v \in M_n$, hence $\|v\|_n \leq 2^{-n}$ and $N(v)^2 \leq \sum_{n \in \mathbb{N}} 2^{-2n} < 1$. This proves $W \subset j(B_V)$. In order to prove the second inclusion recall that the norms $\|\cdot\|_n$ on \mathcal{A} are equivalent to each other. It follows that if $v \in B_V$ then $\|v\|_n < 1$ for all $n \in \mathbb{N}$. That is, for every $n \in \mathbb{N}$, $v \in \lambda_n M_n$ for some $0 < \lambda_n < 1$. By the construction M_n is a convex subset containing the origin. This implies that $\lambda_n M_n \subset M_n$. Hence $j(v) = v \in M_n$ for every $n \in \mathbb{N}$.

Step 2: The construction of the representation (h, α) of (S, X) on V .

The given action $S \times X \rightarrow X$ induces a natural linear norm preserving continuous right action $C(X) \times S \rightarrow C(X)$ on the Banach space $\mathcal{A} = C(X)$. It follows by the construction that W and B are S -invariant subsets in \mathcal{A} . This implies that V is an S -invariant subset of \mathcal{A} and the restricted natural linear action $V \times S \rightarrow V$, $(v, g) \mapsto vg$ satisfies $N(vs) \leq N(v)$. Therefore, the co-homomorphism $h : S \rightarrow \Theta(V)$, $h(s)(v) := vs$ is well defined.

Let $j^* : \mathcal{A}^* \rightarrow V^*$ be the adjoint map of $j : V \rightarrow \mathcal{A}$. Define $\alpha : X \rightarrow V^*$ as follows. For every $x \in X \subset C(X)^*$ set $\alpha(x) = j^*(x)$. Then (h, α) is a representation of (S, X) on the Banach space V .

By the construction, $F \subset W \subset B_V$. Define $\nu : F \hookrightarrow B_V$ as the natural inclusion. Then

$$(9.5) \quad f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall f \in F \quad \forall x \in X.$$

Step 3: Weak continuity of $h : S \rightarrow \Theta(V)$.

By our construction $j^* : C(X)^* \rightarrow V^*$, being the adjoint of the bounded linear operator $j : V \rightarrow C(X)$, is a norm and weak* continuous linear operator. By [16, Lemma 1.2.2] we obtain that $j^*(C(X)^*)$ is norm dense in V^* . Since V is Rosenthal, Haydon's theorem (Fact 5.9.4) gives $Q := cl_{w^*}(co(Y)) = cl_{norm}(co(Y))$, where $Y := j^*(X)$. Now observe that $j^*(P(X)) = Q$. Since $S \times X \rightarrow X$ is separately continuous, every orbit map $\tilde{x} : S \rightarrow X$ is continuous, and each orbit map $\widetilde{j^*(x)} : S \rightarrow j^*(X)$ is weak* continuous. Then also $\widetilde{j^*(z)} : S \rightarrow V^*$ is weak* continuous for each $z \in cl_{norm}(co(j^*(X))) = Q$. Since $sp(Q)$ is norm dense in V^* (and $\|h(s)\| \leq 1$ for each $s \in S$) it easily follows that $\widetilde{j^*(z)} : S \rightarrow V^*$ is weak* continuous for every $z \in V^*$. This is equivalent to the weak continuity of h .

If the action $S \times X \rightarrow X$ is continuous we may assume that h is strongly continuous. Indeed, by the definition of the norm N , we can show that the action of S on V is norm continuous (use the fact that, for each $n \in \mathbb{N}$, the norm $\|\cdot\|_n$ on \mathcal{A} is equivalent to the given norm on \mathcal{A}).

Step 4: V is a Rosenthal space.

Proof. If F is an eventually fragmented family for X then $W := \text{co}(F \cup -F)$ is an eventually fragmented family for X and even for B^* (Proposition 5.22). By the construction of DFJP we get $j : V \rightarrow \mathcal{A}$ such that $cl_{norm} j^*(\mathcal{A}^*) = V^*$ (Fabian). Denote by $M := \cap \{2^n W + \frac{1}{2^n} B\}$. Then $j(B_V) \subset M$. It is easy to see that M is eventually weakly fragmented on X . Since \mathbb{R} is a metrizable uniform space by diagonal arguments we obtain that M is eventually fragmented on X . Therefore M is eventually fragmented also on B^* (again Proposition 5.22). In order to show that V is Rosenthal it is equivalent to show that B_V is eventually fragmented on B_{V^*} (Fact 5.9). It is equivalent to show that for every infinite subset C_0 of B_V there exists a (countable) infinite subset $C \subset C_0$ which is fragmented on B_{V^*} , or equivalently, that (B_{V^*}, ρ_C) is separable. Equivalently, that (V^*, ρ_C) is separable. Since $j(C_0) \subset M$ is an infinite subset (recall that j is injective) we obtain that there exists an infinite subset $j(C)$ which is a fragmented family on B_{A^*} . Equivalently, $(B_{A^*}, \rho_{j(C)})$ is separable. Equivalently, $(\mathcal{A}^*, \rho_{j(C)})$ is separable (Theorem 5.12). By the definition of the adjoint operator ($\langle j(c), v^* \rangle = \langle c, j^*(v^*) \rangle$) we obtain that $(j^*(\mathcal{A}^*), \rho_C)$ is separable. Then its closure is also separable. That is, $cl_{\rho_C}(j^*(\mathcal{A}^*))$ is also ρ_C -separable. Since C is a bounded subset, clearly, $cl_{\rho_C}(j^*(\mathcal{A}^*)) \supset cl_{norm}(j^*(\mathcal{A}^*)) = V^*$. Therefore, (V^*, ρ_C) is separable, as desired. \square

If the compact space X is metrizable then $C(X)$ is separable and it is also easy to see that (V, N) is separable. \square

This proves Theorem 9.4 and hence also Theorem 9.3.1.

For the “Asplund case” (when F is fragmented on X) use

Step 4’: V is an Asplund space.

The main idea is that the corresponding results of [49, Section 7] and [22, Section 9] can be adopted here, thus obtaining a modification of Theorem 9.4 which replaces a Rosenthal space by an Asplund space, and a “Rosenthal family F ” for X by an “Asplund set”. The latter means that for every countable subset $A \subset F$ the pseudometric ρ_A on X defined by

$$\rho_A(x, y) := \sup_{f \in A} |f(x) - f(y)|, \quad x, y \in X$$

is separable. By [16, Lemma 1.5.3] this is equivalent to saying that $(C(X)^*, \rho_A)$ is separable. Now $\text{co}(F \cup -F)$ is an Asplund set for B^* by [16, Lemma 1.4.3]. The rest is similar to the proof of [49, Theorem 7.7]. Checking the weak continuity of h one can apply a similar idea (using again Haydon’s theorem as in (1)).

Finally note that if X is metrizable then in (1) and (2) “approximable” can be replaced by “representable” using an l_2 -sum of a sequence of separable Banach spaces (see Lemma 5.11.3). \square

Note that, in Definition 9.1, when the family F separates points on X it follows that the map α is necessarily an injection. In view of this remark, Theorem 9.2 implies Theorem 9.3 and also the following useful result.

Theorem 9.5. *A compact S -system X is RN (WRN, Eberlein) iff there exists a norm bounded S -invariant fragmented (resp.: eventually fragmented, DLP) family $F \subset C(X)$ which separates points of X .*

Similar to Theorem 4.26 one may show

Theorem 9.6. *Let G be a topological group and $f \in \text{RUC}(G)$. The following are equivalent:*

- (1) $f \in \text{Asp}(G)$ iff f is a matrix coefficient of a strongly continuous Asplund co-representation of G .

- (2) $f \in \text{Tame}(G)$ iff f is a matrix coefficient of a strongly continuous Rosenthal co-representation of G .

9.2. The purely topological case. Note that the definitions and results of Section 9 (for instance, Theorem 9.2) make sense in the purely topological setting, for trivial $S = \{e\}$ actions, yielding characterizations of “small families” of functions, and of RN, WRN and Eberlein compact spaces.

The “only if” parts of these results, in the cases of Eberlein and RN compact spaces (with trivial actions), are consequences of known characterizations of reflexive and Asplund spaces. The Eberlein case yields a well-known result: a compact space X is Eberlein iff there exists a pointwise compact subset $Y \subset C(X)$ which separates the points of X . The RN case is very close to some results of Namioka [55] (up to some reformulations). The case of WRN spaces seems to be new.

Recall that (answering to a question posed by Lindenstrauss) by a classical result of Benjamini-Rudin-Wage, continuous surjective maps preserve the class of Eberlein compact spaces. The same is true for uniformly Eberlein (that is, Hilbert representable) compacta. Recently, Aviles and Koszmider [3] proved that this is not the case for the class RN of Asplund representable compacta, answering a long standing open problem posed by Namioka [55]. The following question seems to be interesting.

Question 9.7. *Is it true that the class of WRN (e.g., Rosenthal representable) compact spaces is closed under continuous onto maps (in the realm of Hausdorff spaces) ?*

Remark 9.8.

- (1) An example of a compact space which is not WRN is $\beta\mathbb{N}$. This was done by Todorčević (private communication).
- (2) Two arrows space is WRN but not RN. More precisely, as we recently established, every compact linearly ordered space is WRN. On the other hand, two arrows space is not RN as it was established by Namioka [55, Example 5.9].
- (3) One may show that a compact space K is WRN iff the Banach space $C(K)$ is *Rosenthal generated* (meaning that there exists a Rosenthal space V and a linear dense (injective) operator $V \rightarrow C(K)$). It is a WRN analog of Stegall’s result for RN compacta.

Corollary 9.9. *Let X be a compact space and $F \subset C(X)$ is a bounded family. If F has DLP on X then F is a fragmented family.*

Proof. Theorem 9.2.3 guarantees that there exists a representation of $(F, \{e\}, X)$ on a reflexive space V . Since V is Asplund we easily obtain by Theorem 5.5.4 that F is fragmented. Alternatively, one may derive the latter from Lemma 5.3.1. \square

10. SOME APPLICATIONS

10.1. When does weak imply strong ? Recall that a Banach space V has the *Kadec property* if the weak and norm topologies coincide on the unit (or some other) sphere of V . Let us say that a subset X of a V is a *Kadec subset* (*light subset* in [46]) if the weak topology coincides with the norm topology. *Light* linear subgroups $G \leq GL(V)$ (with respect to the weak and strong *operator topologies*) can be defined Analogously. Clearly, if G is *orbitwise Kadec* on V that is, all orbits Gv are light in V , then G is necessarily light. The simplest examples are the spheres (orbits of the unitary group $\text{Iso}(H)$) in Hilbert spaces H .

Exercise 10.1. Show that in every Hilbert space H on any sphere S_r the weak and norm topologies are the same.

In general, $\text{Iso}(V)$ need not be light. Indeed, $\text{Iso}(C([0,1]^2))$ is not light, [46]. The following results show that linear actions frequently are “orbitwise Kadec” in the presence of fragmentability properties.

A not necessarily compact G -system X is called *quasiminimal* if $\text{int}(\text{cl}(Gz)) \neq \emptyset$ for every $z \in X$. 1-orbit systems and compact minimal G -systems are quasiminimal.

Theorem 10.2. *Let $G \leq GL(V)$ be a bounded subgroup, X a bounded, (weak, norm)-fragmented G -invariant subset of a Banach space V . Then every, not necessarily closed, quasiminimal G -subspace (e.g., the orbits) Y of X is a Kadec subset.*

Proof. Let $z \in X$. We have to show that for every $\varepsilon > 0$ there exists a w -neighborhood $O(z)$ of z in X such that O is ε -small. Choose $\delta > 0$ such that $(gy_1, gy_2) \in \varepsilon$ for every $(y_1, y_2) \in \delta$ and $g \in G$. Since X is quasiminimal, the set $A := \text{int}(\text{cl}(Gz))$ is non-void. Since X is (τ, norm) -fragmented, we can pick a *non-void* τ -open subset W of X such that $W \subset A$ and W is δ -small in X . Clearly, $W \cap Gz \neq \emptyset$. One can choose $g_0 \in G$ such that $g_0z \in W \cap Gz$. Denote by O the open subset $g_0^{-1}W$ of (X, τ) . Then O is a τ -neighborhood of z and is ε -small. \square

This result together with a characterization of PCP (Theorem 5.10) yield:

Theorem 10.3. [46, 49] *Let V be a Banach space with PCP (e.g., reflexive, RNP, or the dual of Asplund). Then*

- (1) *norm topology = weak topology on every orbit Gv for every norm bounded $G \leq \text{Aut}(V)$.*
- (2) *The weak and the strong operator topologies coincide on $\text{Iso}(V)$.*
- (3) *Every weakly continuous (co)homomorphism $h : G \rightarrow \text{Iso}(V)$ is strongly continuous.*

Some boundedness condition is necessary even for l_2 . In fact, $GL(l_2)$ is not light. The group G may not be replaced in general by semigroups. Indeed, the semigroup $\Theta(l_2)$ is weakly compact but not strongly compact.

Now we turn to the weak* version of the lightness concept. Let V be a Banach space. Let's say that a subset A of the dual V^* is *weak* light* if the weak* and the norm topologies coincide on A . If G is a subgroup of $GL(V^*)$, then the weak* (resp., norm*) topology on G is the weakest topology which makes all orbit maps $\{\tilde{\psi} : G \rightarrow V^* : \psi \in V^*\}$ weak* (resp., norm) continuous.

Definition 10.4. Let $\pi : G \times V \rightarrow V$ be a continuous left action of G on V by linear operators. The *adjoint (or, dual) right* action $\pi^* : V^* \times G \rightarrow V^*$ is defined by $\psi g(v) := \psi(gv)$. The corresponding *adjoint (dual) left* action is $\pi^* : G \times V^* \rightarrow V^*$, where $g\psi(v) := \psi(g^{-1}v)$.

A natural question here is whether the dual action π^* of G on V^* is jointly continuous with respect to the norm topology on V^* . When this is the case we say that the action π (and, also the corresponding representation $h : G \rightarrow \text{Iso}(V)$, when π is an action by linear isometries) is *adjoint continuous*.

Theorem 10.5. [49] *Suppose that V is an Asplund space, $G \leq GL(V)$ is a bounded subgroup, and $X \subset V^*$ is a bounded G -invariant subset.*

- (i) *If (X, w^*) is a quasiminimal (e.g., 1-orbit) G -subset, then X is weak* light.*
- (ii) *The weak* and strong* operator topologies coincide on G (e.g., on $\text{Iso}(V)$).*
- (iii) *Any weakly (hence, also, strongly) continuous group representation $h : G \rightarrow \text{Iso}(V)$ on an Asplund space V is adjoint continuous.*

More generally, this is true for any continuous linear topological group action (not necessarily by isometries).

Theorem 10.6. [49, Corollary 6.9] *Let V be an Asplund Banach space and $\pi : G \times V \rightarrow V$ a linear jointly continuous action. Then the dual action $\pi^* : G \times V^* \rightarrow V^*$ is also jointly continuous.*

The regular representation $\mathbb{T} \rightarrow \text{Iso}(V)$ of the circle group $G := \mathbb{T}$ on $V := C(\mathbb{T})$ is continuous but not adjoint continuous. Consider the Banach space $V := l_1$ and the topological subgroup $G := S(\mathbb{N})$ (“permutations of coordinates”) of $\text{Iso}(l_1)$. Then we have a natural continuous representation of the symmetric topological group $S(\mathbb{N})$ on l_1 which is not adjoint continuous.

One more application is a quick proof of $\text{WAP}(G) \subset UC(G) := \text{LUC}(G) \cap \text{RUC}(G)$, Helmer’s theorem. In fact, we can show more.

Theorem 10.7. [49] $\text{WAP}(G) \subset \text{Asp}(G) \subset UC(G)$ for every semitopological group G .

Proof. Let $f \in \text{Asp}_s(G)$. The function f coincides with a matrix coefficient $m_{v,\psi}$ for a suitable strongly continuous antihomomorphism $h : G \rightarrow \text{Iso}(V)_s$, where V is Asplund. In particular, v is a norm continuous vector. By Theorem 10.5 (or, 10.6) the orbit $G\psi$ is light. Hence, ψ is a norm continuous vector. By Fact 3.3.2, $f = m_{v,\psi}$ is both left and right uniformly continuous. \square

Exercise 10.8. Show that if V is Asplund then $\text{Iso}(V)_w$ (in WOT) is a topological group.

10.2. Ryll-Nardzewski fixed point Theorem.

Theorem 10.9. (Ryll-Nardzewski) *Let V be a locally convex vector space equipped with its uniform structure ξ . Let (Q, τ) be an affine compact S -system such that*

- (1) (Q, τ) is a weakly compact subset in V .
- (2) S is ξ -distal on Q .

Then Q contains a fixed point.

Proof. (Sketch) We can suppose that $Q = cl(coX)$, where X is a compact minimal S -system. Weakly compact set X is ξ -fragmented. By Theorem 10.2, on X the topologies induced by τ and the uniformity ξ agree. So, (S, X) is distal. Therefore, the proof can be reduced to the following theorem of H. Furstenberg.

Theorem 10.10. (Furstenberg) *Every distal compact dynamical system admits an invariant probability measure.*

\square

11. MORE APPLICATIONS FOR TOPOLOGICAL GROUPS

Theorem 11.1. *The group $G := H_+[0, 1]$ is Rosenthal representable.*

Proof. Consider the natural action of G on the closed interval $X := [0, 1]$ and the corresponding enveloping semigroup $E = E(G, X)$. Every element of G is a (strictly) increasing self-homeomorphism of $[0, 1]$. Hence every element $p \in E$ is a nondecreasing function. It follows that E is naturally homeomorphic to a subspace of the Helly compact space (of all nondecreasing selfmaps of $[0, 1]$ in the pointwise topology). Hence E is a Rosenthal compactum. So by the dynamical BFT dichotomy, Fact 8.9, the G -system X is tame. By Theorem 9.3 we have a faithful representation (h, α) of (G, X) on a separable Rosenthal space V . Therefore we obtain a G -embedding $\alpha : X \hookrightarrow (V^*, w^*)$. Then the strongly continuous homomorphism $h : G \rightarrow \text{Iso}(V)^{op}$ is injective. Since $h(G) \times \alpha(X) \rightarrow \alpha(X)$ is continuous (and we may identify X with $\alpha(X)$) it follows, by the minimality properties of the compact open topology, that h is an embedding. Thus $h \circ \text{inv} : G \rightarrow \text{Iso}(V)$ is the required topological group embedding. \square

Remark 11.2. (1) Recall that by [45] continuous group representations on Asplund spaces have the *adjoint continuity property*. In contrast this is not true for Rosenthal spaces. Indeed, assuming the contrary we would have, from Theorem 11.1, that the dual action of the group $H_+[0, 1]$ on V^* is continuous, but this is impossible by the following fact [23, Theorem 10.3] (proved also by Uspenskij (private communication)): every adjoint continuous (co)representation of $H_+[0, 1]$ on a Banach space is trivial.

- (2) There exists a semigroup compactification $\nu : G = H_+[0, 1] \rightarrow P$ into a tame semigroup P such that ν is an embedding. Indeed, the associated enveloping semigroup compactification $j : G \rightarrow E$ of the tame system $(G, [0, 1])$ is tame. Observe that j is a topological embedding because the compact open topology on $j(G) \subset H([0, 1])$ coincides with the pointwise topology.

Let X be a compact dynamical system with the jointly continuous action. Then for every point $x_0 \in X$ the corresponding orbit map $\tilde{x}_0 : G \rightarrow X$ is RUC. If it is also left uniformly continuous then we say that (G, X) is strongly uniformly continuous (SUC).

We are going to show that $H_+[0, 1]$ is SUC-trivial. This will imply that this system is adjoint trivial (by Fact 3.3). Which in turn implies by Theorem 10.5 that it is Asplund-trivial.

Definition 11.3. Let X be a compact G -space. We say that two points $a, b \in X$ are SUC-*proximal* if there exist nets s_i and g_i in G and a point $x_0 \in X$ such that s_i converges to the neutral element e of G , the net $g_i x_0$ converges to a and the net $g_i s_i x_0$ converges to b .

Lemma 11.4. *If the points a and b are SUC-proximal in a G -space X then $a \stackrel{SUC}{\sim} b$.*

Theorem 11.5. *Let $G = H_+[0, 1]$ be the topological group of orientation-preserving homeomorphisms of $[0, 1]$ endowed with the compact open topology. Then G is SUC-trivial.*

Proof. Denote by $j : G \rightarrow G^{SUC}$ and $i : G \rightarrow G^{UC}$ the G -compactifications (i necessarily is proper because $UC(G)$ separates the points and closed subsets) induced by the Banach G -algebras $SUC(G) \subset UC(G)$. There exists a canonical onto G -map $\pi : G^{UC} \rightarrow G^{SUC}$ such that the following diagram of G -maps is commutative:

$$\begin{array}{ccc} G & \xrightarrow{i} & G^{UC} \\ & \searrow j & \downarrow \pi \\ & & G^{SUC} \end{array}$$

We have to show that G^{SUC} is trivial for $G = H_+[0, 1]$. One of the main tools for the proof is the following identification.

Lemma 11.6. [Uspenskij] *The dynamical system G^{UC} is isomorphic to the G -space (G, Ω) . Here Ω denotes the compact space of all curves in $[0, 1] \times [0, 1]$ which connect the points $(0, 0)$ and $(1, 1)$ and “never go down”, equipped with the Hausdorff metric. These are the relations $\omega \subset [0, 1] \times [0, 1]$ where for each $t \in [0, 1]$, $\omega(t)$ is either a point or a vertical closed segment.*

Moreover, the natural action of $G = H_+[0, 1]$ on Ω is $(g\omega)(t) = g(\omega(t))$ (by composition of relations on $[0, 1]$).

We first note that every “zig-zag curve” (i.e. a curve z which consists of a finite number of horizontal and vertical pieces) is an element of Ω . In particular the curves γ_c with exactly one vertical segment defined as $\gamma_c(t) = 0$ for every $t \in [0, c]$, $\gamma_c(c) = \{c\} \times [0, 1]$ and $\gamma(t) = 1$ for every $t \in (c, 1]$, are elements of $\Omega = G^{UC}$. Note that the curve γ_1 is a fixed point for

the left G action. We let $\theta = \pi(\gamma_1)$ be its image in G^{SUC} . Of course θ is a fixed point in G^{SUC} . We will show that $\theta = j(e)$ and since the G -orbit of $j(e)$ is dense in G^{SUC} this will show that G^{SUC} is a singleton.

The idea is to show that zig-zag curves are SUC-proximal in G^{UC} . Then Lemma 11.4 will ensure that their images in G^{SUC} coincide. Choosing a sequence z_n of zig-zag curves which converges in the Hausdorff metric to $i(e)$ in G^{UC} we will have $\pi(z_n) = \pi(\gamma_1) = \theta$ for each n . This will imply that indeed $j(e) = \pi(i(e)) = \pi(\lim_{n \rightarrow \infty} z_n) = \lim_{n \rightarrow \infty} \pi(z_n) = \theta$.

First we show that $\pi(\gamma_1) = \pi(\gamma_c)$ for any $0 < c < 1$. As indicated above, since G^{SUC} is the Gelfand space of the algebra $SUC(G)$, by Lemma 11.4, it suffices to show that the pair γ_1, γ_c is SUC-proximal in G^{UC} . Since $X^{SUC} = X$ for $X := G^{SUC}$ we conclude that $\pi(\gamma_1) = \pi(\gamma_c)$.

Let $p \in G^{UC}$ be the curve defined by $p(t) = t$ in the interval $[0, c]$ and by $p(t) = c$ for every $t \in [c, 1]$. Pick a sequence s_n of elements in G such that s_n converges to e and $s_n c < c$. It is easy to choose a sequence g_n in G such that $g_n s_n c$ converges to 0 and $g_n c$ converges to 1. Then the sequences s_n and g_n are the desired sequences; that is, $g_n p \rightarrow \gamma_c$, $g_n s_n p \rightarrow \gamma_1$ (see the picture below).

Denote $\theta = \pi(\gamma_1) = \pi(\gamma_c)$. Using similar arguments (see the picture below, where $a \stackrel{\pi}{\sim} b$ means $\pi(a) = \pi(b)$) construct a sequence $z_n \in G^{UC}$ of zig-zag curves which converges to $i(e)$ and such that $\pi(z_n) = \theta$ for every n .

In view of the discussion above this construction completes the proof of the theorem. \square

As a corollary we get

Theorem 11.7. $H_+[0, 1]$ is Asplund-trivial (in particular, reflexive-trivial).

Theorem 11.8.

- (1) $L_4[0, 1] \in \mathbf{Ref}_r$;
- (2) $L_4[0, 1] \notin \mathbf{Hilb}_r$.

[Chaatit 1996] The additive group of every separable *stable* (Krivine-Maurey [39]) Banach sp. $\in \mathbf{Ref}_r$.

\implies (1)

(direct) proof of (1):

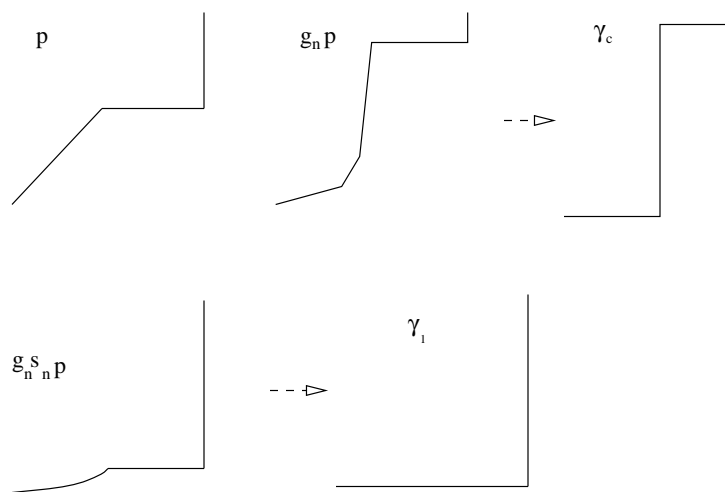
$$\phi : L_{2k} \rightarrow \mathbf{R} \quad v \mapsto e^{-\|v\|} \text{ is wap.}$$

Exercise 11.9. If G admits a left invariant metric with DLP then $G \in \mathbf{Ref}_r$.

Lemma (Grothendieck's DLP) A function $f \in C^b(G)$ is wap iff $\lim_n \lim_m f(g_n h_m) = \lim_m \lim_n f(g_n h_m)$ whenever all the limits exist.

- $(L_{2k}(\mu), \|\cdot\|)$ ($k \in \mathbf{N}$) has the DLP.

$$\begin{aligned} \|u_n + v_m\|^{2k} &= \|u_n\|^{2k} + \sum_{i=1}^{2k-1} C_{2k}^i \int u_n^{2k-i} v_m^i dt + \|v_m\|^{2k} \\ &\int u_n^{2k-i} v_m^i dt = \langle u_n^{2k-i}, v_m^i \rangle \\ u_n^{2k-i} &\in L_{\frac{2k}{2k-i}} \quad v_m^i \in L_{\frac{2k}{i}} = L_{\frac{2k}{2k-i}}^* . \end{aligned}$$



- DLP for $B(V) \times B(V^*) \rightarrow [-1, 1]$ with reflexive V .

(2) $L_4[0, 1] \notin \mathbf{Hilb}_r$

Theorem 11.10 (Aharoni-Maurey-Mityagin 1985). *For $2 < p < \infty$, an infinite-dimensional $L_p(\mu)$ space is not uniformly embedded into a Hilbert space.*

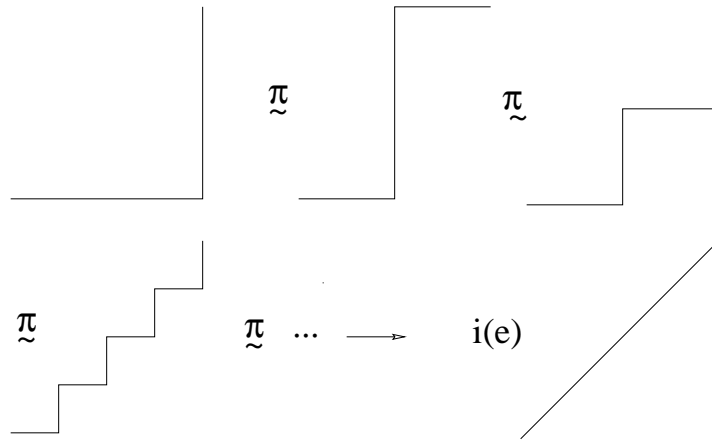
Lemma \forall metric subgroup G of $Is(H)_s$

$$\exists (G, \mathcal{L}) \xrightarrow{unif} H.$$

(a): $\|v_n\| = \frac{1}{2^n}$

(b): $\{\tilde{v}_n : Is(H) \rightarrow H, \quad g \mapsto gv_n\}_{n \in \mathbb{N}}$ generates the left uniformity on $Is(H)$.

$$Is(H)_s \xrightarrow{unif} \prod_n B_{\frac{1}{2^n}} \xrightarrow{unif} \left(\sum_n (H)_n\right)_{l_2\text{-sum}} \longleftrightarrow H$$



$$\implies \boxed{L_p[0, 1] \notin \mathbf{Hilb}_r \quad p > 2} \implies \text{(II)}$$

Remark: l_p ($p > 2$), $c_0 \notin \mathbf{Hilb}_r$

Hint: l_2 is not uniformly universal space for separable Banach spaces (Enflo) in contrast to c_0 (Aharoni).

$$l_p \in \mathbf{Hilb}_r \iff p \leq 2.$$

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