EMBEDDING INFINITE GRAPHS INTO FINITELY GENERATED GROUPS

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Abstract. These are notes for the mini-course "Embedding infinite graphs into finitely generated groups", SSDNM, IMPAN Warszawa, 11-13.03.2015.

The aim of this course is to explain the recent construction of finitely generated groups whose Cayley graphs contain isometrically some infinite sequences of finite graphs. Applications include constructing first examples of finitely generated groups with expanders embedded isometrically into Cayley graphs, first examples of a-T-menable groups without Yu's property A, and some other exotic examples of finitely generated groups. Motivations, sketches of proofs, applications and further remarks will be presented.

1. Overview

The aim of this course is to explain the recent construction [Osa14] of finitely generated groups whose Cayley graphs contain isometrically some infinite sequences of finite graphs. For expanding sequences the resulting groups are not coarsely embeddable into Hilbert spaces and are counterexamples to the Baum-Connes conjecture with coefficients. (Recall that a map $f\colon (X,d_X) \to (Y,d_Y)$ between metric spaces is a *coarse embedding* when $d_Y(f(x_n),f(y_n)) \to \infty$ iff $d_X(x_n,y_n) \to \infty$ for all sequences (x_n), (y_n).) The only other groups with those features are Gromov monsters [Gro03] – groups into which expanders embed weakly. For other families of graphs the technique allows to construct first examples of finitely generated groups with some exotic properties, e.g. a-T-menable groups without Yu's property A.

The course will cover (hopefully, roughly, and up to some shifts) the following subjects:

Day 1. Formulation of the main goal, some motivations, and generalities on the approach. Basics of graphical small cancellation theory: definitions,

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Date: March 15, 2015.

2010 *Mathematics Subject Classification.* 20F69, 20F06, 46B85, 05C15.

Key words and phrases. Small cancellation, coarse embedding, Property A, CAT(0) cubical complex, graph coloring.

Partially supported by Narodowe Centrum Nauki, decision no. DEC-2012/06/A/ST1/00259.

examples, main properties (e.g. isometric embedding of relators). Some history, including a short discussion of Gromov's method [Gro03].

Day 2. A fairly detailed presentation of the proof of the main theorem from [Osa14] – embedding isometrically an infinite sequence of graphs into a finitely generated group. Applications to specific families of graphs. Finitely generated groups containing expanders. Further applications and remarks.

Day 3. Yu's property A and coarse embeddability into a Hilbert space. The construction of groups without property A acting properly on CAT(0) cubical complexes: Small cancellation labellings of graphs with walls, and defining a proper lacunary walling for such groups. Final remarks.

2. The general setting

If not stated otherwise we work with the sequence $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ ***of disjoint finite connected graphs of degree bounded by*** $D > 0$***. We assume that*** $\mathrm{girth}\,\Theta_n \stackrel{n\to\infty}{\longrightarrow} \infty$ ***and*** Θ ***satisfies the following condition:***

$$\mathrm{diam}\,\Theta_n \leqslant A\,\mathrm{girth}\,\Theta_n, \tag{1}$$

where diam ***denotes the diameter,*** girth ***is the length of the shortest simple cycle, and*** A ***is a universal (not depending on*** n***) constant. Furthermore, we fix a small cancellation constant*** $\lambda \in (0, 1/6]$***, and we assume that*** $1 < \lfloor \lambda\,\mathrm{girth}\,\Theta_n \rfloor < \lfloor \lambda\,\mathrm{girth}\,\Theta_{n+1} \rfloor$***.***

Let us make few remarks concerning the assumptions above:

1) Observe that for a sequence $(\Theta_n)_{n\in\mathbb{N}}$ with growing girths, the last assumption can be fulfilled by passing to a subsequence – this is allowed from the point of view of our applications.

2) The growing girth condition is necessary for our approach – for the use of the graphical small cancellation theory; see Section 3.

3) The important case is when the degree $D > 2$. Observe that a sequence of cycles of growing length (girth) is coarsely embeddable into $\mathbb{Z}^2$. Furthermore, for any sequence of bounded degree graphs, it is relatively easy to produce a small cancellation labelling of some of its subdivision; see [AO14, Section 6]. Such subdivisions have many vertices of degree two.

4) The diameter-girth condition (1) is essential in our approach. In Section 6 we present a small cancellation labelling of a sequence of graphs not satisfying this assumption. However, first one finds a labelling of a sequence satisfying (1), and only then one is able to apply some tricks.

3. Graphical small cancellation

The introduction of the graphical small cancellation is usually attributed to Gromov [Gro03], however it appeared implicitly in earlier works.

By a *labeling* (Θ, f) of an undirected graph Θ we mean the graph morphism $f\colon \Theta \to W$ into a bouquet of finitely many loops W, that is a graph with one vertex end several edges. Usually we refer however to the following interpretation of the labeling f. Orient edges of W and decorate every directed edge (loop) by an element of a finite set S. Then the labeling f is determined by the following data: We orient every edge of Θ and we assign to it the corresponding element of the set S or an element of the set $\overline{S}$ of

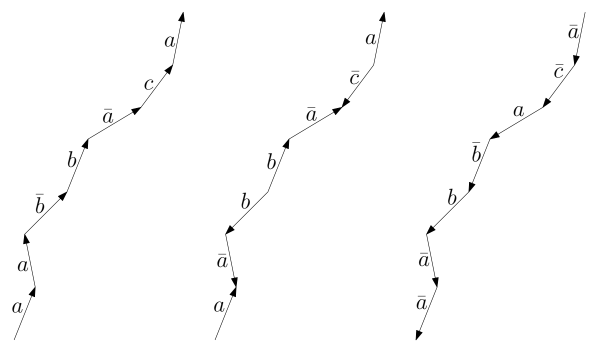

FIGURE 1. Three representations of the same labeling.

formal inverses of elements of S. We call the set $S \cup \overline{S}$ the *(symmetrized) set of labels*, and by $\bar{s}$ we denote the *inverse* of s. Using this interpretation we identify a labeling assigning the label s to an oriented edge vw with the labeling of wv by $\bar{s}$; see Figure 1. The labeling (Θ, f) is *reduced* if $f\colon \Theta \to W$ is locally injective, that is, when labels of two directed edges going out of a vertex are not the same. We consider further only reduced labellings.

A labelling (Θ, f) gives rise to the *graphical presentation*

$$\langle S \mid \Theta \rangle, \tag{2}$$

defining a group $G := \pi_1(W)/\langle\langle \pi_1(\Theta_n)_{n\in\mathbb{N}} \rangle\rangle$. In other words, G is the quotient of the free group $F(S)$ by the normal subgroup generated by words read on cycles of Θ. We call the graphs Θ_n (or the whole graph Θ) *relators*.

A *piece* is a labelled subpath of (Θ, f) appearing in at least two distinct places. The presentation (2) satisfies the $C'(\lambda)$*–small cancellation condition* when for every piece p, and every graph Θ_n containing this piece, we have

$$|p| < \lambda \operatorname{girth} \Theta_n, \tag{3}$$

where $|p|$ denotes the length of p, that is the number of edges in p. Groups defined by small cancellation presentations have many nice properties, and the most important for us will be the following lemma by Gromov.

Lemma 3.1 (see e.g. [Oll06, Theorem 1]). *Every relator (Θ_n, f) embeds isometrically (as a labelled graph) into the Cayley graph of the $C'(1/6)$–small cancellation presentation (2).*

4. The (sketch of the) construction

Our aim is to construct a finitely generated group into which every graph Θ_n embeds isometrically. In view of Lemma 3.1 it is enough to find a $C'(1/6)$–small cancellation presentation (2). In other words, we need to find a small cancellation labelling $(\Theta, m) = ((\Theta_n, m_n))_{n\in\mathbb{N}}$ for Θ.

We construct the small cancellation labelling $(\Theta, m) = ((\Theta_n, m_n))_{n\in\mathbb{N}}$ in three steps. In the first step we construct a labeling $(\Theta, l) = ((\Theta_n, l_n))_{n\in\mathbb{N}}$ such that l_n–labellings of long (relative to $\operatorname{girth}\Theta_n$) paths in Θ_n do not appear in $(\Theta_{n'}, l_{n'})$, for $n \neq n'$. In the second step we construct a labeling $(\Theta, \bar{l}) = ((\Theta_n, \bar{l}_n))_{n\in\mathbb{N}}$ with the property that, for each n, long paths in Θ_n are labeled differently. Finally, the labelling $(\Theta, m) = ((\Theta_n, m_n))_{n\in\mathbb{N}}$ is defined as the product of (Θ, l) and $(\Theta, \bar{l})$. That is, to every directed edge e in Θ_n we assign a pair $(l(e), \bar{l}(e))$. It follows that (Θ, m) is as required.

In the current notes we deal only with the second step – finding $(\Theta, \bar{l})$. The first step – finding (Θ, l) – is very similar. We fix a finite set of labels S (to be specified later) and we will show, for every n separately, that there exists a labelling $(\Theta_n, \bar{l}_n)$ with the property that no labelled path of length at lest $\lambda \operatorname{girth}\Theta_n$ appears in two distinct places in $(\Theta_n, \bar{l}_n)$.

Let us consider what happens when the small cancellation condition is not satisfied. It means, when a given labelling $(\Theta_n, \bar{l}_n)$ admits two long, that is with $k = \lfloor \lambda \operatorname{girth}\Theta_n \rfloor$, paths: $\overline{v} = (v_0, v_1, \ldots, v_k)$ and $\overline{w} = (w_0, w_1, \ldots, w_k)$ labelled the same way. We consider below only the case when the paths (here and further all the paths are simple) are disjoint. Let $(u_0 := v_s, u_1, \ldots, u_r := w_t)$ be a path of minimal length connecting $\overline{v}$ and $\overline{w}$. Further we consider only the case when $s \geqslant t \geqslant k/2$ and the directed edges $w_{i-1}w_i$ and $v_{i-1}v_i$ are both labelled by a_i, for all i; see Figure 2.

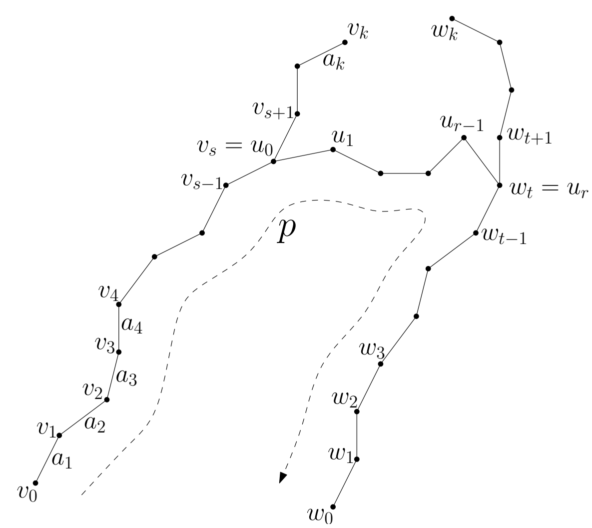

Figure 2. The path p

Consider the path $p := (v_0, \dots, v_s, u_1, \dots, u_{r-1}, w_t, \dots, w_0)$. By (1), its length $|p|$ may be bounded from above by

$$2k + r \leqslant 2\lambda \operatorname{girth} \Theta_n + A \operatorname{girth} \Theta_n = (2\lambda + A) \operatorname{girth} \Theta_n. \tag{4}$$

In its labeling the beginning sub-path of length t is labeled the same way — up to changing orientation — as the ending sub-path of length t, that is, it has the form (where 'repetitive' parts are underlined):

$$(\underline{a_1, a_2, \dots, a_t}, \dots, \underline{\bar{a}_t, \dots, \bar{a}_2, \bar{a}_1}), \tag{5}$$

with

$$t \geqslant k/2 > \frac{\lambda \operatorname{girth} \Theta_n}{4}. \tag{6}$$

(The last inequality is a rough estimate coming from $k > \lambda \operatorname{girth} \Theta_n - 1$.)

Therefore, if the labelling has to satisfy the small cancellation condition, we have to avoid the existence of paths p labelled as in (5). We show below, that there exists a labelling by S, not admitting paths p as above. Similarly – excluding few other cases mentioned above – there exists a $s'(1/6)$–small cancellation labelling.

We use a probabilistic argument. Consider a random labelling of Θ_n by labels from S. Recall the following fundamental tool from combinatorics; see e.g. [AS00].

Lemma 4.1 (Lovász Local Lemma)**.** *Let $A_1, A_2, \dots, A_s$ be events such that each event occurs with probability at most p and such that each event is independent of all the other events except for at most d of them. If $4pd \leqslant 1$ then there is a nonzero probability that none of the events occurs.*

Remarks. (1) The above version of the Lovász Local Lemma is due to Erdös-Lovász [EL75]. In [Osa14] an asymmetric version is used.

(2) The Lovász Local Lemma can be viewed as a natural extension of the fact that $\bigcap_{i=1}^{s} A_s^c \neq \emptyset$ for independent events $A_1, A_2, \dots, A_s$ occurring with probability less than 1 each.

In our situation A_i's are the events when a given path as in (4) is labeled *badly*, that is, as in (5),(6). We analyze now – roughly – the corresponding values of p and d from Lemma 4.1. Observe that the number of labellings of the path p with $|S|$ labels is comparable to

$$|S|^{|p|}. \tag{7}$$

The number of bad labellings (as in (5)) is roughly

$$|S|^{|p|-t}. \tag{8}$$

Therefore we get

$$p \sim |S|^{-t}. \tag{9}$$

On the other hand, the events of labelling badly two paths p_1, p_2 are independent whenever the two paths are disjoint, thus

$$d \sim |p_1| D^{|p_2|}. \tag{10}$$

Therefore, by (9), (10), and (4), (6), we get

$$4pd \lesssim 4(2\lambda + A)\mathrm{girth}\,\Theta_n D^{(2\lambda+A)\mathrm{girth}\,\Theta_n} |S|^{-\frac{\lambda \mathrm{girth}\,\Theta_n}{4}}. \tag{11}$$

It is clear that the left hand side in (11) can be made arbitrary small, by choosing $|S|$ big, with the choice depending on D, A, λ, but not on n. Therefore, for appropriate S, by Lovász Local Lemma, there exists a labelling of Θ_n without bad labellings of subpaths. Similarly, one can prove that there exists a $C'(\lambda)$–small cancellation labelling $(\Theta, \bar{l})$. This concludes the construction.

5. Applications

There exist expander families Θ satisfying our assumptions (as in Section 2); see e.g. [Gro03, AD08]. Therefore, applying the construction for such Θ, we obtain the first examples of groups as follows.

Theorem 5.1. *There exist finitely generated groups with expanders embedded isometrically into their Cayley graphs.*

For the celebrated Gromov construction (of Gromov monsters) [Gro03] only a weak embedding (a coarse notion weaker than the coarse embedding) of expanders in groups is established; see [AD08] and the detailed discussion in [Osa14, Section 2.4]. Besides the Gromov monsters, our groups are the only examples of groups with the following exotic properties.

Corollary 5.2. *If Θ is an expanding sequence of graphs then the group $\langle S \,|\, \Theta \rangle$ is not coarsely embeddable into a Hilbert space, and it does not satisfy the Baum-Connes conjecture with coefficients.*

The weak embedding of expanders established in [Gro03] (see [AD08]) is not enough for some analyses of the failure of the Baum-Connes conjectures. There – see e.g. [WY12, BGW13, FS14] – the groups as in Theorem 5.1 are of use. The isometric embedding of expanders is also used by D. Hume [Hum14] for showing that there is uncountably many quasi-isometry classes of groups that are not coarsely embeddable into the Hilbert space.

Sapir [Sap14] developed a technique of embedding groups with combinatorially aspherical recursive presentation complexes into groups with finite combinatorially aspherical presentation complexes. By embedding a group as in Theorem 5.1 into a finitely presented group we obtain the first examples of such groups coarsely containing expanders. Furthermore, Sapir's embedding is quasi-isometric, and gives rise to embedding (via the Davis trick) into the fundamental group of an aspherical manifold.

Corollary 5.3. *There exist closed aspherical manifolds of dimension* 4 *and higher whose fundamental groups contain quasi-isometrically embedded expanders.*

The group G defined by the presentation (2) is the limit of finitely presented groups G_i defined by presentations $\langle S \mid (\Theta_n)_{n=1}^{i} \rangle$. For Θ being a family of D–regular graphs with $D > 2$, we obtain the first examples of groups as follows, answering a question of Osin-Świątkowski; see e.g. [Dra08, Problem 4.5].

Corollary 5.4. *There exists a sequence* $G_1 \twoheadrightarrow G_2 \twoheadrightarrow G_3 \twoheadrightarrow \cdots$ *of finitely presented groups with the following properties. For all* i*, asdim*$(G_i) = 2$*, and the asymptotic dimension of the limit group* G *is infinite.*

Note that despite the group G above has infinite asymptotic dimension, it behaves in many ways as a two-dimensional group – see e.g. [OŚ13].

6. PW not A groups

We present here the construction of finitely generated groups without Yu's property A (in other words: non-exact groups) acting properly on CAT(0) cubical complexes (in particular, coarsely embeddable into a Hilbert space).

Recall that a uniformly discrete metric space (X, d) has *property A* if for every $\epsilon > 0$ and $R > 0$ there exists a collection of finite subsets $\{A_x\}_{x\in X}$, $A_x \subseteq X \times \mathbb{N}$ for every $x \in X$, and a constant $S > 0$ such that

(1) $\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} \leqslant \epsilon$ when $d(x, y) \leqslant R$, and
(2) $A_x \subseteq B(x, S) \times \mathbb{N}$.

Property A implies coarse embeddability into a Hilbert space, and the converse was an open question for groups. For spaces, the negative answer is due to [Now07] and [AGŠ12] (we use further a counterexample from [Ost12]). We construct our examples so that there is a natural structure of a *space with walls* ("cuts" such that any two points are separated by finitely many of them; see [HP98]) on the group (the vertex set of the Cayley graph). We show then that the wall pseudo-metric is proper (with respect to the word metric). It follows (see [Nic04, CN05]) that the group acts properly on a CAT(0) cubical complex (in other words: it has property PW). On the other hand, the Cayley graph contains isometrically D–regular graphs of growing girths, with $D > 2$. By a result of Willett [Wil11], it follows that the group have no property A.

The construction of walls relies on earlier works of D. Wise [Wis04, Wis11] (see also the notes [Wis12]), and their development in the case of infinitely presented small cancellation groups [AO15, AO14]. Let Θ be a sequence of D–regular graphs as in Section 2, with $D > 2$. Let (Θ, m) be its $C'(1/6)$–small cancellation labelling as in Section 4. We define now a new sequence $(\widehat{\Theta}, \widehat{m})$ of labelled graphs with some walls. The labeled graph $(\widehat{\Theta}, \widehat{m})$ is the labeled graph covering defined as follows: For every n, $\widehat{\Theta}_n$ is the $\mathbb{Z}_2$–homology cover of Θ_n. As observed by Wise (see [Wis11, Section 9] and

[Wis12, Section 10.3]), every $\widehat{\Theta}_n$ is then equipped with a structure of graph with walls – a wall corresponds to edges in $\widehat{\Theta}_n$ being preimages of a given edge in Θ_n. That is, removing open edges from the preimage of a given edge in Θ_n disconnects $\widehat{\Theta}_n$ into two connected components defining the corresponding cut on the set of vertices. The group defined by the graphical presentation (corresponding to the labelling $(\widehat{\Theta}, \widehat{m})$)

$$\langle S \mid \widehat{\Theta} \rangle, \tag{12}$$

contains isometric copies of $(\widehat{\Theta}, \widehat{m})$ in the Cayley graph.

Remark. The presentation (12) is not a small cancellation presentation in the sense of Section 3. It satisfies a modified $C'(1/6)$–small cancellation condition, where we do not distinguish (labelled) paths up to the (labelled) graph automorphism.

The walls in the Cayley graph of the presentation (12) correspond to sets of edges defining walls in each (isometrically embedded) relator $\widehat{\Theta}_n$. One has to check that such this defines a wall, and that the wall pseudo-metric is proper. This is guaranteed by the *proper lacunary walling condition* – saying that edges in the same wall are "far away" in a relator. To satisfy this condition it may be necessary for $\widehat{\Theta}_n$ to be a high multiplicity iterated $\mathbb{Z}_2$–homology cover of Θ_n. Having this satisfied we obtain our goal.

Theorem 6.1. *The group defined by the graphical presentation (12) has no property A and acts properly on a CAT(0) cubical complex.*

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