Boundedness vs. blow-up in the Keller-Segel system

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Abstract

The fully parabolic Keller-Segel chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

is considered under homogeneous Neumann boundary conditions in bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 1$.
We demonstrate rigorous analytical techniques which can be used to identify situations when solutions either remain bounded, or exhibit a blow-up phenomenon. In the latter case, which is of particular interest in various applications, we especially focus on the occurrence of blow-up within finite time.

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1 Introduction and preliminary estimates

The Keller-Segel model is introduced and some applications outlined. The question whether or not blow-up occurs is brought into focus.
As a preparation for the subsequent analysis, some basic functional inequalities are recalled. In particular, these include sharp estimates describing the smoothing action of the Neumann heat semigroup in bounded domains.

2 Local existence, uniqueness and extensibility

Based on the Banach fixed point theorem, a local well-posedeness theory is established within the framework of classical solutions. Here the initial data are supposed to satisfy the mild assumptions $u_0 \equiv u(\cdot, 0) \in C^0(\bar{\Omega})$ and $v_0 \equiv v(\cdot, 0) \in W^{1,\theta}(\Omega)$ for some $\theta > n$. The norms in these spaces are shown to appear also in an extensibility criterion, saying that if the maximal existence time T_{max} of a solution is finite then $\|u(\cdot, t)\|_{C^0(\bar{\Omega})} + \|v(\cdot, t)\|_{W^{1,\theta}(\Omega)}$ must blow up at $t = T_{max}$.
Apart from that, a natural energy identity associated with the Keller-Segel system is derived.

3 A refined extensibility criterion. Boundedness in the case $n = 1$

A refined extensibility criterion is derived, ensuring that whenever $p \geq 1$ is such that $p > \frac{n}{2}$, assuming boundedness of $\int_\Omega u^p(\cdot, t)$ throughout some time interval $(0, T)$ is sufficient for the solution to allow for an extension beyond this interval. A first application thereof yields boundedness of all solutions in the spatially one-dimensional case.

4 Subcritical mass solutions when $n = 2$: Application of the Moser-Trudinger inequality

The Moser-Trudinger inequality is applied to show that in planar domains, whenever the initial data $u_0 = u(\cdot, 0)$ satisfy the subcriticality condition $\int_\Omega u_0 < 4\pi$, then the solution satisfies $\int_\Omega u \ln u(\cdot, t) \leq c$ throughout its existence interval for some $c > 0$. Moreover, along such trajectories the above energy functional is shown to remain uniformly bounded.

5 Subcritical mass solutions when $n = 2$: Boundedness

Based on the estimates gained in the previous lecture, it is shown that in the case $n = 2$, under the assumption $\int_\Omega u_0 < 4\pi$ there exists $c > 0$ such that $\int_\Omega u^2(\cdot, t) \leq c$ for all $t \in (0, T_{max})$. As a consequence, the solution must be global and bounded in any such case.

6 A sufficient condition for boundedness in the higher-dimensional case

It is shown that in the case $n \geq 3$, global existence and boundedness of solutions can be enforced using an alternative smallness condition on the initial data, in particular involving the norm of u_0 in $L^p(\Omega)$ for some $p > \frac{n}{2}$. This will be achieved through a detailed analysis using the smoothing properties of the heat semigroup. To underline the strength of this approach, it is also demonstrated that as a by-product a rather precise information on the large time behavior of such small-data solutions can be obtained.

7 Finite-time blow-up in parabolic-elliptic simplifications

Some simplified Keller-Segel models are introduced, and it is indicated how these can be made accessible to a number of mathematical tools. In particular, analyzing the time evolution of certain second moments of solutions allows to conclude that some large-mass solutions in the respective two-dimensional models must blow up within finite time.

8 Unbounded solutions in fully parabolic systems: An energy-based contradictory strategy

Returning to the original parabolic system, it is proved that whenever a solution thereof is global and bounded, it approaches a corresponding steady state having its energy below the energy of the initial data. On the basis of this observation a strategy to detect unbounded solutions is developed.

9 Unbounded solutions in fully parabolic systems: A priori estimates for stationary solutions

Following the strategy introduced in the previous lecture, a priori estimates from below are derived for the values of the energy functional when evaluated at rather arbitrary stationary solutions. It is thereby shown that indeed – in an appropriate sense – ‘many’ radially symmetric unbounded solutions exist whenever Ω is a ball in $\mathbb{R}^n$ for some $n \geq 3$.

10 Finite-time blow-up: A novel view upon dissipation

In light of the above approach, the use of the energy identity is revised. This leads to a refined strategy, aiming at a more efficient use of the energy identity by quantitatively estimating the dissipation rate. As a first preliminary step toward this, a pointwise upper estimate for the second component v of solutions is derived in the radially symmetric setting.

11 Finite-time blow-up: Estimating $\int_\Omega uv$

It is shown that a crucial issue is to derive upper bounds for $\int_\Omega uv$, and it is seen how this relates to a corresponding estimate for solutions to a certain elliptic-hyperbolic system. Moreover, an inequality relating $\int_\Omega uv$ to $\int_\Omega |\nabla v|^2$ is derived.

12 Finite-time blow-up: Bounds via domain splitting in the case $n \ge 3$

An appropriate estimate for $\int_\Omega |\nabla v|^2$ is derived which will make it possible to complete the argument from the previous lecture. For this purpose, the domain Ω is suitably decomposed into an inner ball and an outer annulus. In the case $n \ge 3$, different testing procedures are applied in these two regions.

13 Finite-time blow-up as a generic phenomenon when $n \ge 3$

Combining the results collected above, an estimate of the form

$$\int_\Omega uv \le C \cdot \left\{ \left\| \Delta v - v + u \right\|_{L^2(\Omega)}^{2\theta} + \left\| \frac{\nabla u}{\sqrt{u}} - \sqrt{u} \nabla v \right\|_{L^2(\Omega)} + 1 \right\}$$

is seen to hold along trajectories, where $C > 0$ and $\theta \in (0,1)$ are appropriate constants. This allows to conclude that finite-time blow-up indeed occurs, and that moreover in the radially symmetric framework such explosions indeed can be regarded as a generic phenomenon.