Analysis and geometry on groups

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1 Introduction

These notes provide an introduction to problems in the topic under a very general name analysis and geometry on groups. Two concepts are central to these lectures: amenability and property (T). The first one was introduced by von Neumann in the twenties of the last century and for decades played a central role in functional analysis and geometric group theory. Its origin is a Banach-Tarski paradoxical decomposition but the scope of its applications became quite wide ranging from random walks, spectra of operators to fixed point properties just to mention few.

Property (T) was introduced by Kazhdan in late sixties in order to prove that lattices in higher rank simple Lie groups are finitely generated. Later property (T) became a fundamental property for groups with applications to measure theory, combinatorics, operator algebras, etc.

2 Amenability

In 1929 von Neumann defined the notion of amenability which became the fundamental notion in group theory.

2.1 Amenable groups

Definition 2.1. *A group* Γ *is* amenable *if there exists* $\mu : 2^{\Gamma} \longrightarrow [0;1]$ *such that :*

(a) $\mu(A \cup B) = \mu(A) + \mu(B)$ *for any pair* (A, B) *of disjoint subsets* Γ*;*

(b) $\mu(\Gamma) = 1$*;*

(c) $\mu(gA) = \mu(A)$ *for every subset* A *of* Γ *and* g *in* Γ*.*

Remark 2.2.

If Γ *is finite, necessarily* $\mu(A) = \frac{|A|}{|\Gamma|}$.

It can be shown that a group is amenable if and only if all finitely generated subgroups are amenable. There this notion is particularly interesting for countable groups.

Originally this notion was studied to understand the paradoxal decomposition of the sphere S^2. There are 8 disjoint subsets $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$ of the sphere S^2 and elements g_i, h_i of $SO(3)$ such that:

$$S^2 = A_1 \cup \ldots \cup A_4 \cup B_1 \cup \ldots \cup B_4$$

$$S^2 = g_1(A_1) \cup \ldots \cup g_4(A_4) \qquad\qquad S^2 = h_1(B_1) \cup \ldots \cup h_4(B_4)$$

Banach, in 1922 proved that S^1 does not admit a paradoxal decomposition as S^2.

Definition 2.3. *A word in letters* a*,* a^{-1}*,* b*, y* b^{-1} *is called irreducible if* $a\,a^{-1}$*,* $a^{-1}\,a$*,* $b\,b^{-1}$ *or* $b^{-1}\,b$ *never appear. A free group on two letters* $F_2 = < a, b >$ *is a subset of irreducible words in* a*,* a^{-1}*,* b*, and* b^{-1}*, with composition and reduction. Its neutral element is an empty set denoted* e*.*

Proposition 2.4. F_2*, a free group of rank 2 is not amenable.*

Proof. To prove it one can present a paradoxal decomposition similar to the paradoxal decomposition of the sphereS^2. This implies that there is no such a mean. Indeed:

$$A_+ = \{\text{words which start with } a\} \qquad A_- = \{\text{words which start with } a^{-1}\}$$
$$B_+ = \{\text{words which start with } b\} \qquad B_- = \{\text{words which start with } b^{-1}\}$$

Then

$$F_2 = A_+ \cup A_- \cup B_+ \cup B_- \cup \{e\}$$

and

$$a^{-1}A_+ \cup A_- = F_2$$

and

$$b^{-1}B_+ \cup B_- = F_2.$$

Applying the mean μ :

$$1 = \mu(F_2) = \mu(A_+ \cup A_- \cup B_+ \cup B_- \cup \{e\})$$

given that the mean is invariant by translations and that the mean of any element is zero, in particular $\mu(\{e\}) = 0$.

As μ is a mean invariant by translations $\mu(A_+) = \mu(a^{-1}A_+)$. Also,

$$1 = \mu(a^{-1}A_+) + \mu(A_-) + \mu(b^{-1}B_+) + \mu(B_-) = \mu(F_2) + \mu(F_2) = 2$$

This gives a desired contradiction. □

Remark 2.5. *It is known that a group is amenable if and only if it does not admit any paradoxical decomposition.*

Proposition 2.6. *The group $\mathbb{Z}$ is amenable. (Banach 1922)*

Proof. The idea is to define $\mu(A)$ as a limit $\lim_{n\to+\infty} \frac{|A\cap[-n;n]|}{2n+1}$. The problem is that this limit does not always exist. One can appeal to non trivial ultra filters but this is essentially equivalent to showing that $\mathbb{Z}$ is amenable.

Instead we define μ using a functional denoted $\mu : l^\infty(\mathbb{Z}) \to \mathbb{R}$ as well. We obtain the valuee $\mu(A)$ defining: $\mu(A) = \mu(\chi_A)$. To get the mean we need to apply the normalization

$$\mu(c.\chi_{\mathbb{Z}}) = c.$$

Let $g \in l^\infty(\mathbb{Z})$, and g_n the function g translated by n. More precisely, g_n is defined:

$$g_n(k) = g(n+k)$$

Therefore we can define

$$H = \left\langle \sum_{\text{finite}} (g_i - (g_i)_{n_i}), n_i \in \mathbb{Z}, g_i \in l^\infty \right\rangle \subseteq l^\infty(\mathbb{Z})$$

and we impose

$$\mu(h) = 0$$

for any h in H.

Thanks to the Hahn-Banach theorem we can extend μ to the whole space. To apply this theorem we need to prove that

$$\|\mu_{|\mathbb{C}\chi_{\mathbb{Z}} \oplus H}\| \leq 1$$

This is equivalent to

$$\inf_{n \in \mathbb{Z}} h(n) \leq 0$$

for any $h \in H$. This is easy to prove for $\mathbb{Z}$. □

Remark 2.7. *The same proof shows that any abelian group is amenable.*

In general, if a group Γ contains F_2 it is nonamenable. Indeed one can obtain a paradoxal decomposition of Γ from the one for F_2.

A natural question is if a converse (Γ non-amenable $\Longrightarrow F_2 \subseteq \Gamma$) is true. It reminaid an open problem for some time. Finally in 80ies it was shown by Olshanski that the answer is negative. A counterexample is the following:

$$B(2, 665) = < a, b | w^{665}(a, b) = 1 >$$

where the expression $w^{665}(a, b) = 1$ means that any word to the power 665 defines a trivial element.

A theorem due to Adyan and Novikov shows that this group is infinite. Their proof also shows that this group is non-amenable.

Remark 2.8. *It is known that the groups $B(2, n)$ for $n = 2, 3, 4$ and 6 are finite. For 5 and from 7 to 665 nothing is known.*

Theorem 2.9. *(Følner, '50) Let* Γ *be a countable group. It is amenable if and only if there is a sequence of finite subsets* A_n *of* Γ *such that:*

$$\lim_{n\to\infty} \frac{|A_n \Delta g A_n|}{|A_n|} = 0 \text{ for any } g \text{ in } \Gamma$$

where gA_n *is a translation of* A_n *by* g.

Definition 2.10. *A group* Γ *is said to be of finite type if it contains a finite subset* $S \subseteq \Gamma$ *which generates the group.*

One can prove that a group is amenable if and only if all subgroups of finite type are amenable. Therefore we will study only such groups.

Definition 2.11. *A group* G *generated by a finite set* S *is said of subexponential growth if* $b(n) = |S^n|$ *grows slower than any exponential function.*

Proposition 2.12. *If* G *is a group generated by a finite set* S *is of subexponential growth than there exists a sequence of integers* (n_k) *such that* S^{n_k} *is a Følner sequence.*

Proof. Suppose not. In this case there is a constant $C > 0$ such that for every n

$$|\partial S^n| \geqslant C|S^n|.$$

But, $|S^{n+1}| - |S^n| = |S^{n+1} \setminus S^n| = |\partial S^n| \geqslant C|S^n|$, and thus

$$|S^{n+1}| \geqslant (1+C)|S^n| \geqslant (1+C)^{n+1}.$$

□

Open problem: Is S^n is a Følner sequence (without passing to a subsequence)?

2.2 Automata groups

It follows from the work of von Neumann that the groups of subexponential growth are amenable and that this class is closed under following elementary operations: extensions, quotients, subgroups and direct limits.

Before the construction of the group generated by an automaton described later, all amenable groups could be obtained from groups of subexponential growth using elementary operations described above.

Let SG_0 be the class of groups such that all finitely generated subgroups are of subexponential growth. Suppose that $\alpha > 0$ is an ordinal and we have defined SG_β for any ordinal $\beta < \alpha$. Now if α is a limit ordinal let

$$SG_\alpha = \bigcup_{\beta<\alpha} SG_\beta.$$

If α is not a limit ordinal let SG_α be the classe of groups which can be obtained from groups in $SG_{\alpha-1}$ using either extenstions or direct limits. Let

$$SG = \bigcup_{\alpha} SG_\alpha.$$

The groups in this class are called subexponentially amenable.

SG is the smallest class of groups which contains groups of subexponential growth and is closed under elementary operations.

Definition of groups generated by automata

Automata groups were invented by Aleshin and play an important role in the theory of infinite groups. The automata which we consider are finite, reversible and have the same input and output alphabets, say $D = \{0, 1, \dots, d-1\}$ for a certain integer $d > 1$. To such an automaton A are associated a finite set of states Q, a transition function $\phi : Q \times D \to Q$ and the exit function $\psi : Q \times D \to D$. The automaton A is caracterized by a quadruple (D, Q, ϕ, ψ).

The automaton A is inversible if, for every $q \in Q$, the function $\psi(q, \cdot) : D \to D$ is a bijection.

In this case, $\psi(q, \cdot)$ can be identified with an element σ_q of the symmetric group S_d on $d = |D|$ symbols.

There is a convenient way to represent a finite automaton by a marked graph $\Gamma(A)$ which vertices correspond to elements of Q.

Two states $q, s \in Q$ are connected by an arrow labelled by $i \in D$ if $\phi(q, i) = s$; each vertex $q \in Q$ is labelled by a corresponding element σ_q of the symmetric group.

The automata we just defined are non-initial. To make them initial we need to mark some state $q \in Q$ as the initial state. The initial automaton $A_q = (D, Q, \phi, \psi, q)$ acts on the right on the finite and infinite sequences over D in the following way. For every symbol $x \in D$ the automaton immediately gives $y = \psi(q, x)$ and changes its initial state to $\phi(q, x)$.

By joning the exit of A_q to the input of another automaton $B_s = (S, \alpha, \beta, s)$, we get an application which corresponds to the automaton called the composition of A_q and B_s and is denoted by $A_q \star B_s$.

This automaton is formally described as the automaton with the set of the states $Q \times S$ and the transition and exit functions Φ, Ψ defined by

$$\Phi((x,y),i) = (\phi(x,i), \alpha(y, \psi(x,i))),$$

$$\Psi((x,y),i) = \beta(y, \psi(x,i))$$

and the initial state (q, s).

The composition $A \star B$ of two non-initial automata is defined by the same formulas for input and output functions but without indicating the initial state.

Two initial automata are equivalent if they define the same application. There is an algorithm to minimize the number of states.

The automaton which produces the identity map on the set of sequences is called trivial. If A is an invertible then for every state q the automaton A_q admits an inverse automaton A_q^{-1} such that $A_q \star A_q^{-1}$, $A_q^{-1} \star A_q$ are equivalent to the trivial one. The inverse automaton can be formally described as the automaton $(Q, \widetilde{\phi}, \widetilde{\psi}, q)$ were $\widetilde{\phi}(s,i) = \phi(s, \sigma_s^{-1}(i))$, $\widetilde{\psi}(s,i) = \sigma_s^{-1}(i)$ for $s \in Q$. The equivalence classes of finite invertible automata over the alphabet D constitute a group called a group of finite automata which depends on D. Every set of finite automata generates a subgroup of this group.

Now let A be an invertible automaton. Let $Q = \{q_1, \ldots, q_t\}$ be the set of states of A and let $A_{q_1}, \ldots, A_{q_t}$ be the set of initial automata which can be obtained from A. The group $G(A) = \langle A_{q_1}, \ldots, A_{q_t} \rangle$ is called the group generated or determined by A.

Automata groups and wreath products

There is a relation between automata groups and wreath products. For a group of the form $G(A)$ one has the following interpretation.

Let $q \in Q$ be a state of A and let $\sigma_q \in S_d$ be the permutation associated to this state. For every symbol $i \in D$ we denote $A_{q,i}$ the initial automaton having as the initial state $\phi(q,i)$ (then $A_{q,i}$ for $i = 0, 1, \ldots, d-1$ runs over the set of initial automata which are neighbors of A_q, i.e. such that the graph $\Gamma(A)$ has an arrow from q to $\phi(q,i)$).

Let G and F be the groups of finite type such that F be a group of permutation of the set X (we are interested in the case where F is the symmetric group S_d and X is the set $\{0, 1, \ldots, d-1\}$). We define the wreath product $G \wr F$ of these groups as follows. The elements of $G \wr F$ are the couples (g, γ) where $g : X \to G$ is a function such that $g(x)$ is different from the identity element of G, denoted Id, only for a finite number of elements x of X, and where γ is an element of F. The multiplication in $G \wr F$ is defined by:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where

$$g_3(x) = g_1(x)g_2(\gamma_1^{-1}(x)) \quad \text{for } x \in X.$$

We write the elements of the group $G \wr S_d$ as $(a_0, \ldots, a_{d-1})\sigma$, where $a_0, \ldots, a_{d-1} \in G$ and $\sigma \in S_d$.

The group $G = G(A)$ admits the embedding into a wreath product $G \wr S_d$ via the application

$$A_q \to (A_{q,0}, \ldots, A_{q,d-1})\sigma_q,$$

where $q \in Q$. The right expression is called a wreath decomposition of A. We write $A_q = (A_{q,0}, \ldots, A_{q,d-1})\sigma_q$.

For simplicity we denote a the generator of A_a of the group generated by the automaton A.

Let us consider a particularly simple example of a group generated by a finite automaton. Consider the alphabet consisting of two letters 0 and 1. Denote by ε the non trivial element of S_2. Let G be a group generated by a and b defined by

$$a = (b, id)\varepsilon$$

$$b = (a, id).$$

Theorem 2.13. *The G generated by the automaton defined above is not subexponentially amenable, i.e. $G \notin SG$.*

Proof. Suppose that $G \in SG_\alpha$ for α minimal. Then α cannot be 0 as G has exponential growth (the semi-group generated by a and b is free). Moreover α is not a limit ordinal as if $G \in SG_\alpha$ for a limit ordinal then $G \in SG_\beta$ for an ordinal $\beta < \alpha$. Also G is not a direct limit (of an increasing sequence of groups) as it is finitely generated. Thus there exist N, $H \in SG_{\alpha-1}$ such that the following sequence is exact:

$$1 \to N \to G \to H \to 1.$$

For the group G, any normal subgroup $N \triangleleft G$ which is not trivial, has the following property: there exists a subgroup of N with G as a quotient. Each class SG_α is closed with respect to quotients and subgroups. We deduce that $G \in SG_{\alpha-1}$. Contradiction. □

To show amenablity of G one uses a criterion of Kesten concerning random walks on G.

2.3 Random walks

We define an operator $M : l^2(\Gamma) \longrightarrow l^2(\Gamma)$ by:

$$Mf(g) = \frac{1}{|S|}\sum_{s\in S} f(s\,g)$$

We will consider a special case where $S = S^{-1}$. M is self adjoint and we have the following theorem :

Theorem 2.14. *(Kesten 1959) Γ is amenable if and only if $\|M\| = 1$.*

Remark 2.15.

It is easy to see that $\|M\| \leqslant 1$. One has many different characterizations of the notion of amenability (which show that it is a fundamental notion) but the previous one is particularly important.

It is easy to prove that amenability implies that $\|M\| = 1$.

Indeed for a Følner sequence A_n, we take $\chi_{A_n} \in l^2(\Gamma)$ and $\frac{\|M\chi_{A_n}\|}{\|\chi_{A_n}\|} \to 1$, which proves the result.

Definition 2.16. *Consider Γ a group of finite type and a subset S. We say Cay(Γ, S) is a Cayley graph if the vertices of this graph are elements $\gamma \in \Gamma$ with edges $(\gamma, s\,\gamma)$ for $s \in S$ and $\gamma \in \Gamma$.*

Remark 2.17. *Cayley graphs depend on the choice of generators.*

Example 2.18.

In case of $\mathbb{Z}^2$, we can choose $S = \{(0, \pm 1), (\pm 1, 0)\}$. The Cayley graph in this case is a square grid in the plane.

In case of the group $F_2 =< a, b >$, one can take $S = \{a^{\pm 1}, b^{\pm 1}\}$. The graph has degree 4 and is without loops. It is called a tree, i.e. has no cycles.

Definition 2.19.

If G is a group of finite type, a finite subset S which generates G is called symmetric if $s \in S \implies s^{-1} \in S$.

For a group G of finite type and a symmetric subset S which generates G, a random walk on G is simple if all elements of S are equidistributed.

Theorem 2.20. *Let G be a group if finite type and S a finite symmetric subset which generates G. For a simple random walk, let $p_n(id, id)$ be a probability to come back to the identity after n steps. The group G is amenable if and only if:*

$$\lim_{n\to\infty} \sqrt[2n]{p_{2n}(id, id)} = 1.$$

3 Property (T)

3.1 Expanders graphs and applications

Definition 3.1.

Let X be a finite graph and A a subset of X. We define the boundary *of A and denote it ∂A as the arrows with one extremity in A and another in A^c.*

We define an isoperimetric constant *for a graph X as*

$$h(X) = \min\left\{\frac{|\partial A|}{|A|} : A \subseteq X, 1 \leqslant |A| \leqslant \frac{|X|}{2}\right\}$$

Remark 3.2. *In case of infinite graphs we consider finite subsets A of X, and we forget the condition about the cardinality of A.*

Example 3.3.

If $X = Cay(\mathbb{Z}, \{\pm 1\})$ clearly $h(X) = 0$. It is enough to consider the intervals of integers $\{-n, -n+1, \cdots, n-1, n\}$ (whose boundary has cardinality 2) and n tends to infinity.

In general, for any amenable group $h(X) = 0$. Indeed, if A_n is a Følner sequence, then $\frac{|\partial A_n|}{|A_n|} \longrightarrow 0$, as $|\partial A_n| \leqslant \sum_{s\in S} |A_n \Delta s A_n|$ and S is finite.

If $X = Cay(F_2, \{a^{\pm 1}, b^{\pm 1}\})$, then $h(X) = 2$.

Definition 3.4. *A sequence of finite graphs X_n ($|X_n| < \infty$) of degree k (where $k \geqslant 3$) is* a sequence of expanders *if $|X_n| \to \infty$ and there exists $c > 0$ such that $h(X_n) \geqslant c > 0$.*

One cannot produce expanders from quotients of an amenable group as:

Proposition 3.5. *Let Γ be an amenable group generated by a finite subset S, and Γ_n a sequence of finite quotients of Γ.*

Then $Cay(\Gamma_n, S)$ is not a sequence of expanders.

Example 3.6. *To give an example where Γ and Γ_n satisfy the conditions consider $\Gamma = \mathbb{Z}$ and $\Gamma_n = \mathbb{Z}/n\mathbb{Z}$.*

In order to convince ourselves that the notion is non trivial we show the existence of expanding graphs.

Actually we prove that almost all graphs are expanders. One can compare this proof with the proof of existence of non-algebraic numbers, when one shows that almost all real numbers are non-algebraic. This is quite easy, however showing that a particular number is non-algebraic might be very hard.

The proof will be based on the theory of random graphs.

Definition 3.7. *The sets of graphs $X(n, k)$ are defined as follows:*

- *Its vertices are $\{0, 1\} \times \{1, \cdots, n\}$, which can be represented on two lines of n points each. Thus there are $2n$ vertices in each graph $X(n, k)$.*
- *The arrows are defined using k permutations $\pi_1, \cdots, \pi_k \in S_n$ and there is an arrow between $(0, i)$ and $(1, j)$ if there is a certain k such that $\pi_k(i) = j$.*

The graphs thus obtained are of degree k. Even we obtain isomorphic graphs we will not identify them.

The following was proven by Pimsker.

Theorem 3.8. *Suppose $k \geqslant 5$. Then*

$$\lim_{n\to\infty} \frac{\#\{X \in X(n, k) : h(X) \geqslant 1/2\}}{\#X(n, k)} = 1.$$

To prove this theorem we will use another definition of the isoperimetric constant just in the context of graphs $X(n, k)$, which is easier to manipulate here.

Definition 3.9.

We define $\partial' A' = \{x \in X \setminus A' : \text{exists an edge } (x, y) : y \in A'\}$

Let $X \in X(n,k)$. *We define* I *the first row of vertices of* X, *i.e. the points* $\{(0,1),\cdots,(0,n)\}$, *y* O *the second line. Let us define :*

$$h'(X) = \min\left\{\frac{|\partial' A'|}{|A'|} : A' \subseteq I, |A'| \leqslant \frac{|I|}{2} = \frac{n}{2}\right\}$$

It is easy to see that:

Proposition 3.10. *Let* $X \in X(n,k)$, *then* $h(X) \geqslant h'(X) - 1$.

Proof. Instead of showing directly $h(X) \geq \frac{1}{2}$, we show that $h'(X) \geq \frac{3}{2}$.

The number of all permutations $(\pi_1,\cdots,\pi_n)$ is $(n!)^k$.

We estimate the number of permutations which lead to X such that $h'(X) < \frac{3}{2}$. If $h'(X) \leq \frac{3}{2}$, there exists $A \subseteq I$ with $|A| < \frac{1}{2}n$ and $B \subseteq O$ with $|B| = \frac{3}{2}|A|$ such that :

$$\partial' A \subseteq B$$

Therefore we can prove:

$$\sum_{\substack{A\subseteq I \\ |A|\leq \frac{n}{2}}} \sum_{\substack{B\subseteq O \\ |B|=\frac{3}{2}|A|}} \left(\frac{|B|!(n-|A|)!}{(|B|-|A|)!}\right)^k$$

It is not difficult to prove that this expression divided by $(n!)^k$ tends to 0, which end the proof of the theorem. □

3.2 Property (T)

Definition 3.11.

Let $\pi : G \longrightarrow B(\mathcal{H})$ *be a representation of the group* G *on* $\mathcal{H}$. *This representation is* unitary *if* $\pi(g) \in \mathcal{U}(\mathcal{H})$ *for any* g *in* G.

We say that a representation π almost has invariant vectors *if for any* $\varepsilon > 0$ *and any compact subset* K *of* G, *there exists a vector* ξ *of* $\mathcal{H}$ *:*

$$\|\pi(k)\xi - \xi\| < \varepsilon\|\xi\| \text{ for any } k \text{ in } K.$$

Remark 3.12. *If* G *is a discrete group, we substitute the condition* K *compact with* K *finite.*

Example 3.13. *Let $G = \mathbb{Z}$, $\mathcal{H} = l^2(\mathbb{Z})$ and consider the regular representation λ:*

$$\lambda(n)f(m) = f(m+n) \text{ for all } f \in l^2(\mathbb{Z}) \text{ and } n \in \mathbb{Z}.$$

λ almost has invariant vectors. Indeed, let $\xi_n = \chi_{\{-n,\cdots,n\}}$. Consider a finite subset K of $\mathbb{Z}$. Then

$$\lim_{n\to\infty} \frac{\|\xi_n - \lambda(k)\xi_n\|}{\|\xi_n\|} = 0 \text{ for all } k \text{ in } K.$$

But the representation λ does not have any non zero invariant vector. Otherwise this vector would be invariant by λ, i.e. this vector would be a constant function with l^2 norm finite.

Proposition 3.14. *The following conditions are equivalent:*

- *(a) The group G is amenable;*
- *(b) The regular representation of G almost has invariant vectors.*

Proof. The implication $(a) \implies (b)$ is very easy if one uses the Følner sequence. Indeed let $\xi_n = \chi_{A_n}$ and almost invariance follows from the definition of the Følner sequence.

To show the other implication we will construct Følner sequence. Let K be finite or compact subset of G. Consider an almost invariant vector (of norma 1) for a regular representation, i.e. $f \in l^2(G)$ such that $\sum_{g\in G} f(g)^2 = 1$ and $\sum_{g\in G} |f_k(g) - f(g)|^2 \leqslant \varepsilon$ for all k in K. (Here f_k represents f translated by k.)

Let $F = f^2$. Therefore $F \in l^1(G)$, and $\|F\| = 1$. Fix k in K. We can write:

$$\begin{aligned}
\|F_k - F\|_1 &= \sum_{g\in G} |f_k(g)^2 - f(g)^2| = \sum_{g\in G} |f_k(g) - f(g)|\,|f_k(g) + f(g)| \\
&\leqslant \left(\sum_{g\in G} (f_k(g) - f(g))^2\right)^{1/2} \left(\sum_{g\in G} |f_k(g) + f(g)|^2\right)^{1/2} \\
&\leqslant \varepsilon^{1/2} 2\|f\|_2 = 2\varepsilon^{1/2} = \varepsilon'
\end{aligned}$$

and we can assume the quantity is arbitrary small.

For an a positive, let $\mathcal{U}_a = \{g \in G : F(g) \geqslant a\}$. Therefore:

$$1 = \sum_{g \in G} F(g) = \int_0^\infty |\mathcal{U}_a| da$$

and:

$$||F_k - F||_1 = \sum_{g \in G} |F_k(g) - F(g)| = \int_0^\infty |\mathcal{U}_a \Delta k \mathcal{U}_a| da \leqslant \varepsilon'$$

We deduce: $\int_0^\infty |\mathcal{U}_a \Delta k \mathcal{U}_a| da \leqslant \varepsilon' \int_0^\infty |\mathcal{U}_a| da$ and:

$$\int_0^\infty \sum_{k \in K} |\mathcal{U}_a \Delta k \mathcal{U}_a| da \leqslant \varepsilon' |K| \int_0^\infty |\mathcal{U}_a| da$$

and therefore there exists a such that: $\sum_{k \in K} |\mathcal{U}_a \Delta k \mathcal{U}_a| \leqslant \varepsilon' |K| \, |\mathcal{U}_a|$.

Finally let $\varepsilon'' = \varepsilon' |K|$. Therefore:

$$|\mathcal{U}_a \Delta k \mathcal{U}_a| \leqslant \varepsilon'' |\mathcal{U}_a| \text{ for all } k \text{ in } K.$$

□

Definition 3.15. *(Kazhdan 1967) G has* property (T) *if any unitary representation of G which almost has invariant vectors, has an invariant non-zero vector.*

Example 3.16. *$\mathbb{Z}$ does not have property (T). More generally an amenable group has almost invariant vectors but has invariant vectors if and only if it is finite.*

Proposition 3.17. *Let G be a finitely generated group, and S a finite subset which generates G. Suppose G has property (T). If G_n is a sequence of finite quotients G there is a constant $c > 0$ such that*

$$h\left(Cay(G_n, S)\right) \geqslant c,$$

i.e. $Cay(G_n, S)$ is a sequence of expanders.

Proof. Suppose that such a constant c does not exist. Let us construct a representation of G which almost has invariant vectors without invariant vectors.

If such a constant c does not exist, for any $\varepsilon > 0$ there exists a group G_n such that $h(\mathrm{Cay}(G_n, S)) \leqslant \varepsilon$, i.e. there exists a subset $A_n \subseteq G_n$ with $|A_n| \leqslant \frac{|G_n|}{2}$ such that for all s of S,

$$\frac{|A_n \Delta s A_n|}{|A_n|} \leqslant \varepsilon$$

Consider $\chi_{A_n} \in l^2(G_n)$. Therefore

$$\|\chi_{A_n} - s\chi_{A_n}\|_2^2 = \|\chi_{A_n} - \chi_{s\,A_n}\|_2^2 \leqslant \varepsilon\|\chi_{A_n}\|_2^2 \tag{1}$$

i.e. the vector χ_{A_n} is almost invariant. The problem is that χ_{G_n} is an invariant vector.

To avoid this problem, consider the vector space

$$l_0^2(G_n) = \{f \in l^2(G_n), \sum_{g\in G} f(g) = 0\}$$

Let a be a positive constant such that $|A_n| = a|A_n^c|$. Thus $\chi_{A_n} - a\chi_{A_n^c} \in l_0^2(G)$. We can estimate a : $|A_n| \leqslant \frac{|G_n|}{2}$ implies that $a \leqslant 1$.

On the other hand one can write $\chi_{A_n^c} = \chi_{G_n} - \chi_{A_n}$. And so $\chi_{A_n} - a\chi_{A_n^c} = (1+a)\chi_{A_n} - a\chi_{G_n}$.

As (1) for this vector, one writes:

$$\begin{aligned}\|(1+a)\chi_{A_n} - a\chi_{G_n} - ((1+a)\chi_{s\,A_n} - a\chi_{s\,G_n})\| &= (1+a)\,\|\chi_{A_n} - \chi_{s\,A_n}\| \\ &\leqslant \varepsilon(1+a)\|\chi_{A_n}\| \\ &\leqslant 2\varepsilon\|\chi_{A_n} - a\chi_{A_n^c}\|\end{aligned}$$

The first inequality follows from (1). The second from $a \leqslant 1$ and from $\|\chi_{A_n}\| \leqslant \|\chi_{A_n} - a\chi_{A_n^c}\|$. The last one follows as χ_{A_n} and $\chi_{A_n^c}$ are orthogonal in $l^2(G_n)$.

In this way we have constructed a representation of G which almost has invariant vectors and no invariant vectors which contradicts the fact that G has property (T). □

Kazhdan introduced property (T), in order to study properties of discrete subgroups of Lie groups. Consider the following problem:

> If Γ is a discrete subgroup of $\mathrm{SL}(n,\mathbb{R})$ with $\mathrm{Vol}(\mathrm{SL}(n,\mathbb{R})/\Gamma) < \infty$, we ask if Γ is of finite type.

If $n = 2$, it follows from the work of Poincaré that the answer is positive. In order to prove the same statement for $n \geq 3$, Kazhdan proved the following

Proposition 3.18. *(Kazhdan) If G is countable and has property (T), then G is finitely generated.*

Proof. Let us enumerate the elements of G : $G = \{g_1, g_2, \cdots\}$. Now define G_n as the subgroup of G generated by $\{g_1, g_2, \cdots, g_n\}$. If G is not finitely generated, for all n, $|G : G_n| = \infty$. Note that G/G_n is not always a group.

Consider $\oplus_{n\in\mathbb{N}}\, l^2(G/G_n)$. For all n, G acts on G/G_n. For any $n_0 \in \mathbb{N}$, we can consider $\delta_{n_0} \in \oplus l^2(G/G_{n_0})$ which can be defined:

$$\delta_{n_0} = (0, \cdots, 0, \underbrace{\delta_{G_{n_0}}}_{\in\, l^2(G/G_{n_0})}, 0, \cdots)$$

where $\delta_{G_{n_0}}$ is a function equal to 1 for G_{n_0} and 0 for other.

Clearly, δ_n is invariant by G_n. But for any finite subset K of G, exists n such that $K \subseteq G_n$. Therefore this representation almost has invariant vectors without invariant vectors which contradicts property (T). □

What remains to prove is property (T). This is done using the following :

Theorem 3.19. *If G is a Lie group and $\Gamma \subseteq G$ is a lattice (Γ is discrete and Vol$(G/\Gamma) < \infty$) then G has property (T) if and only if Γ has property (T).*

Proof. To prove the implication which is interesting for us, one uses the induced representation of Γ in G. □

Proposition 3.20. *If G has property (T) then $|G/G'| < \infty$, where G' is the abelianisation of G.*

Proof. If G/G' is infinite we have the homomorphism $\phi : G \longrightarrow \mathbb{Z}$. Given that $\mathbb{Z}$ does not have property (T), one can construct a representation of G which almost has invariant vectors without having non-zero invariant vectors. □

We recall the following problem from twenties of the last century:

> Is the Lebesgue measure the only finitely additive measure on S^n, invariant by $SO(n+1)$, defined for Lebesgue measurable subsets of total measure 1?

For $n = 1$, Banach in the twenties showed the answer is negative.

For $n \geqslant 4$, Margulis and Sullivan proved that the answer is affirmative. Their proofs used property (T).

Drinfeld finally proved that the answer is also positive for the most difficult cases $n = 3$ and 2.

The uniqueness is shown using the following property: there are k elements ρ_i of $SO(n+1)$ and $\varepsilon > 0$ such that for any $f \in L_0^2(S^n)$

$$\max_i \|f - \rho_i f\|_2 \geqslant \varepsilon \|f\|_2,$$

which implies that there are no almost invariant vectors.

It is still an open problem to prove that this property is true for rotations ρ_i chosen at random.