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SOME APPLICATIONS OF SET THEORY IN BANACH SPACES AND OPERATOR ALGEBRAS

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I hereby declare that the dissertation is my own work.

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The dissertation is ready to be reviewed.

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Streszczenie

Rozprawa poświęcona jest wybranym problemom w analizie funkcjonalnej, których rozwiązania opierają się na metodach teorii mnogości i topologii. Omawiamy cztery tematy obejmujące zagadnienia takie jak niezmienniki przestrzeni Banacha, zbieżność miar Radona oraz istnienie zanurzeń pewnych C*-algebr w algebrę Calkina.

W pierwszej części rozprawy badamy σ-ideały podzbiorów przestrzeni Banacha generowane przez hiperpłaszczyzny i analizujemy ich standardowe niezmienniki kardynalne: addytywność, liczba pokryciowa, jednorodność i kofinalność. Obliczamy ich wartości dla ośrodkowych przestrzeni Banacha oraz pokazujemy, że niesprzecznie zależą one tylko od gęstości dla wszystkich przestrzeni Banacha. Pozostałe pytania sprowadzają się do rozstrzygnięcia, czy dla każdej nieośrodkowej przestrzeni Banacha X następujące zdania są dowodliwe w ZFC:

- X można pokryć przy pomocy ω_1 hiperpłaszczyzn,
- wszystkie podzbiory X mocy mniejszej niż $\mathrm{cf}([\mathrm{dens}(X)]^\omega)$ można pokryć przeliczalnie wieloma hiperpłaszyznami.

Pokazujemy także, że odpowiedzi na powyższe są twierdzące, jeśli ograniczymy się do jednej z wielu dobrze zbadanych klas przestrzeni Banacha. Pierwsze pytanie związane jest z problemem, czy każda zwarta przestrzeń Hausdorffa z małą przekątną jest metryzowalna, a drugie z dużymi liczbami kardynalnymi.

Drugi temat dotyczy przestrzeni Banacha funkcji ciągłych na przestrzeniach zwartych. Pokazujemy, że jeśli K jest ośrodkową i spójną przestrzenią zwartą, $C(K)$ ma mało operatorów (tzn. każdy ograniczony operator liniowy $T\colon C(K) \to C(K)$ jest postaci $T(f) = fg + S(f)$, gdzie S jest słabo zwarty oraz $g \in C(K)$) oraz przestrzeń $C(K)$ jest izomorficzna z przestrzenią $C(L)$, to K i L są homeomorficzne z dokładnością do skończenie wielu punktów. Następnie, dla każdej liczby naturalnej $n > 0$ konstruujemy, przy założeniu zasady karo Jensena ($\diamondsuit$), przestrzeń zwartą K mającą opisane powyżej własności oraz wymiar pokryciowy równy n. Wnioskujemy, że jeśli L jest przestrzenią zwartą taką, że $C(K)$ i $C(L)$ są izomorficzne, to $\dim L = n$.

Trzeci temat dotyczy teorio-miarowych własności algebr Boole'a oraz powiązanych z nimi przestrzeni Banacha. Definiujemy σ-scentrowane pojęcie forcingu, które forsuje istnienie algebry Boole'a z własnością Grothendiecka i bez własności Nikodyma. W szczególności dowodzimy, że istnienie takiej algebry jest niesprzeczne z negacją hipotezy continuum. Skonstruowana przez nas algebra składa się z borelowskich podzbiorów

zbioru Cantora oraz ma moc równą ω_1. Pokazujemy też, jak usprawnić konstrukcję takiej algebry otrzymanej przez Talagranda przy założeniu hipotezy continuum korzystając z naszej metody.

Ostatnia część rozprawy poświęcona jest algebrze Calkina $\mathcal{Q}(\ell_2)$ tj. algebrze ograniczonych operatorów na ℓ_2 podzielonej przez ideał operatorów zwartych. Pokazujemy, że w modelu Cohena nie istnieje *-zanurzenie algebry $\ell_\infty(\mathcal{Q}(\ell_2)))$ w algebrę $\mathcal{Q}(\ell_2)$. Wnioskujemy z tego, że w modelu Cohena korona stabilizacji algebry $\mathcal{Q}(\ell_2)$ nie jest izomorficzna z $\mathcal{Q}(\ell_2)$.

Słowa kluczowe: algebra Calkina, forcing, hiperpłaszczyzna, mało operatorów, miara Radona, przestrzeń Banacha, teoria mnogości, własność Grothendiecka, własność Nikodyma, wymiar pokryciowy.

Abstract

The dissertation is devoted to selected problems in functional analysis whose solutions rely on set-theoretic and topological methods. We discuss four topics involving issues such as invariants of Banach spaces, convergence of Radon measures or the existence of embeddings of various C*-algebras into the Calkin algebra.

In the first part we study the σ-ideals of subsets of Banach spaces generated by hyperplanes and investigate their standard cardinal characteristics: the additivity, the covering number, the uniformity and the cofinality. We determine their values for separable Banach spaces, and we show that it is consistent that they depend only on the density for all Banach spaces. The remaining questions can be reduced to deciding if the following can be proved in ZFC for every nonseparable Banach space X:

- X can be covered by ω_1-many of its hyperplanes,
- all subsets of X of cardinalities less than $\mathrm{cf}([\mathrm{dens}(X)]^\omega)$ can be covered by countably many hyperplanes.

We also answer these questions in the affirmative in many well-investigated classes of Banach spaces. The first question is related to the problem whether every compact Hausdorff space which has a small diagonal is metrizable and the second to large cardinals.

The second topic concerns Banach spaces of continuous functions on compact spaces. We show that if K is a separable connected compact space, $C(K)$ has few operators (i.e. every bounded linear operator $T\colon C(K) \to C(K)$ is of the form $T(f) = fg + S(f)$, where S is weakly compact and $g \in C(K)$) and $C(K)$ is isomorphic to $C(L)$ for some compact space L, then K and L are homeomorphic modulo finitely many points. Next, for every natural number $n > 0$ we construct, assuming Jensen's diamond principle ($\diamondsuit$), a compact space K that has the covering dimension equal to n and possesses the above mentioned properties. We conclude that if L is a compact space such that $C(L)$ is isomorphic to $C(K)$, then $\dim L = n$.

The third topic concerns measure-theoretic properties of Boolean algebras and related Banach spaces. We define a σ-centered notion of forcing that forces the existence of a Boolean algebra with the Grothendieck property and without the Nikodym property. In particular, we prove that the existence of such an algebra is consistent with the negation of the continuum hypothesis. The algebra we construct consists of Borel subsets of the Cantor set and has cardinality ω_1. We also show how to apply our method to streamline Talagrand's construction of such an algebra under the continuum hypothesis.

The last part of the dissertation is devoted to the Calkin algebra $\mathcal{Q}(\ell_2)$ i.e. the C*-algebra of bounded operators on ℓ_2 divided by the ideal of compact operators. We show that in the Cohen model there is no *-embedding of ℓ_∞-sum of Calkin algebras into $\mathcal{Q}(\ell_2)$. We conclude that in the Cohen model the corona of the stabilization of $\mathcal{Q}(\ell_2)$ is not isomorphic to $\mathcal{Q}(\ell_2)$.

Keywords: Banach space, Calkin algebra, covering dimension, few operators, forcing, Grothendieck property, hyperplane, Nikodym property, Radon measure, set theory.

Contents

Chapter 1

Introduction

1.1 Overview

The dissertation focuses on applications of set theory and general topology in functional analysis. These include results on cardinal invariants of Banach spaces, constructions of Banach spaces with special properties and the consistency of the non-existence of some embeddings in the category of C*-algebras. Most of the results concern nonseparable Banach spaces. Set-theoretic methods involve infinitary combinatorics, consistency proofs by forcing and the use of additional axioms.

The history of applications of set theory and topology in the theory of Banach spaces goes back to the foundations of functional analysis. It became clear from the very beginning that even the most basic results strongly rely on the use of the axiom of choice and basic topological principles such as the Baire category theorem or Tychonoff's theorem.

Along with the development of set theory it naturally turned out that many properties of classical Banach spaces are closely related to combinatorial and topological structures. For instance, the Banach space ℓ_∞ of bounded sequences of real numbers is isometric to the space $C(\beta\mathbb{N})$ of continuous real-valued functions on the Čech-Stone compactification of the natural numbers. Thus, the analysis of $\beta\mathbb{N}$ and the Boolean algebra of its clopen subsets (which is isomorphic to $\mathcal{P}(\mathbb{N})$) is a useful tool for investigating properties of ℓ_∞ and related spaces (see e.g. [37, 64]).

More generally, the properties of Banach spaces of the form $C(K)$ (where K is a compact Hausdorff space) may be deduced from the topological properties of K. For instance, K is metrizable if and only if $C(K)$ is separable. If K is extremally disconnected (which is equivalent to being projective in the category of compact Hausdorff spaces [54]), then $C(K)$ is an injective Banach space [76] (however, the problem if every injective Banach space is isomorphic to $C(K)$ for K extremally disconnected remains open).

Another advantage of Banach spaces of the form $C(K)$ is the description of bounded functionals as Radon measures on K coming from the Riesz representation theorem [119, Theorem 18.4.1]. Moreover, if K is zero-dimensional, then Radon measures on K are in the natural correspondence with finitely additive bounded measures on the Boolean algebra Clop(K) consisting of clopen subsets of K. These facts will be crucial for many of the results contained in the dissertation.

Among other examples of useful set-theoretic tools, we can mention almost disjoint families i.e. uncountable families of subsets of $\mathbb{N}$ in which intersections of two distinct members are finite. Such families are a source of interesting examples of Banach spaces. For example, they were used in the construction of the famous Johnson-Lindenstrauss space JL_2, which is the first example of a Banach space that is not WCG, but whose dual is WCG [70]. They also appear in the context of few operators and decompostions [86]. Haydon used almost disjoint families to show that Banach spaces of continuous functions on the Stone spaces of Boolean algebras with the Subsequential Completeness Property have the Grothendieck property [67, Proposition 1B]. We use Haydon's idea in Chapter 3 in the proof of Theorem 3.4.8.

At the end of the last century it turned out that the answers for many natural questions concerning nonseparable Banach spaces are independent of ZFC. Most known examples include, among others, Kaplansky's conjecture asking if there is a discontinuous homomorphism from a Banach algebra of the form $C(K)$ (with pointwise multiplication) [24], the question whether every nonseparable Banach space admits an uncountable biorthogonal system [65] or the question whether the ideal of compact operators on a separable Hilbert space may be written as the sum of two properly smaller ideals [112]. There are also statements that are known to be consistent, but still open in ZFC. For instance, Drewnowski and Roberts showed that the continuum hypothesis (CH) implies that the Banach space ℓ_∞/c_0 is primary (i.e. for every decomposition $\ell_\infty/c_0 = X \oplus Y$ at least one of the spaces X, Y is isomorphic to ℓ_∞/c_0) [37]. Many of our main results consist of proofs of consistency or independence of ZFC involving the use of additional axioms or the method of forcing (see e.g. Theorem 2.1.3, Theorem 3.4.9, Theorem 4.5.15).

Recently set-theoretic methods received a lot of attention also in the category of C*-algebras. First examples come from the works of Akemann and Anderson [1, 2, 4–6]. Most of the results concern objects such as the algebra $\mathcal{B}(H)$ of bounded operators on a separable Hilbert space H and the Calkin algebra $\mathcal{Q}(H)$ of bounded operators on H modulo compact operators, which are treated as the non-commutative analogues of $\beta\mathbb{N}$ and $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ (or ℓ_∞ and ℓ_∞/c_0) respectively. For instance, due to works of Phillips, Weaver and Farah [41, 106] the existence of outer automorphisms of the Calkin algebra is independent of ZFC. Another issue is the universality in the category of C*-algebras. It follows from CH that the Calkin algebra is universal in the class of C*-algberas of density not bigger than $\mathfrak{c}$ [46], while there are models of ZFC in which this statement is false [133]. In Chapter 5 we discuss possible *-embeddings of ℓ_∞-sums of Calkin algebras into the Calkin algebra in the Cohen model.

The dissertation is divided into four independent parts, each presented in one of the following chapters.

Chapter 2: Coverings of Banach spaces and their subsets by hyperplanes

In this chapter we introduce and investigate some cardinal characteristics of σ-ideals generated by hyperplanes in Banach spaces.

We say that a subspace of a Banach space X is a hyperplane of X if it is the kernel of a non-zero bounded linear functional of X. For a Banach space X of dimension bigger than 1 we consider the ideal $\mathcal{I}(X)$ consisting of all subsets of X that may be covered by countably many hyperplanes and investigate its standard cardinal characteristics (which are motivated by the characteristics of the ideal of Lebesgue measure zero sets and the ideal of meager sets of the reals, cf. [14, p. 12]):

- $\mathfrak{add}(X)$ = the minimal cardinality of a family of sets from $\mathcal{I}(X)$ whose union is not in $\mathcal{I}(X)$,
- $\mathfrak{cov}(X)$ = the minimal cardinality of a family of sets from $\mathcal{I}(X)$ whose union is equal to X,
- $\mathfrak{non}(X)$ = the minimal cardinality of a subset of X that is not in $\mathcal{I}(X)$,
- $\mathfrak{cof}(X)$ = the minimal cardinality of a family of sets from $\mathcal{I}(X)$ such that each member of $\mathcal{I}(X)$ is contained in some element of that family.

It turns out that the values of $\mathfrak{add}$ and $\mathfrak{cof}$ are trivial in the context of the considered ideals. Namely, for any Banach space X of dimension bigger than 1 we have $\mathfrak{add}(X) = \omega_1$ and $\mathfrak{cof}(X) = |X^*|$, where X^* is the dual space of X. Moreover, if X is separable, then $\mathfrak{non}(X) = \omega_1$ and $\mathfrak{cov}(X) = \mathfrak{c}$ (Theorem 2.1.1).

We determine the values of $\mathfrak{cov}$ and $\mathfrak{non}$ assuming additional axioms (the Generalized Continuum Hypothesis GCH or Martin's Maximum MM):

Theorem 2.1.3. *Assume* GCH *or* MM. *Let X be a nonseparable Banach space. Then*

(1) $\mathfrak{cov}(X) = \omega_1$,
(2) $\mathfrak{non}(X) = \mathrm{dens}(X)$ *if* $\mathrm{cf}(\mathrm{dens}(X)) > \omega$,
(3) $\mathfrak{non}(X) = \mathrm{dens}(X)^+$ *if* $\mathrm{cf}(\mathrm{dens}(X)) = \omega$.

Moreover, the same is consistent with any possible size of the continuum $\mathfrak{c}$. If violations of the above equalities concerning $\mathfrak{non}$ are consistent, then so is the existence of a measurable cardinal.

We also calculate $\mathfrak{cov}$ and $\mathfrak{non}$ and for many classes of nonseparable Banach spaces in ZFC (Theorem 2.1.6 and Theorem 2.1.7).

Surprisingly, in all cases we know the considered characteristics depend only on the density of the corresponding Banach spaces. We conjecture that this holds in ZFC for all Banach spaces.

Another interesting issues are the connections between the covering number and an open problem concerning small diagonals in non-metrizable compact spaces (see Proposition 2.4.10), and the problem of the existence of overcomplete sets in nonseparable Banach spaces (Theorem 2.1.3).

This chapter covers the content of the paper [56] (joint work with Piotr Koszmider).

Chapter 3: A Banach space $C(K)$ reading the dimension of K

This chapter is devoted to the question whether the dimension of a compact space K may be an isomorphic invariant of the Banach space $C(K)$ of continuous functions on K.

The classical theorem of Miljutin says that if K and L are uncountable compact metric spaces, then the Banach spaces $C(K)$ and $C(L)$ are isomorphic. In particular, $C([0,1]^n)$ and $C([0,1]^m)$ are isomorphic for any $n, m > 0$, so in general the dimension of the underlying compact space is not an invariant of given Banach space of continuous functions. Koszmider constructed an example of a compact space K such that $C(K)$ is not isomorphic to any $C(L)$ for L zero-dimensional [80]. We show that under the assumption of Jensen's diamond principle ($\diamondsuit$) this result may be improved. Namely, we show the following:

Theorem 3.4.9. *Assume $\diamondsuit$. Then for every $k \in \omega \cup \{\infty\}$ there is a compact Hausdorff space K such that $\dim(K) = k$ and whenever $C(K) \sim C(L)$, $\dim(L) = k$.*

The proof of this theorem consists of two parts. First, we show that if K is a perfect separable compact Hausdorff space such that $C(K)$ has few operators (i.e. every bounded operator $T\colon C(K) \to C(K)$ is of the form $T(f) = gf + S(f)$, where $S\colon C(K) \to C(K)$ is weakly compact), then every L such that $C(K) \sim C(L)$ differs from K only on a finite set (Theorem 3.2.19). More precisely: there are open subsets $U \subseteq K, V \subseteq L$ and finite sets $E \subseteq K, F \subseteq L$ such that U, V are homeomorphic and $K = U \cup E, L = V \cup F$. It follows that if $C(L) \sim C(K)$, then $\dim K = \dim L$.

Then using $\diamondsuit$ for every $k \in \omega \cup \{\infty\}$ we construct a compact Hausdorff space of dimension k with the above mentioned properties (see Theorem 3.4.8).

This chapter covers the content of the paper [55].

Chapter 4: Grothendieck vs Nikodym

This chapter focuses on a longstanding open problem concerning the existence of a Boolean algebra with the Grothendieck property, but without the Nikodym property.

We say that a Boolean algebra $\mathbb{B}$ has the Grothendieck property if the Banach space $C(\mathrm{St}(\mathbb{B}))$ of continuous real-valued functions on the Stone space of $\mathbb{B}$ has the Grothendieck property (i.e. the weak*-convergence of sequences in $C(\mathrm{St}(\mathbb{B}))^*$ is equivalent to the weak convergence). We say that $\mathbb{B}$ has the Nikodym property if every pointwise convergent (here by points we mean elements of $\mathbb{B}$) sequence of finitely additive bounded measures on $\mathbb{B}$ is bounded in norm.

In 1984 Talagrand constructed assuming CH an example of a Boolean algebra with the Grothendieck property and without the Nikodym property. His algebra consists of Borel subsets of the Cantor set with certain symmetry property (we call such sets *balanced*). We modify Talagrand's approach, which allows us to obtain a consistent example of such Boolean algebra in a model satisfying $\neg\mathsf{CH}$ (Theorem 4.5.15). More precisely, we define a σ-centered notion of forcing $\mathbb{P}$ that forces the existence of a Boolean algebra of cardinality ω_1 with the Grothendieck property and without the Nikodym property. In the model obtained from $\mathbb{P}$ we have $\mathfrak{p} = \mathfrak{s} = \mathrm{cov}(\mathcal{M}) = \omega_1$ (Corollary 4.5.14). We also show how to construct a balanced Boolean algebra with the Grothendieck property under CH using our modification of Talagrand's method (Theorem 4.4.8)

This chapter covers the content of the paper [57] (joint work with Agnieszka Widz).

Chapter 5: The Calkin algebra in the Cohen model

In this chapter we discuss the problem of the existence of *-embeddings of some C*-algebras of density $\mathfrak{c}$ into the Calkin algebra i.e. the quotient algebra $\mathcal{Q}(\ell_2) = \mathcal{B}(\ell_2)/\mathcal{K}(\ell_2)$ of bounded operators on ℓ_2 modulo compact operators.

Recently, the algebra $\mathcal{Q}(\ell_2)$ has received a lot of attention in the context of applications of set theory, since it shares some important properties with $\mathcal{P}(\mathbb{N})/Fin$. For instance, CH implies that $\mathcal{Q}(\ell_2)$ is universal in the class of C*-algebras of the density continuum [46]. Inspired by a result of [20] which says, that in the Cohen model $\ell_\infty(\ell_\infty/c_0)$ does not embed into ℓ_∞/c_0 as a Banach space, we show:

Theorem 5.3.3. *In the Cohen model there is no *-embedding of $\ell_\infty(c_0(\omega_2))$ into $\mathcal{Q}(\ell_2)$. In particular, there is no *-embedding of $\ell_\infty(\mathcal{Q}(\ell_2))$ into $\mathcal{Q}(\ell_2)$.*

As a consequence we get that $\mathcal{Q}(\ell_2)$ is not isomorphic to the corona of the stabilization of the Calkin algebra.

Theorem 5.3.8. *In the Cohen model there is no *-embedding of $\mathcal{Q}(\mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2))$ into $\mathcal{Q}(\ell_2)$.*

1.2 Notation and terminology

Most of the notation that we use should be standard. For unexplained terminology check [38, 39, 69].

Set theory

For the purpose of the thesis we denote by $\mathbb{N}$ the set of positive integers and we put $\omega = \mathbb{N} \cup \{0\}$. The symbols $\mathbb{Q}$ and $\mathbb{R}$ denote the rationals and the reals respectively. The cardinality of a set A is denoted by $|A|$. For $n \in \mathbb{N}$, ω_n stands for the n-th uncountable cardinal, ω_ω is the smallest cardinal which is greater than ω_n for each $n \in \mathbb{N}$. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$ and is called the continuum. If α is an ordinal number, then $\mathrm{cf}(\alpha)$ denotes its cofinality. *Lim* stands for the class of all limit ordinals. Odd and Even stand for the classes of odd and even ordinals respectively. A subset $S \subseteq \omega_1$ is called stationary, if it has non-empty intersection with every closed and unbounded subset of ω_1.

The symbol $[A]^{<\omega}$ denotes the family of all finite subsets of A, $[A]^\omega$ is the family of countable subsets of A, $\mathrm{cf}([A]^\omega)$ denotes the cofinality of $[A]^\omega$ considered as the set partially ordered by inclusion, that is the minimal cardinality of a family of countable subsets of A such that any countable subset of A is included in an element of the family.

By $f \restriction A$ we mean the restriction of a function f to a set A. If f is a partial function, then $f \restriction A = f \restriction (\mathrm{dom}(f) \cap A)$, where $\mathrm{dom}(f)$ is the domain of f. The symbol $\sum_{n\in\omega} f_n$ will always denote the pointwise sum of functions f_n (if the sum exists).

Axioms

ZFC denotes Zermelo-Fraenkel set theory with the axiom of choice. We say that a sentence φ is relatively consistent with a set of axioms if its negation $\neg\varphi$ cannot be proven from those axioms unless assuming ZFC leads to a contradiction. We usually skip the word "relatively". A sentence φ is independent of ZFC if both $\varphi, \neg\varphi$ are consistent with ZFC. The continuum hypothesis CH means '$\mathfrak{c} = \omega_1$'. The generalized continuum hypothesis GCH means $2^\kappa = \kappa^+$ for every cardinal κ. MM stands for Martin's Maximum and PFA for Proper Forcing Axiom. It is known that MM implies PFA and PFA implies $\mathfrak{c} = \omega_2$ (for the definitions of MM and PFA and proofs of mentioned facts check [69]). Jensen's diamond principle ($\diamondsuit$) stands for the following sentence (for other equivalent formulations see [28]): there is a sequence of sets $A \subseteq \alpha$ for $\alpha < \omega_1$ such that for any subset $A \subseteq \omega_1$ the set $\{\alpha : A \cap \alpha = A_\alpha\}$ is stationary in ω_1. It is a well-known fact, that $\diamondsuit$ implies CH.

General topology

All topological spaces we consider are Hausdorff. We denote the closure of A by $\overline{A}$. For a topological space X the set $\Delta(X) = \{(x, x) : x \in X\}$ is called the diagonal of X. The covering dimension (also known as Lebesgue covering dimension or topological dimension, [38, Definition 1.6.7]) of X is denoted by $\dim X$. The set X' is the subset of X consisting of non-isolated points in X. A sequence $(x_n)_{n\in\omega}$ is said to be non-trivial, if it is not eventually constant. We say that a topological space X is c.c.c. if every family of pairwise disjoint open subsets of X is countable. We say that X is scattered if there every subset $Y \subseteq X$ contains an isolated point in Y. By basic open subset of $[0,1]^{\omega_1}$ we mean the product $\prod_{\alpha<\omega_1} U_\alpha$ where each $U_\alpha \subseteq [0,1]$ is a relatively open interval with rational endpoints and $U_\alpha = [0,1]$ for all but finitely many α's.

The Cantor set

For a sequence s, its m-th term will be denoted by s_m. By the Cantor set we mean the set $C = \{-1,1\}^{\mathbb{N}}$ of the sequences with values in $\{-1,1\}$ with the usual product topology. The set $\{-1,1\}^n$ consists of all sequences of length n with values in $\{-1,1\}$.

For $s \in \{-1,1\}^n$ we put

$$\langle s\rangle = \{x \in C : x \restriction n = s\},$$

where $x \restriction n$ is the sequence of first n elements of x. The family of all Borel subsets of C will be denoted by $\mathrm{Bor}(C)$. For a set $Z \subseteq C$ the symbol χ_Z stands for the characteristic function of Z.

The symbol λ will denote the normalized Haar measure on C (considered as a group with coordinate-wise multiplication). In particular, $\lambda(\langle s\rangle) = 1/2^n$ for $s \in \{-1,1\}^n$.

Boolean algebras.

For the basic terminology concerning Boolean algebras see [25, 53, 78]. We will focus on Boolean algebras consisting of Borel subsets of the Cantor set endowed with the standard operations $\cup, \cap, \setminus$. The symmetric difference of sets A and B will be denoted by $A\triangle B$. For $n \in \mathbb{N}$, $\mathbb{A}_n$ is the finite subalgebra of $\mathrm{Bor}(C)$ generated by $\{\langle s\rangle : s \in \{-1,1\}^n\}$. The family of all clopen subsets of C will be denoted by $\mathrm{Clop}(C)$. Note that $\mathrm{Clop}(C) = \bigcup_{n\in\mathbb{N}} \mathbb{A}_n$. For a Boolean algebra $\mathbb{A}$ we denote by $\mathrm{at}(\mathbb{A})$ the set of its atoms. In particular, $\mathrm{at}(\mathbb{A}_n) = \{\langle s\rangle : s \in \{-1,1\}^n\}$. The Stone space of $\mathbb{A}$ will be denoted by $\mathrm{St}(\mathbb{A})$. For $A \in \mathbb{A}$ we denote by $[A]$ the corresponding clopen subset of $\mathrm{St}(\mathbb{A})$. A family $\{H_n\}_{n\in\mathbb{N}} \subseteq \mathbb{A}$ is called an antichain, if $H_n \cap H_m = \varnothing$ for $n \neq m$. For a Boolean algebra $\mathbb{B}$ and a subset $B \subseteq C$ we put

$$\mathcal{F}(\mathbb{B}, B) = \{A \cap B, A\setminus B : A \in \mathbb{B}\}.$$

Banach spaces

We will use one symbol $\|\cdot\|$ to denote norms in all considered Banach spaces - this should not lead to misunderstandings. For a Banach space X, its density $\mathrm{dens}(X)$ is the minimal cardinality of a dense subset in X (in the norm topology). X^* stands for the Banach space of bounded linear functionals on X (with the operator norm). For $S \subseteq X$ by $\mathrm{span}(S)$ we denote the smallest linear subspace of X containing S and

$\overline{\mathrm{span}(S)}$ stands for its closure. The symbol $\ker(x^*)$ denotes the kernel of a functional $x^* \in X^*$. If $x_i \in X, x_i^* \in X^*$ for $i \in I$ are such that $x_i^*(x_j) = \delta_{i,j}$, then $(x_i, x_i^*)_{i\in I}$ is called a biorthogonal system. If moreover $X = \overline{\mathrm{span}\{x_i : i \in I\}}$, then such a system is called fundamental. For the definition and various characterizations of WLD spaces see [60].

For a compact space K we denote by $C(K)$ the Banach space of real-valued continuous functions on K with the standard supremum norm. If $x \in K$, then $\delta_x \in C(K)^*$ is defined by $\delta_x(f) = f(x)$. $C_I(K)$ denotes the subset of $C(K)$ of functions with the range included in the interval $[0, 1]$. For Banach spaces X and Y, a bounded linear operator $T: X \to Y$ is said to be weakly compact if the closure of $T[B_X]$ is compact in the weak topology in Y (here B_X stands for the unit ball in X). The symbol $X \sim Y$ means that X and Y are isomorphic as Banach spaces. $\mathcal{B}(X)$ denotes the algebra of all bounded operators on a Banach space X (with the operator norm). An operator $T: C(K) \to C(L)$ is multiplicative, if $T(fg) = T(f)T(g)$.

For any set A by $c_0(A)$ we denote the Banach space of functions $f: A \to \mathbb{R}$ such that for each $\varepsilon > 0$ there is finitely many $a \in A$ with $|f(a)| > \varepsilon$ with the supremum norm. For $1 \le p < \infty$ by $\ell_p(A)$ we denote the Banach space of functions $f: A \to \mathbb{R}$ such that $\|f\|^p = \sum_{a\in A} |f(a)|^p < \infty$. By $\ell_\infty(A)$ we mean the Banach space of bounded functions $f: A \to \mathbb{R}$ with the supremum norm. The symbol $\ell_\infty^c(A)$ denotes the subspace of $\ell_\infty(A)$ consisting of functions with countable supports (where the support of $f \in \ell_\infty(A)$ is the set $\{a \in A : f(a) \neq 0\}$). We also write $c_0(\mathbb{N}) = c_0, \ell_p(\mathbb{N}) = \ell_p$ and $\ell_\infty(\mathbb{N}) = \ell_\infty$. We will denote by $\mathcal{L}_2(C)$ the real Hilbert space of square-integrable (with respect to λ) functions on C with the inner product

$$\langle f, g\rangle = \int_C fg d\lambda.$$

Radon measures on compact spaces

For a compact space K we will identify the space of bounded linear functionals on $C(K)$ with the space $M(K)$ of Radon measures on K (the identification is given by the Riesz representation theorem). For every $\alpha < \omega_1$ we have an embedding $E_\alpha: C([0, 1]^\alpha) \to C([0, 1]^{\omega_1})$ given by $E_\alpha(f) = f \circ \pi_\alpha$, where $\pi_\alpha: [0, 1]^{\omega_1} \to [0, 1]^\alpha$ is the natural projection. For a Radon measure μ on $[0, 1]^{\omega_1}$ we will denote by $\mu \restriction C([0, 1]^\alpha)$ the restriction of μ treated as a functional on $C([0, 1]^{\omega_1})$ to the subspace $E_\alpha[C([0, 1]^\alpha)]$. Equivalently, $\mu \restriction C([0, 1]^\alpha)$ is a measure on $[0, 1]^\alpha$ given by

$$\mu \restriction C([0, 1]^\alpha)(A) = \mu(\pi_\alpha^{-1}(A)).$$

For any measure μ we denote by $|\mu|$ its variation.

We say that a sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq M(K)$ converges weakly, if it is convergent in the weak topology of the Banach space $M(K)$. We say that $(\mu_n)_{n\in\mathbb{N}} \subseteq M(K)$ is weak*-convergent, if it converges in the weak* topology, where $M(K)$ is treated as the dual space to $C(K)$.

For a Radon measure μ on K a Borel set $F \subseteq K$ is a Borel support of μ, if $\mu(X) = 0$ for every Borel $X \subseteq K \setminus F$. We say that a sequence $(\mu_n)_{n\in\mathbb{N}}$ of Radon measures has pairwise disjoint Borel supports, if there are pairwise disjoint Borel sets $(F_n)_{n\in\mathbb{N}} \subseteq K$ such that F_n is a Borel support of μ_n for every $n \in \mathbb{N}$. Note that, unlike the support of a measure, a Borel support is not unique.

Measures on Boolean algebras

For the general theory of measures on Boolean algebras see [119, Chapter V]. Throughout Chapter 4 we will discuss canonical measures (witnesses to the lack of the Nikodym property) φ_n for $n \in \mathbb{N}$, given by the formula

$$\varphi_n(A) = \int_A \delta_n d\lambda$$

for $A \in \mathrm{Bor}(C)$, where $\delta_n \colon C \to \{-1, 1\}, \delta_n(x) = x_n$.

In what follows a measure on a Boolean algebra $\mathbb{A}$ is always a finitely additive signed bounded measure on $\mathbb{A}$. We will call such measures concisely "measures on $\mathbb{A}$". If μ is a measure on a Boolean algebra $\mathbb{A}$ and $\mathbb{B} \subseteq \mathbb{A}$ is a subalgebra, then $\mu \restriction \mathbb{B}$ denotes the restriction of μ to $\mathbb{B}$. For a measure μ on $\mathbb{A}$ we define its variation $|\mu|$ as a measure on $\mathbb{A}$ given by

$$|\mu|(X) = \sup\{|\mu(A)| + |\mu(B)| : A, B \in \mathbb{A}, A, B \subseteq X, A \cap B = \varnothing\},$$

and its norm (total variation) as

$$\|\mu\| = |\mu|(1),$$

where 1 is the biggest element of $\mathbb{A}$. Note that for every $n \in \mathbb{N}$ we have $|\varphi_n| = \lambda$ and $\|\varphi_n\| = 1$. If μ is non-negative and $\|\mu\| = 1$, then μ is called a probability measure.

Every measure on a Boolean algebra $\mathbb{A}$ extends uniquely to a Radon measure on the space $\mathrm{St}(\mathbb{A})$ (see [119, Section 18.7]). If μ is a measure on a Boolean algebra $\mathbb{A}$, then $\widetilde{\mu}$ denotes the corresponding Radon measure on $\mathrm{St}(\mathbb{A})$. In particular, $\|\widetilde{\mu}\| = \|\mu\|$ and $|\widetilde{\mu}| = |\mu|$.

A sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on a Boolean algebra $\mathbb{A}$ is said to be pointwise convergent, if there is a measure μ on $\mathbb{A}$ such that $\mu_n(A) \xrightarrow{n\to\infty} \mu(A)$ for every $A \in \mathbb{A}$. It is a well-known fact, that a sequence of Radon measures $(\widetilde{\mu}_n)_{n\in\mathbb{N}}$ on $\mathrm{St}(\mathbb{A})$ is weak*-convergent if and only if the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in the norm and pointwise convergent on $\mathbb{A}$.

Forcing

Most of the notation concerning forcing should be standard. For the unexplained terminology see [14, 69, 89]. The universe of sets will be denoted by V. For a forcing notion $\mathbb{P}$ we denote by $V^{\mathbb{P}}$ a generic extension of V obtained by forcing with $\mathbb{P}$. The evaluation of a constant c in the class V is denoted by c^V.

For the purpose of Section 4.5 we will identify Borel subsets of C with their codes with respect to some absolute coding (see [69, Section 25]). We say that a notion of forcing $\mathbb{P}$ is σ-centered , if there are countably many families $\mathcal{A}_n \subseteq \mathbb{P}$ for $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n = \mathbb{P}$ and for every $n \in \mathbb{N}$ and every finite set $\{p_i\}_{i\in I} \subseteq \mathcal{A}_n$ there is $p \in \mathbb{P}$ so that $p \leq p_i$ for all $i \in I$.

A $\mathbb{P}$-name $\dot{x}$ is a called nice name for a subset of $M \in V$, if it is of the form $\dot{x} = \bigcup_{m\in M}\{\check{m}\} \times A_m$, where each A_m is an antichain in $\mathbb{P}$ and $\check{m}$ is the canonical name for m. Given an automorphism $\sigma\colon \mathbb{P} \to \mathbb{P}$ and a $\mathbb{P}$-name $\dot{x} = \{(\dot{y}_i, p_i) : i \in I\}$ we denote $\sigma(\dot{x}) = \{(\sigma(\dot{y}_i), \sigma(p_i)) : i \in I\}$ (cf. [69, p. 221]). In particular, we have $\sigma(\check{m}) = \check{m}$ for $m \in V$.

C*-algebras

For basic terminology and information on C*-algebras see [15, 43]. By ℓ_2 we denote the Hilbert space of square-summable sequences of complex numbers with the standard inner product $\langle (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\rangle = \sum_{n=1}^{\infty} a_n\bar{b}_n$. The symbol $\mathcal{B}(\ell_2)$ denotes the C*-algebra of bounded operators on ℓ_2 (with the standard operator norm). The ideal of compact operators in $\mathcal{B}(\ell_2)$ is denoted by $\mathcal{K}(\ell_2)$. We define the Calkin algebra as the quotient $\mathcal{Q}(\ell_2) = \mathcal{B}(\ell_2)/\mathcal{K}(\ell_2)$. For a C*-algebra $\mathcal{A}$ we denote by $\mathcal{M}(\mathcal{A})$ its multiplier algebra and define its corona as $\mathcal{Q}(\mathcal{A}) = \mathcal{M}(\mathcal{A})/\mathcal{A}$ (see Definition 5.3.4).

We say that an element $p \in \mathcal{A}$ of a C*-algebra $\mathcal{A}$ is a projection, if $p = p^2 = p^*$. The set of all projections in $\mathcal{A}$ is denoted by $\mathrm{Proj}(\mathcal{A})$ and forms a poset with the ordering given by $p \leq q$ if and only if $pq = p$. If $P, Q \in \mathrm{Proj}(\mathcal{B}(\ell_2))$, then $P \leq^{\mathcal{K}} Q$ means that $P - PQ \in \mathcal{K}(\ell_2)$ or - equivalently - that $\pi(P) \leq \pi(Q)$, where $\pi\colon \mathcal{B}(\ell_2) \to \mathcal{Q}(\ell_2)$ is the canonical quotient map.

Chapter 2

Coverings of Banach spaces and their subsets by hyperplanes

2.1 Introduction

All Banach spaces considered in this chapter are of dimension bigger than 1 and over the reals. A hyperplane of a Banach space X is a one-codimensional closed subspace of X. It is easy to see that it is nowhere dense in X. The family of all hyperplanes of X will be denoted by $\mathcal{H}(X)$. Given a Banach space X one can define the hyperplane ideal $\mathcal{I}$ of X as

$$\mathcal{I}(X) = \{Y \subseteq X : \exists \mathcal{F} \subseteq \mathcal{H}(X) \ \ Y \subseteq \bigcup \mathcal{F},\ \mathcal{F} \text{ countable}\}.$$

That is, $\mathcal{I}(X)$ is the family of all subsets of X which can be covered by countably many hyperplanes of X. By the Baire category theorem $X \notin \mathcal{I}(X)$ for any Banach space X. We consider the standard cardinal characteristics of the ideal $\mathcal{I}(X)$:

- $\mathfrak{add}(X)$ is the minimal cardinality of a family of sets from $\mathcal{I}(X)$ whose union is not in $\mathcal{I}(X)$,
- $\mathfrak{cov}(X)$ is the minimal cardinality of a family of sets from $\mathcal{I}(X)$ whose union is equal to X,
- $\mathfrak{non}(X)$ is the minimal cardinality of a subset of X that is not in $\mathcal{I}(X)$,
- $\mathfrak{cof}(X)$ is the minimal cardinality of a family of sets from $\mathcal{I}(X)$ such that each member of $\mathcal{I}(X)$ is contained in some element of that family.

Such cardinal characteristics are standard tools for investigating the combinatorial properties of a σ-ideal. The most known case are their applications to the understanding of the ideal of Lebesgue measure zero sets and the ideal of meager sets of the reals (see e.g. [14]). It is easy to observe that if the ideal is proper and contains all singletons we have the following inequalities: $\mathfrak{add} \leq \mathfrak{cov} \leq \mathfrak{cof}$ and $\mathfrak{add} \leq \mathfrak{non} \leq \mathfrak{cof}$. The purpose of this chapter is to investigate the possible values of the above cardinals for the ideal $\mathcal{I}(X)$ and understand how they depend on X. A somewhat surprising conclusion is that the values depend almost entirely only on the density of X and the X^* or even are fixed for all separable and all nonseparable Banach spaces. The first result presented in

Section 2.3.1 describes these values for all separable Banach spaces. It is an immediate consequence of appropriately formulated result from [77]:

Theorem 2.1.1. *Suppose that X is a separable Banach space of dimension bigger than 1. Then the following equalities hold:*

- $\mathfrak{add}(X) = \omega_1$,
- $\mathfrak{non}(X) = \omega_1$,
- $\mathfrak{cov}(X) = \mathfrak{c}$,
- $\mathfrak{cof}(X) = \mathfrak{c}$.

In fact the values of $\mathfrak{add}$ and $\mathfrak{cof}$ are always trivial (Propositions 2.3.2, 2.3.3) due to an elementary fact that $H \subseteq G$ implies $H = G$ for any two $G, H \in \mathcal{H}(X)$ any X (Proposition 2.2.1). The results from Section 2.3 provide also much information about the general case including the nonseparable case:

Theorem 2.1.2. *Suppose that X is a Banach space of dimension bigger than 1. Then the following equalities and inequalities hold:*

- $\mathfrak{add}(X) = \omega_1$,
- $\omega_1 \leq \mathfrak{cov}(X) \leq \mathfrak{c}$,
- $\mathrm{dens}(X) \leq \mathfrak{non}(X) \leq \mathrm{cf}([\mathrm{dens}(X)]^\omega)$,
- $\mathfrak{cof}(X) = |X^*|$.

Proof. Propositions 2.3.2, 2.3.3, 2.3.4, 2.3.5. □

So the interesting cardinal characteristics are $\mathfrak{cov}$ and $\mathfrak{non}$. First we note that making additional (but diverse) set theoretic assumptions (which are known to be independent of ZFC) the values of $\mathfrak{cov}$ and $\mathfrak{non}$ are completely determined by the density of the space or even fixed. For $\mathfrak{non}$ this follows just from results on cardinal arithmetic and Theorem 2.1.2:

Theorem 2.1.3. *Assume the Generalized Continuum Hypothesis* GCH *or Martin's Maximum* MM*. Let X be a nonseparable Banach space. Then*

(1) $\mathfrak{cov}(X) = \omega_1$,
(2) $\mathfrak{non}(X) = \mathrm{dens}(X)$ *if* $\mathrm{cf}(\mathrm{dens}(X)) > \omega$,
(3) $\mathfrak{non}(X) = \mathrm{dens}(X)^+$ *if* $\mathrm{cf}(\mathrm{dens}(X)) = \omega$.

Moreover the same is consistent with any possible size of the continuum $\mathfrak{c}$. If violations of the above equalities concerning $\mathfrak{non}$ are consistent, then so is the existence of a measurable cardinal.

Proof. Propositions 2.3.5, 2.4.10, 2.5.4, 2.5.5, 2.5.6 and the fact that MM implies PFA. □

Not only consistent set-theoretic hypotheses determine the values of $\mathfrak{cov}$. Also a well-known topological statement which is unknown to be provable but known to be consistent fixes the value of $\mathfrak{cov}$.

Theorem 2.1.4. *Assume that all compact Hausdorff spaces with a small diagonal are metrizable. Let X be a nonseparable Banach space. Then*

$$\mathfrak{cov}(X) = \omega_1.$$

Proof. Lemma 2.4.7. □

For the definition of a space with a small diagonal see Definition 2.4.5. In fact a weaker natural topological hypothesis has the same impact on $\mathfrak{cov}$ (see Question 2.6.2). The following main questions remain open:

Question 2.1.5. *Can one prove in ZFC any of the following sentences?*

(1) Every nonseparable Banach space can be covered by ω_1 of its hyperplanes.
(2) In any Banach space X of dimension bigger than 1 each subset of cardinality smaller than $\mathrm{cf}([\mathrm{dens}(X)]^\omega)$ can be covered by countably many hyperplanes.

The positive answer to the above questions would settle the values of $\mathfrak{cov}$ an $\mathfrak{non}$ in ZFC as in Theorem 2.1.3. Note that by Theorem 2.1.2 (3) in every infinite dimensional Banach space X there is a subset of cardinality $\mathrm{cf}([\mathrm{dens}(X)]^\omega)$ which cannot be covered by countably many hyperplanes. Attempting to prove the sentences of Question 2.1.5 for all nonseparable Banach spaces we manage to prove them in many cases:

Theorem 2.1.6. *Suppose that X is any nonseparable Banach space belonging to one of the following classes:*

(1) X admits a fundamental biorthogonal system,
(2) X is of the form $C(K)$ for K scattered, Hausdorff compact,
(3) X contains an isomorphic copy of $\ell_1(\omega_1)$,
(4) The dual ball B_{X^*} of X^* has uncountable tightness in the weak topology.*

Then X can be covered by ω_1 hyperplanes, i.e. $\mathfrak{cov}(X) = \omega_1$.

Proof. Propositions 2.4.4, 2.4.9, 2.4.8, Lemmas 2.4.6, 2.4.7. □

Note that this implies that spaces like $c_0(\kappa)$, $\ell_p(\kappa)$ for $1 \leq p < \infty$, and any $\kappa > 1$, reflexive spaces, WLD spaces (by (1)), $\ell_\infty(\kappa)$, $L_\infty(\{0,1\}^\kappa)$ for any $\kappa > 1$, (by (3)) satisfy the conclusion of the above theorem.

Theorem 2.1.7. *Suppose that X is any Banach space of dimension bigger than 1 belonging to one of the following classes:*

(1) X admits a fundamental biorthogonal system,
(2) X has density ω_n for some $n \in \mathbb{N}$.

Then each subset of X of cardinality smaller than $\mathrm{cf}([\mathrm{dens}(X)]^\omega)$ can be covered by countably many hyperplanes, i.e. $\mathfrak{non}(X) = \mathrm{cf}([\mathrm{dens}(X)]^\omega)$.

Proof. Propositions 2.5.1 and 2.5.3. □

Note that (1) above implies that spaces like $c_0(\kappa)$, $\ell_p(\kappa)$ for $1 \leq p < \infty$, spaces $\ell_\infty(\kappa)$, $L_\infty(\{0,1\}^\kappa)$ for any $\kappa > 1$ (by a result of [27] since $\ell_2(dens(X))$ is a quotient of such spaces X), reflexive spaces, WLD spaces satisfy the conclusion of the above theorem. Note that a Banach space X of density ω_n for $n \in \mathbb{N}$ may have cardinality arbitrarily bigger than ω_n as $|X| = \mathrm{dens}(X)^\omega = \omega_n^\omega = \mathfrak{c} \cdot \omega_n$ by Proposition 2.2.2.

Let us also note one application of our results. Recall that a subset Y of a Banach space X is overcomplete ([115], [85]) if $|Y| = \mathrm{dens}(X)$ and every subset $Z \subseteq Y$ of cardinality $\mathrm{dens}(X)$ is linearly dense in X. The following constitutes a progress on Question 39 from [85].

Theorem 2.1.8. *Assume the Proper Forcing Axiom* PFA*. Let X be a Banach space such that $\mathrm{cf}(\mathrm{dens}(X)) > \omega_1$. Then X does not admit an overcomplete set. Moreover this statement is consistent with any possible size of the continuum $\mathfrak{c}$.*

Proof. By Theorem 2.1.3 the hypothesis implies that every nonseparable Banach space X can be covered by ω_1 many hyperplanes $\{H_\alpha : \alpha < \omega_1\}$. If $Y \subseteq X$ and $|Y| = \mathrm{dens}(X)$, then by $\mathrm{cf}(\mathrm{dens}(X)) > \omega_1$ there is $\alpha < \omega_1$ such that $|H_\alpha \cap Y| = \mathrm{dens}(X)$, so $Z = H_\alpha \cap Y$ witnesses that Y is not overcomplete. □

The structure of the chapter is the following. Section 2.2 contains preliminaries. Section 2.3 establishes Theorems 2.1.1 and 2.1.2. Section 2.4 includes progress on Question 2.1.5 (1) and arrives at Theorems 2.1.3 (1), 2.1.4 and 2.1.6. Section 2.5 includes progress on Question 2.1.5 (2) and arrives at Theorems 2.1.3 (2), (3) and 2.1.7. The last Section 2.6 discusses the perspectives for further research and states additional questions.

No knowledge of logic or higher set-theory is required from the reader to follow the chapter. This is because all consistency results are obtained by applying consistency results already present in the literature.

2.2 Preliminaries

2.2.1 Hyperplanes

Let us recall here some elementary and well-known facts concerning hyperplanes in Banach spaces.

Lemma 2.2.1. *Suppose that X is a Banach space. Then the following hold.*

(1) If H, G are hyperplanes of X and $H \subseteq G$, then $H = G$.
(2) If a hyperplane H is contained in a countable union $\bigcup_{i\in\mathbb{N}} H_i$ of hyperplanes H_i, then $H = H_i$ for some $i \in \mathbb{N}$.

Proof. (1) Every hyperplane in a Banach space X is a kernel of some non-zero bounded functional and kernels of $f, g \in X^*$ are different if and only if f and g are linearly independent (3.1.13, 3.1.14 of [119]).

For (2) assume that $H \not\subseteq H_i$ for any $i \in \omega$. Then $H_i \cap H$ are nowhere dense in H. Hence by the Baire category theorem $\bigcup_{i\in\omega} H_i \cap H$ has empty interior in H, which leads to contradiction with $H = \bigcup_{i\in\omega} H_i \cap H$. Now use (1). $\square$

2.2.2 Cardinalities of Banach spaces

Let us recall here some well-known facts concerning cardinalities of Banach spaces. The first one follows from the Lemma 2.8 of [13] and the fact that $(\kappa^\omega)^\omega = \kappa^\omega$.

Proposition 2.2.2. *If X is a Banach space, then* $\mathrm{dens}(X)^\omega = |X|^\omega = |X|$.

Proposition 2.2.3. *If X is a Banach space of dimension bigger than* 1*, then* $|X^*| = |\mathcal{H}(X)|$.

Proof. Every hyperplane in a Banach space X is a kernel of some non-zero bounded functional and kernels of $f, g \in X^*$ are different if and only if f and g are linearly independent (3.1.13, 3.1.14 of [119]). So $|X^*| = \mathfrak{c} \cdot |\mathcal{H}(X)|$. If $f, g \in X^*$ are linearly independent, then the kernels of $f + \lambda g$ are different for different choices of $\lambda \in \mathbb{R}\setminus\{0\}$. So $\mathfrak{c} \leq |\mathcal{H}(X)|$ and so $|X^*| = |\mathcal{H}(X)|$. $\square$

Note that $|X^*|$ is not determined by $|X|$ or $\mathrm{dens}(X)$. By Proposition 2.2.2 we have $\mathrm{dens}(c_0(\mathfrak{c})) = |c_0(\mathfrak{c})| = \mathrm{dens}(\ell_1(\mathfrak{c})) = |\ell_1(\mathfrak{c})| = \mathfrak{c}$ and $\mathrm{dens}(\ell_\infty(\mathfrak{c})) = |\ell_\infty(\mathfrak{c})| = 2^{\mathfrak{c}}$ while $c_0(\mathfrak{c})^* = \ell_1(\mathfrak{c})$ and $\ell_1^*(\mathfrak{c}) = \ell_\infty(\mathfrak{c})$.

2.2.3 Ideals

Proposition 2.2.4. *Let X be a Banach space of dimension bigger than* 1*. Then* $\mathfrak{add}(X) \leq \mathfrak{cov}(X) \leq \mathfrak{cof}(X)$ *and* $\mathfrak{add}(X) \leq \mathfrak{non}(X) \leq \mathfrak{cof}(X)$*.*

Proof. This is elementary. Since $\mathcal{I}(X)$ contains all singletons and is a σ-ideal, Lemma 1.3.2 of [14] applies. $\square$

2.3 Basic results on the values of the cardinal characteristics

2.3.1 Separable Banach spaces

It turns out that the values of our cardinal characteristics on separable Banach spaces are the same. We include the proof of the following result for the convenience of the reader.

Proposition 2.3.1 ([77, Theorem 2.4])**.** *Let X be a separable Banach space. Then there exists a set $Y \subseteq X$ of cardinality $\mathfrak{c}$ such that for every hyperplane H of X the set $H \cap Y$ is finite.*

Proof. Let $\{x_n : n \in \mathbb{N}\} \subseteq X$ be linearly dense in X and consist of norm one vectors. Let

$$y_\lambda = \sum_{n\in\mathbb{N}} \lambda^n x_n$$

for each $\lambda \in (0, 1/2)$. We claim that $Y = \{y_\lambda : \lambda \in (0, 1/2)\}$ satisfies the theorem. Let H be a hyperplane $x^* \in X^*$ be the norm one nonzero linear bounded functional whose kernel is H. We have $\limsup_{n\to\infty} \sqrt[n]{|x^*(x_n)|} \leq \sup_{n\in\mathbb{N}} \sqrt[n]{|x^*(x_n)|} \leq 1$ and so the formula

$$f(\lambda) = \sum_{n\in\mathbb{N}} x^*(x_n)\lambda^n$$

defines an analytic function on $(-1, 1)$. $f \equiv 0$ on $(-1, 1)$ only if $x^*(x_n) = 0$ for each $n \in \mathbb{N}$, which is not the case since x^* is not the zero functional on X. By the properties of analytic functions f cannot have infinitely many zeros in $(0, 1/2)$, which means that $0 = f(\lambda) = x^*(\sum_{n\in B} \lambda^n x_n) = x^*(y_\lambda)$ only for finitely many $\lambda \in (0, 1)$ as required. □

Theorem 2.1.1. *Suppose that X is a separable Banach space of dimension bigger than 1. Then the following equalities hold:*

- $\mathfrak{add}(X) = \omega_1$,
- $\mathfrak{non}(X) = \omega_1$,
- $\mathfrak{cov}(X) = \mathfrak{c}$,
- $\mathfrak{cof}(X) = \mathfrak{c}$.

Proof. $|\mathcal{H}(X)| \leq \mathfrak{c}$ if X is separable as hyperplanes are determined by continuous functionals and such are determined by their values on a dense set. So by Proposition 2.2.4 it is enough to prove that $\mathfrak{non}(X) = \omega_1$ and $\mathfrak{cov}(X) = \mathfrak{c}$. Let Y be the set from Proposition 2.3.1 and $Y' \subseteq Y$ any set such that $|Y'| = \omega_1$. If Y' is covered by countably many hyperplanes $\{H_n\}_{n\in\mathbb{N}}$, then there is $n \in \mathbb{N}$ for which H_n contains an infinite subset $Z \subseteq Y'$, so $H_n = \overline{\text{span}(Z)} = X$, which is a contradiction. Hence $\mathfrak{non}(X) = \omega_1$.

Assume now that X is covered by $\kappa < \mathfrak{c}$ sets from $\mathcal{I}(X)$. Then X is covered by κ hyperplanes, so there is a hyperplane H containing an infinite subset of Y and again we get a contradiction. Hence $\mathfrak{cov}(X) = \mathfrak{c}$. □

Note that the first and last equations are also special cases of Propositions 2.3.2 and 2.3.3.

2.3.2 General Banach spaces

Proposition 2.3.2. *Let X be a Banach space of dimension bigger than 1. Then*

$$\mathfrak{add}(X) = \omega_1.$$

Proof. It is clear that $\mathfrak{add}(X) \geq \omega_1$. If $f, g \in X^*$ are linearly independent, then the kernels of $f + \lambda g$ are different hyperplanes for different choices of $\lambda \in \mathbb{R}\backslash\{0\}$. So let $\mathcal{F}$ be any collection of ω_1-many distinct hyperplanes. We have $\mathcal{F} \subseteq \mathcal{I}$. However $\bigcup \mathcal{F} \notin \mathcal{I}$ because otherwise if $\{H_i : i \in \mathbb{N}\} \subseteq \mathcal{H}$ and $\bigcup \mathcal{F} \subseteq \bigcup_{i\in\mathbb{N}} H_i$, then for every $H \in \mathcal{F}$ we have $H = H_i$ for some $i \in \mathbb{N}$ by Proposition 2.2.1 which contradicts the fact that $\mathcal{F}$ is uncountable. □

Proposition 2.3.3. *Let X be a Banach space of dimension bigger than 1. Then*

$$\mathfrak{cof}(X) = |X^*|.$$

Proof. Let $\mathcal{F}$ be a cofinal family in $\mathcal{I}(X)$. Without losing generality we can assume that $\mathcal{F}$ consists of countable sums of hyperplanes. By Lemma 2.2.1 every set in $\mathcal{F}$ contains only countably many hyperplanes, so $|\mathcal{F}| \geq |X^*|$. Moreover $|\mathcal{F}|$ is not greater than the cardinality of the family of all countable sets of hyperplanes which is equal to $|X^*|^\omega = |X^*|$ by Proposition 2.2.2. Thus $|\mathcal{F}| = |X^*|$. □

Proposition 2.3.4. *Let X be a Banach space of dimension bigger than 1. Then*

$$\mathrm{dens}(X) \leq \mathfrak{non}(X) \leq \mathrm{cf}([\mathrm{dens}(X)]^\omega).$$

If $\mathrm{cf}(\mathrm{dens}(X)) = \omega$, *then* $\mathrm{dens}(X) < \mathfrak{non}(X)$.

Proof. Assume that $Y \subseteq X$ and $|Y| < \mathrm{dens}(X)$. Then $\overline{\mathrm{span}(Y)}$ is a proper subspace of X and so it is contained in some hyperplane and hence $Y \in \mathcal{I}$, so $\mathrm{dens}(X) \leq \mathfrak{non}(X)$.

Let $\{x_\alpha : \alpha < \mathrm{dens}(X)\}$ be a dense subset of X. Let $\mathcal{F} \subseteq [\mathrm{dens}(X)]^\omega$ be a family which is cofinal in $[\mathrm{dens}(X)]^\omega$ and of cardinality $\mathrm{cf}([\mathrm{dens}(X)]^\omega)$. By Proposition 2.3.1 for each $F \in \mathcal{F}$ the subspace $X_F = \overline{\mathrm{span}\{x_\alpha : \alpha \in F\}} \subseteq X$ contains a subset Y_F such that $|Y_F| = \omega_1$ and it cannot be covered by countably many hyperplanes in X_F. Put $Y = \bigcup_{F \in \mathcal{F}} Y_F$. We claim that $Y \notin \mathcal{I}(X)$ and $|Y| = [\mathrm{dens}(X)]^\omega$. If Y were covered by countably many hyperplanes H_n of X, there would be $F \in \mathcal{F}$ such that $H_n \cap X_F \neq X_F$ for all $n \in \mathbb{N}$ which is a contradiction with the choice of Y_F. Hence $Y \notin \mathcal{I}(X)$. Also $|Y| = \omega_1 \cdot |\mathcal{F}| = \omega_1 \cdot \mathrm{cf}([\mathrm{dens}(X)]^\omega) = \mathrm{cf}([\mathrm{dens}(X)]^\omega)$ as $\mathrm{dens}(X)$ is uncountable.

Now assume that $\mathrm{cf}(\mathrm{dens}(X)) = \omega$. If $Y \subseteq X$ and $|Y| = \mathrm{dens}(X), Y = \bigcup_{n \in \mathbb{N}} Y_i$ with $|Y_i| < \mathrm{dens}(X)$, then every Y_i is contained in some closed subspace of X and hence in a hyperplane H_i for $i \in \mathbb{N}$. Thus $Y \in \mathcal{I}$. □

Proposition 2.3.5. *Let X be a Banach space of dimension bigger than 1. Then*

$$\omega_1 \leq \mathfrak{cov}(X) \leq \mathfrak{c}.$$

In particular, under CH, $\mathfrak{cov}(X) = \omega_1$ *for every nonseparable Banach space X. If* $\mathrm{cf}(\mathrm{dens}(X)) > \omega$, *then* $\mathfrak{cov}(X) \leq \mathrm{cf}(\mathrm{dens}(X))$. *In particular if* $\mathrm{dens}(X) = \omega_1$, *then* $\mathfrak{cov}(X) = \omega_1$.

Proof. Since $\mathcal{I}(X)$ is a σ-ideal, we have $\omega_1 \leq \mathfrak{cov}(X)$. Let $f, g \in X^*$ be linearly independent. Then for every $x \in X$ there are $(a, b) \in \mathbb{R} \setminus \{(0,0)\}$ such that $af(x) + bg(x) = 0$. Thus the family of hyperplanes $\{\ker af + bg : (a, b) \in \mathbb{R}^2 \setminus \{(0,0)\}\}$ of cardinality $\mathfrak{c}$ covers X.

Let $\kappa = \mathrm{dens}(X)$ and let $\{x_\alpha : \alpha < \kappa\}$ be a dense subset of X. Let $\kappa = \sup\{\alpha_\xi : \xi < \mathrm{cf}(\kappa)\}$. Let $X_\xi = \overline{\mathrm{span}\{x_\alpha : \alpha < \alpha_\xi\}}$ for $\xi < \mathrm{cf}(\kappa)$. Each X_ξ is a proper subspace of X since the density of X is $\kappa > \alpha_\xi$. Also every element $x \in X$ is in the closure of a countable subset of $\{x_\alpha : \alpha < \kappa\}$, and so by the uncountable cofinality of κ we conclude that $x \in X_\xi$ for some $\xi < \mathrm{cf}(\kappa)$. □

2.4 Covering nonseparable Banach spaces with ω_1 hyperplanes

By Proposition 2.3.5 and Theorem 2.1.1 if we assume CH we have $\mathfrak{cov}(X) = \omega_1$ for all Banach spaces X. In this section we investigate whether $\mathfrak{cov}(X) = \omega_1$ may hold for all nonseparable Banach spaces without this assumption (Note that by Theorem 2.1.1 if CH fails, then $\mathfrak{cov}(X) > \omega_1$ for all separable Banach spaces). We prove that the value of $\mathfrak{cov}$ is indeed ω_1 for many classes of nonseparable Banach spaces (Propositions 2.4.4, 2.4.8, 2.4.9) and that consistently it holds for all Banach spaces in the presence of diverse negations of CH (Proposition 2.4.10). The deepest observations rely heavily on set-theoretic topological results of [71], [36], [35] concerning small diagonals and countable tightness in compact Hausdorff spaces (Definition 2.4.5).

Lemma 2.4.1. *Suppose that X, Y are Banach spaces and $T\colon X \to Y$ is a bounded linear operator whose range is dense in Y. Then $\mathfrak{cov}(Y) \le \mathfrak{cov}(X)$.*

Proof. If $0 \neq y^* \in Y^*$, then $T^*(y^*) \neq 0$ because the range of T is dense in Y, so a covering of Y by hyperplanes induces a covering of X by hyperplanes which is of the same cardinality which proves $\mathfrak{cov}(X) \le \mathfrak{cov}(Y)$. □

Lemma 2.4.2. *For every nonseparable Banach space X there is a linear bounded operator $T\colon X \to \ell_\infty(\omega_1)$ with nonseparable range. In particular, all values of the cardinal characteristic $\mathfrak{cov}$ on nonseparable Banach spaces are bounded by the values on nonseparable subspaces of $\ell_\infty(\omega_1)$.*

Proof. Every Banach space is isometric to a subspace of $C(K) \subseteq \ell_\infty(K)$, where $K = B_{X^*}$. So we may assume that $X \subseteq \ell_\infty(\kappa)$ for some uncountable cardinal κ. As X is nonseparable, it contains an uncountable discrete set D. This fact is witnessed by the coordinates from some set $A \subseteq \kappa$ of cardinality ω_1. That is there exist $\varepsilon > 0$ such that for every distinct $d, d' \in D$ we have is $|d(\alpha) - d'(\alpha)| > \varepsilon$ for some $\alpha \in A$. Consider the restriction operator $R\colon X \to \ell_\infty(A)$. It is clear that the range is nonseparable by the choice of A. To conclude the last part of the lemma take any nonseparable Banach space X and consider the operator T as in the first part of the lemma and let Y be the closure of the range of T. Using Lemma 2.4.1 we conclude that $\mathfrak{cov}(X) \le \mathfrak{cov}(Y)$. □

Let us now prove a simple but useful:

Lemma 2.4.3. *Let X be a Banach space. The following conditions are equivalent:*

(1) $\mathfrak{cov}(X) = \omega_1$.
(2) *X is a union of ω_1 hyperplanes.*

(3) There is $A \subseteq X^* \setminus \{0\}$ of cardinality ω_1 such that for every $x \in X$ there is $x^* \in A$ such that $x^*(x) = 0$.

(4) There is a bounded linear operator $T\colon X \to \ell_\infty(\omega_1)$ such that

 (1) for every $\alpha < \omega_1$ there is $x \in X$ such that $T(x)(\alpha) \neq 0$.

 (2) for every $x \in X$ there is $\alpha < \omega_1$ such that $T(x)(\alpha) = 0$.

Proof. The equivalence of the first three items is clear. Assume (3) and let us prove (4). Let $\{H_\alpha : \alpha < \omega_1\}$ be the hyperplanes that cover X and let $x^*_\alpha \in X^*$ be such that H_α is the kernel of x^*_α and $\|x^*_\alpha\| = 1$ for all $\alpha < \omega_1$. Let $T(x)(\alpha) = x^*_\alpha(x)$. Condition (a) follows from the fact that $x^*_\alpha \neq 0$ and condition (b) from the fact that H_αs cover X.

Now assume (4) and let us prove (3). Condition (a) implies that $x^*_\alpha = T^*(\delta_\alpha)$ is a nonzero element of X^*, and so its kernel is a hyperplane. Condition (b) implies that the kernels of x^*_αs cover X. □

Proposition 2.4.4. *Let X be a nonseparable Banach space. Each of the following sentences implies the next.*

(1) X admits a fundamental biorthogonal system.

(2) There is a bounded linear operator $T\colon X \to c_0(\omega_1)$ with nonseparable range (i.e. X is not half-pcc in the terminology of [33]).

(3) $\mathfrak{cov}(X) = \omega_1$.

In particular for every nonseparable WLD Banach space X we have $\mathfrak{cov}(X) = \omega_1$.

Proof. Let $\{(x_\alpha, x^*_\alpha) : \alpha < \kappa\}$ be a fundamental biorthogonal system. Define $T\colon X \to \ell_\infty(\omega_1)$ by $T(x)(\alpha) = x^*_\alpha(x)$. As $T(x_\alpha) = \chi_{\{\alpha\}} \restriction \omega_1 \in c_0(\omega_1)$ and X is the closure of the linear span of $\{x_\alpha : \alpha < \omega_1\}$ we conclude that $T[X] \subseteq c_0(\omega_1)$. $T(x_\alpha) = \chi_{\{\alpha\}}$ for $\alpha < \omega_1$ witnesses the fact that the range is nonseparable.

Now assume (2). As the range of T is nonseparable, by passing to an uncountable set of coordinates we may assume that for all $\alpha < \omega_1$ there is $x \in X$ such that $T(x)(\alpha) \neq 0$. So item (4) of Lemma 2.4.3 is satisfied, and hence $\mathfrak{cov}(X) = \omega_1$.

To make the final observation use the fact that WLD Banach spaces admit fundamental biorthogonal systems e.g. by the results of [134]. □

Note that the paper [33] contains many results on properties of Banach spaces X which imply item (2) of Lemma 2.4.4, for example this happens when X^* contains a nonmetrizable weakly compact subset. To obtain more Banach spaces X satisfying $\mathfrak{cov}(X)$ we need some topological considerations. First recall the following:

Definition 2.4.5. Let K be a compact Hausdorff space.

(1) We say that K has a small diagonal if for every uncountable subset A of $K^2 \setminus \Delta(K)$ there is an uncountable $B \subseteq A$ whose closure is disjoint from $\Delta(K)$.

(2) We say that K has countable tightness (is countably determined) if whenever $K \ni x \in \overline{A}$ for $A \subseteq K$, then there is a countable $B \subseteq A$ such that $x \in \overline{B}$.

In the following lemma the implication from (3) to (4) is the result of [71].

Lemma 2.4.6. *Suppose that K is a compact Hausdorff space. Each of the following sentences implies the next.*

(1) For every $A \subseteq K$ of cardinality ω_1 there is a continuous $f\colon K \to \mathbb{R}$ such that $f \restriction A$ is injective.

(2) For every $A = \{(x_\alpha, y_\alpha) : \alpha < \omega_1\} \subseteq K^2 \setminus \Delta(K)$ of cardinality ω_1 there is a continuous $f\colon K \to \mathbb{R}$ such that

$$\{\alpha : f(x_\alpha) \neq f(y_\alpha)\}$$

is uncountable.

(3) K has a small diagonal.

(4) K is countably tight.

Proof. Assume (1). Let $A \subset K^2$ be a set of cardinality ω_1 disjoint from the diagonal. Let $A = \{(x_\alpha, y_\alpha) : \alpha < \omega_1\}$. Put $L = \{x_\alpha, y_\alpha : \alpha < \omega_1\}$ and let $f\colon K \to \mathbb{R}$ be continuous and $f \restriction L$ injective. Then $f(x_\alpha) \neq f(y_\alpha)$ for each $\alpha < \omega_1$, so we obtain (2).

Assume (2). Let $A \subset K^2$ be uncountable. We may assume that A is of cardinality ω_1 and so $A = \{(x_\alpha, y_\alpha) : \alpha < \omega_1\}$. Let $f\colon K \to \mathbb{R}$ be continuous and such that

$$\{\alpha : f(x_\alpha) \neq f(y_\alpha)\}$$

is uncountable. Then $g\colon K^2 \to \mathbb{R}$ defined by $g(x, y) = |f(x) - f(y)|$ is continuous and $g \restriction A$ is non-zero. Hence there exist $\varepsilon > 0$ and an uncountable subset $A' \subseteq A$ such that $g(a) > \varepsilon$ for $a \in A'$. It follows that the closure of A' is disjoint from diagonal as $g \restriction \Delta(K) = 0$ which completes the proof of (3).

For the last implication see the proof of [71, Corollary 2.3]. □

It is easy to see that the one-point compactification of an uncountable discrete space is countably tight but does not have a small diagonal, so (4) does not imply (3). We do not know if the other implications reverse (cf. Question 2.6.2).

Lemma 2.4.7. *Let X be a Banach space. Each of the following sentences implies the next.*

(1) The dual ball B_{X^*} does not have a small diagonal in the weak topology.*

(2) There is $\{x^*_\alpha : \alpha < \omega_1\} \subseteq B_{X^*} \setminus \{0\}$ such that $\{\alpha : x^*_\alpha(x) \neq 0\}$ is at most countable for each $x \in X$.

(3) There is $A \subseteq B_{X^*}$ of cardinality ω_1 such that $\delta_x \restriction A$ is not injective for each $x \in X$, where $\delta_x \in C(B_{X^*})$ is given by $\delta_x(x^*) = x^*(x)$.

(4) $\mathfrak{cov}(X) = \omega_1$.

Proof. Suppose (1). By the implication from (2) to (3) of Lemma 2.4.6 there is $A = \{(y^*_\alpha, z^*_\alpha) : \alpha < \omega_1\} \subseteq B^2_{X^*} \setminus \Delta(B_{X^*})$ of cardinality ω_1 such that for every continuous $f\colon K \to \mathbb{R}$ the set $\{\alpha : f(y^*_\alpha) \neq f(z^*_\alpha)\}$ is countable. Of course $x \in X$ defines a continuous function on the dual ball in the weak* topology so $\{\alpha : (y^*_\alpha - z^*_\alpha)(x)\}$ is countable for all $x \in X$. So we put $x^*_\alpha = y^*_\alpha - z^*_\alpha$ and we obtain (2).

Assume (2). Put $A = \{x^*_\alpha : \alpha < \omega_1\}$. Then for each $x \in X$ image of $\delta_x \restriction A$ is countable, so $\delta_x \restriction A$ is not injective since $|A| > \omega$.

Now suppose (3). Consider $\mu_\alpha = x_\alpha - y_\alpha \in X^*$, where $\{\{x_\alpha, y_\alpha\} : \alpha < \omega_1\} = [A]^2$. Then the kernels of μ_α's cover X. ☐

Since it is consistent that all nonmetrizable compact spaces do not have small diagonals, it is also consistent that sentences (1)-(4) from Lemma 2.4.7 are equivalent in the class of nonseparable Banach spaces. It is still an open question whether this holds in ZFC.

Proposition 2.4.8. *If X is a Banach space which contains an isomorphic copy of $\ell_1(\omega_1)$, then $\mathfrak{cov}(X) = \omega_1$. In particular this holds for any space which contains ℓ_∞ like $\ell_\infty(\kappa)$, $L_\infty(\{0,1\}^\kappa)$, ℓ_∞/c_0 etc.*

Proof. By the main result of [128], if a Banach space X contains $\ell_1(\omega_1)$, then there is a continuous surjection $\Phi\colon B_{X^*} \to [0,1]^{\omega_1}$, where B_{X^*} is considered with the weak* topology. As countable tightness is preserved by continuous map and $[0,1]^{\omega_1}$ is not countably tight (consider $1_{[0,\omega_1)} \in \overline{\{1_{[0,\alpha)} : \alpha < \omega_1\}}$) we conclude that B_{X^*} is considered with the weak* topology is not countably tight. By Lemma 2.4.6 B_{X^*} does not have a small diagonal, and so by Lemma 2.4.7 we conclude that $\mathfrak{cov}(X) = \omega_1$. ☐

Proposition 2.4.9. *If K is compact nonmetrizable and scattered, then $\mathfrak{cov}(C(K)) = \omega_1$.*

Proof. K must be uncountable. Let $A \subseteq K$ be any subset of cardinality ω_1. As a continuous image of a scattered compact space is scattered compact we conclude that for any continuous $f\colon K \to \mathbb{R}$ the image of f is countable and so $f \restriction A$ is not injective which implies that $\delta_f \restriction \{\delta_x : x \in A\}$ is not injective. Hence $C(K)$ satisfy condition (3) of Lemma 2.4.7. ☐

Proposition 2.4.10.

(1) PFA *implies that every nonseparable Banach space X satisfies $\mathfrak{cov}(X) = \omega_1$.*

(2) It is consistent with any possible size of the continuum, that every nonseparable Banach space X satisfies $\mathfrak{cov}(X) = \omega_1$.

Proof. It is shown in [36] that assuming PFA every compact Hausdorff space with a small diagonal is metrizable. So by Lemma 2.4.7 we conclude that $\mathfrak{cov}(X) = \omega_1$ for every nonseparable Banach space X under PFA. Similarly Theorem 5.8 from [35] shows that it is consistent with any possible size of the continuum (in models obtained from CH model by adding Cohen reals) that each compact space with countable tightness has a small diagonal if and only if it is metrizable. However, non-countably-tight compact spaces cannot have a small diagonal (in ZFC) by the result of [71] that is the implication from (3) to (4) in Lemma 2.4.6. So by Lemma 2.4.7 we conclude that $\mathfrak{cov}(X) = \omega_1$ for every nonseparable Banach space X in these models as well. ☐

2.5 Covering small subsets of Banach spaces by countably many hyperplanes

By Proposition 2.3.4 if X is a Banach space of dimension bigger than 1 the value of $\mathfrak{non}(X)$ (i.e. the minimal cardinality of a set not covered by countably many hyperplanes) is in the interval $[\mathrm{dens}(X), \mathrm{cf}([\mathrm{dens}(X)]^\omega)]$ if $\mathrm{cf}(\mathrm{dens}(X))$ is uncountable and is in the interval $[\mathrm{dens}(X)^+, \mathrm{cf}([\mathrm{dens}(X)]^\omega)]$ if $\mathrm{cf}(\mathrm{dens}(X))$ is countable. As we will see below, just purely set-theoretic known results imply that under many assumptions these intervals reduce to singletons and so the values of $\mathfrak{non}(X)$ are completely determined by $\mathrm{dens}(X)$. It remains open, however, if $\mathfrak{non}(X) = \mathrm{cf}([\mathrm{dens}(X)]^\omega)]$ for every nonseparable Banach space without any extra set-theoretic assumptions.

Proposition 2.5.1. *If X is a Banach space with density ω_n for $n \in \mathbb{N}\backslash\{0\}$, then $\mathfrak{non}(X) = \mathrm{dens}(X) = \mathrm{cf}([\mathrm{dens}(X)]^\omega)$.*

Proof. By induction on $n \in \mathbb{N}$, using the decomposition of ω_n into smaller ordinals one proves that $\mathrm{cf}([\omega_n]^\omega) = \omega_n$. Now Proposition 2.3.4 implies that $\mathfrak{non}(X) = \omega_n$. □

Proposition 2.5.2. *Let X be a Banach space of density κ and dimension bigger than 1. Suppose that there are functionals $\{x_\alpha^* : \alpha < \kappa\} \subseteq X^*$ such that for every $x \in X$ the set $Z_x = \{\alpha : x_\alpha^*(x) \neq 0\}$ is countable. Then $\mathfrak{non}(X) = \mathrm{cf}([\kappa]^\omega)$.*

Proof. Let $\lambda < \mathrm{cf}([\kappa]^\omega)$ and $Y = \{x_\alpha : \alpha < \lambda\} \subseteq X$. By the assumption the family $\mathcal{Z} = \{Z_x : x \in Y\}$ is not cofinal in $[\kappa]^\omega$. Pick $A \in [\kappa]^\omega$, which is not included in any element of $\mathcal{Z}$. Then for every $x \in Y$ there is $\alpha \in A$ such that $x_\alpha^*(x) = 0$, so x is in kernel of x_α^*. Thus kernels of x_α^*s for $\alpha \in A$ cover Y, which proves that $\mathfrak{non}(X) \geq \mathrm{cf}([\kappa]^\omega)$. The inequality $\mathfrak{non}(X) \leq \mathrm{cf}([\kappa]^\omega)$ is true by Proposition 2.3.4. □

Proposition 2.5.3. *If a Banach space X of density κ and dimension bigger than 1 admits a fundamental biorthogonal system, then $\mathfrak{non}(X) = \mathrm{cf}([\kappa]^\omega)$.*

Proof. Let $\{x_\alpha, x_\alpha^*\}_{\alpha<\kappa}$ be a fundamental biorthogonal system. For every x pick a countable set $L_x \subset \kappa$ such that $x \in \overline{\mathrm{span}\{x_\alpha : \alpha \in L_x\}}$. Then $Z_x = \{\alpha : x_\alpha^*(x) \neq 0\} \subseteq L_x$, so Z_x is also countable. Hence x_α^*s satisfy conditions of Proposition 2.5.2. □

Proposition 2.5.4. *Assume that $\kappa^\omega = \kappa$ for all regular $\kappa > \omega_\omega$. Let X be a Banach space of dimension bigger than 1. Then*

(1) If $\mathrm{cf}(\mathrm{dens}(X)) = \omega$, then $\mathfrak{non}(X) = \mathrm{dens}(X)^+$,
(2) If $\mathrm{cf}(\mathrm{dens}(X)) > \omega$, then $\mathfrak{non}(X) = \mathrm{dens}(X)$.

In particular the above equations hold under GCH *or* MM.

Proof. By assumption we have $\mathrm{cf}([\kappa]^\omega) = \kappa$ for all regular $\kappa > \omega_\omega$. If κ is singular of uncountable cofinality then $\kappa^\omega = \Sigma_{\mu<\kappa}\, \mu^\omega = \Sigma_{\mu<\kappa}\, \mu = \kappa$ so $\mathrm{cf}([\kappa]^\omega) = \kappa$. If $\mathrm{cf}(\kappa) = \omega$ then $\kappa^\omega > \kappa$ and $\kappa^\omega \leq (\kappa^+)^\omega = \kappa^+$ so $\mathrm{cf}([\kappa]^\omega) \leq \kappa^+$. Hence for $\mathrm{dens}(X) \geq \omega_\omega$ the equalities (1) and (2) follow from Proposition 2.3.4. The case when $\mathrm{dens}(X) = \omega_n$ for some $n \in \mathbb{N}$ is covered by Proposition 2.5.1.

For a limit cardinal κ of uncountable cofinality under GCH we have

$$\kappa^\omega = \Sigma_{\lambda<\kappa}\lambda^\omega \leq \Sigma_{\lambda<\kappa}\lambda^+ \leq \kappa^2 = \kappa$$

so $\kappa^\omega = \kappa$. For successor cardinals we have $(\kappa^+)^\omega = \kappa^\omega\kappa^+ \leq (\kappa^+)^2 = \kappa^+$.

If MM holds, then by Theorem 37.13 of [69] we have $\kappa^\omega = \kappa^{\omega_1} = \kappa$ for each regular $\kappa > \omega_1$.

□

Proposition 2.5.5. *It is consistent with any possible size of the continuum that for every Banach space X of dimension bigger than 1 we have*

(1) *If* $\mathrm{cf}(\mathrm{dens}(X)) = \omega$, *then* $\mathfrak{non}(X) = \mathrm{dens}(X)^+$,
(2) *If* $\mathrm{cf}(\mathrm{dens}(X)) > \omega$, *then* $\mathfrak{non}(X) = \mathrm{dens}(X)$.

Proof. Start with a model V of GCH and increase the continuum using a c.c.c. forcing (e.g. add Cohen reals). The cardinals and their cofinalities do not change. Moreover $[\kappa]^\omega \cap V$ is cofinal in $[\kappa]^\omega$ as any countable set of ordinals in a c.c.c. extension is included in a countable set in the ground model, so the calculations from the proof of Proposition 2.5.4 remain true. □

Proposition 2.5.6. *For every Banach space X of dimension bigger than 1 we have*

(1) *If* $\mathrm{cf}(\mathrm{dens}(X)) = \omega$, *then* $\mathfrak{non}(X) = \mathrm{dens}(X)^+$,
(2) *If* $\mathrm{cf}(\mathrm{dens}(X)) > \omega$, *then* $\mathfrak{non}(X) = \mathrm{dens}(X)$,

unless there is a measurable cardinal in an inner model.

Proof. If there is no measurable cardinal in an inner model, then there is an inner model M which satisfies GCH and satisfies the covering lemma i.e. $[\kappa]^{\omega_1} \cap M$ is cofinal in $[\kappa]^{\omega_1}$ for each cardinal κ (see [32]). This implies that $[\kappa]^\omega \cap M$ is cofinal in $[\kappa]^\omega$ for each cardinal κ. So since M satisfies GCH, Proposition 2.5.4 implies the theorem. (For a similar argument see the proof of Theorem 13.3 (d) in [111].) □

Recall that assuming the existence of a suitably large cardinal the consistency of $2^{\omega_n} < \omega_\omega$ and $2^{\omega_\omega} = \omega_{\omega+k}$ for any $n \in \mathbb{N}$ and $k > 1$ was proved in [93] (this problem was also considered with weaker assumptions in [59]). In this case $\mathrm{cf}([\omega_\omega]^\omega) = \omega_{\omega+k}$ because $[\omega_\omega]^\omega = \bigcup\{[A]^\omega : A \in \mathcal{F}\}$ for any cofinal family in $[\omega_\omega]^\omega$ and $|\bigcup\{[A]^\omega : A \in \mathcal{F}\}| \leq \mathfrak{c} \cdot |\mathcal{F}| = |\mathcal{F}|$ as $\mathfrak{c} < \omega_\omega$. It follows that $\mathrm{cf}([\omega_{\omega+m}]^\omega) \geq \omega_{\omega+k}$ for $0 \leq m < k$. So not only the existence of Banach spaces of density ω_ω which assume the value of $\mathfrak{non}$ smaller than in Propositions 2.5.4, 2.5.5 if $k \geq 3$ but also of a regular density $\omega_{\omega+1}$ is not excluded by cardinal arithmetic in the considered model.

2.6 Final remarks

2.6.1 Densities of quotients of Banach spaces

The famous Separable Quotient Problem asks if every infinite dimensional Banach space has a separable infinite dimensional quotient. In the direction of bounding the densities of quotients of Banach spaces, one can easily prove that every Banach X space has a infinite dimensional quotient whose density is not bigger then $\mathfrak{c}$. In this light the following is natural to ask:

Question 2.6.1. *Is it true in ZFC that every nonseparable Banach space has a quotient of density ω_1?*

By Propositions 2.3.5 and 2.4.1 the positive answer to question 2.6.1 would imply that $\mathfrak{cov}(X) = \omega_1$ for every Banach space X. It would also imply that the Separable Quotient Problem consistently has positive answer since it is proved in [130] that it is consistent that all Banach spaces of density ω_1 have infinite dimensional separable quotients. In fact, for this it would be enough to obtain the consistency of the positive answer to Question 2.6.1 with the additional set-theoretic assumptions of [130], like the PFA.

2.6.2 Banach spaces with no fundamental biorthogonal systems

Theorems 2.1.6 and 2.1.7 determine the values of $\mathfrak{cov}$ and $\mathfrak{non}$ for Banach spaces admitting fundamental biorthogonal systems. So looking for spaces witnessing different values of $\mathfrak{cov}$ or $\mathfrak{non}$ we should understand better spaces not admitting such systems. The first and classical example of such a space is the subspace $\ell_\infty^c(\mathfrak{c}^+)$ of $\ell_\infty(\mathfrak{c}^+)$ consisting of elements with countable supports ([58], [110]). However it contains a copy of ℓ_∞ and so $\ell_1(\omega_1)$ so $\mathfrak{cov}(\ell_\infty^c(\lambda)) = \omega_1$ for any infinite λ by Theorem 2.1.6. Moreover Proposition 2.5.2 implies that $\mathfrak{non}(\ell_\infty^c(\lambda)) = \mathrm{cf}([\lambda]^\omega)$ for any $\lambda > \mathfrak{c}^+$ as $\mathrm{dens}(\ell_\infty^c(\lambda)) = \lambda$ in such a case. Other reason for not admitting a fundamental biorthogonal system in a nonseparable space is not admitting any uncountable biorthogonal system: The Kunen line and the examples of [91], [83], [18] have all density ω_1, so they have $\mathfrak{cov} = \omega_1$ by Proposition 2.3.5. The only known Banach space of density bigger than ω_1 with no uncountable biorthogonal systems is that of [19]. However it is of the form $C(K)$ with K scattered so Theorem 2.1.6 implies that $\mathfrak{cov} = \omega_1$. It also has density ω_2, so Theorem 2.1.7 implies that its $\mathfrak{non}$ is $\omega_2 = \mathrm{cf}([\omega_2]^\omega)$.

2.6.3 A question on compact Hausdorff spaces

By Lemma 2.4.7 positive answer to the following question would imply that $\mathfrak{cov}(X) = \omega_1$ for every nonseparable Banach space X:

Question 2.6.2. *Is it provable that every nonmetrizable compact Hausdorff space K admits a subspace $L \subseteq K$ of cardinality ω_1 such that for no $f \in C(K)$ the restriction $f \restriction L$ is injective?*

Recall that it was proved in [34] that every nonmetrizable compact Hausdorff space admits a subspace of size ω_1 which is nonmetrizable. Moreover the above result and Proposition 11 of [36] imply that every nonmetrizable compact Hausdorff space K admits a subspace $L \subseteq K$ of cardinality ω_1 such that for no $f \in C(K)$ we have $f^{-1}[\{f(x)\}] = \{x\}$ for all $x \in L$.

Chapter 3

A Banach space $C(K)$ reading the dimension of K

In [80] Koszmider showed that there is a compact Hausdorff space K such that whenever L is compact Hausdorff and the Banach spaces $C(K)$ and $C(L)$ are isomorphic, the dimension of L is greater than zero. In the light of this result Pełczyński asked, whether there is a compact space K with $\dim(K) = k$ for given $k \in \omega\setminus\{0\}$, such that if $C(K) \sim C(L)$, then $\dim(L) \geq k$ ([84, Problem 4]). We show that the answer to this question is positive, if we assume Jensen's diamond principle ($\diamondsuit$). Namely, we prove the following:

Theorem 3.4.9. *Assume $\diamondsuit$. Then for every $k \in \omega \cup \{\infty\}$ there is a compact Hausdorff space K such that $\dim K = k$ and whenever $C(K) \sim C(L)$, $\dim L = k$.*

Note that typically the dimension of K is not an invariant of the Banach space $C(K)$ under isomorphisms. For instance, the classical result by Miljutin says that if K, L are compact metrizable uncountable spaces, then the Banach spaces $C(K)$ and $C(L)$ are isomorphic ([95]). This also shows that $C(K)$ with the desired property cannot admit any complemented copy of $C(L)$ where L is compact, metrizable and uncountable (indeed, if $C(K) \sim X \oplus C(L)$, then $C(K) \sim X \oplus C(L) \oplus C([0,1]^n) \sim C(K) \oplus C([0,1]^n)$ for any $n \in \omega$). Another result by Pełczyński says that if G is an infinite compact topological group of weight κ, then $C(G)$ is isomorphic to $C(\{0,1\}^\kappa)$ ([102]).

On the other hand the space $C(K)$ remembers many topological and set-theoretic properties of K. For example Cengiz showed that if $C(K) \sim C(L)$, then K and L have the same cardinalities ([21]). If K is scattered, then by Pełczyński-Semadeni theorem L is scattered as well ([103]). In this case both spaces must be zero-dimensional. If K is an Eberlein compact, then L is also Eberlein ([98]). If K is a Corson compact and L is homogeneous, then L is Corson ([108]).

Although the isomorphic structure of $C(K)$ does not remember the dimension of K, the metric structure of $C(K)$ contains such information, since by the Banach-Stone theorem K and L are homeomorphic, whenever $C(K)$ and $C(L)$ are isometric. Similar results were obtained by Gelfand, Kolmogorov and Kaplansky in the category of rings of functions on compact spaces and in the category of Banach lattices ([52, 75]). It is also worthy to mention that the covering dimension of K is an invariant for the space $C_p(K)$ of continuous functions on K with the pointwise topology ([104]).

The key property of the space K that we construct to prove Theorem 3.4.9 is the fact that the Banach space $C(K)$ has few operators i.e. every bounded operator $T\colon C(K) \to C(K)$ is of the from $T = gI + S$, where $g \in C(K)$ and S is weakly compact. Schlackow showed that if the Banach space $C(K)$ has few operators, $C(K) \sim C(L)$ and both spaces K, L are perfect, then K and L are homeomorphic ([117]). We improve this result under the assumption that K is separable and connected.

Theorem 3.2.19. *Suppose that K is a separable connected compact Hausdorff space such that $C(K)$ has few operators and L is a compact Hausdorff space such that $C(K) \sim C(L)$.*

Then K and L are homeomorphic modulo finite set i.e. there are open subsets $U \subseteq K, V \subseteq L$ and finite sets $E \subseteq K, F \subseteq L$ such that U, V are homeomorphic and $K = U \cup E, L = V \cup F$.

The first example (under the continuum hypothesis) of a Banach space $C(K)$ with few operators appeared in the work of Koszmider ([80]). Later, Plebanek showed how to remove the use of CH from such constructions ([107]). Considered spaces have many interesting properties (cf. [84, Theorem 13]) e.g. $C(K)$ is indecomposable Banach space, it is not isomorphic to any of its proper subspaces nor any proper quotient, it is a Grothendieck space, K is strongly rigid (i.e. identity and constant functions are the only continuous functions on K) and does not include non-trivial convergent sequences. For more examples and properties of Banach spaces $C(K)$ with few operators see [11, 40, 82, 84, 87, 88].

In the further part of the chapter we show how to construct a Banach space $C(K)$ with few operators, where K has arbitrarily given dimension. Theorem 3.4.9 is an almost immediate consequence of Theorem 3.2.19 and the following theorem.

Theorem 3.4.8. *Assume $\diamondsuit$. For each $k > 0$ there is a compact Hausdorff, separable, connected space K such that $C(K)$ has few operators and $\dim K = k$.*

Our construction is a modification of one of the spaces K from [80, Theorem 6.1], which is a separable connected compact space such that $C(K)$ has few operators. The original space is constructed as an inverse limit of metrizable compact spaces $(K_\alpha)_{\alpha<\omega_1}$, where on intermediate steps we add suprema to countable families of functions in the lattice $C(K_\alpha)$ for $\alpha < \omega_1$, using the notion of strong extension. However, the considered families of functions are very general, which leads to the problem that described operation may rise the dimension of given space and the final space is infinite-dimensional. We show that under $\diamondsuit$ we are able to limit the choice of functions in the way that we can control the dimension of the spaces at each step. In order to control the dimension we introduce the notion of essential-preserving maps. Similar ideas were studied in Fedorchuk's work ([48–50]). For instance, Fedorchuk considered maps that are ring-like, monotonic and surjective, which implies that they are essential-preserving (however, those notions are much stronger and are not applicable in our context).

One may also consider other notions of dimension such as small or large inductive dimension. However, since Theorem 3.1.12 does not work if we replace the covering dimension with one of the inductive dimensions, we do not know if the spaces we constructed have finite inductive dimensions.

The structure of the chapter is the following. Section 3.1 contains necessary results about covering dimension. In section 3.2 we prove Theorem 3.2.19 characterizing properties of spaces $C(K)$ with few operators preserved under isomorphisms. In Section 3.3 we develop tools for controlling dimension in some inverse limits of systems of compact spaces. Section 3.4 contains the description of the construction leading to the main theorem of the chapter. The last section includes remarks and open questions.

3.1 Covering dimension

This section is devoted to the basic properties of covering dimension and its behavior in inverse limits of compact spaces. We start with several basic definitions. Recall that for a family $\mathcal{A}$ of sets we define its order as the largest integer n such that $\mathcal{A}$ contains $n+1$ sets with non-empty intersection. If there is no such n, then we say that the order of $\mathcal{A}$ is ∞.

Definition 3.1.1. [38, Definition 1.6.7] Let X be a topological space. We say that **covering dimension** of X (denoted by $\dim X$) is not greater than n, if every finite open cover of X has a finite open refinement of order at most n. We say that $\dim X = n$ if $\dim X \leq n$, but not $\dim X \leq n-1$. If there is no n such that $\dim X = n$, then we say that $\dim X = \infty$.

Definition 3.1.2. [38, Definition 1.1.3] Let X be a topological space. A closed set $P \subseteq X$ is a **partition** between A and B if there are disjoint open sets $U \supseteq A, V \supseteq B$ such that $X \backslash P = U \cup V$.

Definition 3.1.3. [22, p. 16] A family $\{(A_i, B_i) : i = 1, 2, \ldots, n\}$ of pairs of disjoint closed subsets of a space X is called **essential** if for every family $\{C_i : i = 1, 2, \ldots, n\}$ such that for each $i \leq n$ the set C_i is a partition between A_i and B_i we have

$$\bigcap_{i=1}^{n} C_i \neq \varnothing.$$

For the proof of the following theorems see [22, Lemma 3.2, Theorem 3.3].

Theorem 3.1.4. *For a normal space X the following conditions are equivalent:*

(1) a family $\{(A_i, B_i) : i = 1, 2, \ldots, n\}$ of pairs of disjoint closed sets is not essential in X,

(2) *for each $i = 1, 2, \dots n$ there are disjoint open sets U_i, V_i such that $A_i \subseteq U_i, B_i \subseteq V_i$ and*

$$\bigcup_{i=1}^{n} (U_i \cup V_i) = X,$$

(3) *for each $i = 1, 2, \dots n$ there are disjoint closed sets C_i, D_i such that $A_i \subseteq C_i, B_i \subseteq D_i$ and*

$$\bigcup_{i=1}^{n} (C_i \cup D_i) = X.$$

Theorem 3.1.5. *For a normal space X the following conditions are equivalent:*

(1) $\dim X \geq n$,
(2) *there is an essential family in X consisting of n pairs.*

Definition 3.1.6. Let $\pi \colon L \to K$ be a continuous function between compact Hausdorff spaces. We will say that π is **essential-preserving** if for every family $\{(A_i, B_i) : i = 1, 2, \dots, n\}$ essential in K, the family $\{(\pi^{-1}(A_i), \pi^{-1}(B_i)) : i = 1, 2, \dots, n\}$ is essential in L.

Note that Theorem 3.1.5 immediately implies that if $\pi \colon L \to K$ is essential-preserving, then $\dim L \geq \dim K$.

Lemma 3.1.7. *[22, Lemma 16.1] Assume that K_γ is an inverse limit of a system $\{K_\alpha : \alpha < \gamma\}$, where K_α are compact Hausdorff spaces. If A, B are closed disjoint subsets of K_γ then there is $\alpha < \gamma$ such that $\pi^\gamma_\alpha[A], \pi^\gamma_\alpha[B]$ are disjoint subsets of K_α, where π^γ_α stands for the canonical projection from K_γ into K_α.*

Theorem 3.1.8. *Let $\{K_\alpha : \alpha < \gamma\}$ be an inverse system of compact Hausdorff spaces with inverse limit K_γ such that for each limit ordinal $\beta < \gamma$, K_β is an inverse limit of $\{K_\alpha : \alpha < \beta\}$. Assume that for each $\alpha < \gamma$ the map $\pi^{\alpha+1}_\alpha \colon K_{\alpha+1} \to K_\alpha$ is surjective and essential-preserving. Then the canonical projection $\pi^\gamma_1 \colon K_\gamma \to K_1$ is essential-preserving. In particular* $\dim K_\gamma \geq \dim K_1$.

Proof. We will prove by induction on α that $\pi^\alpha_1 \colon K_\alpha \to K_1$ is essential-preserving. For successor ordinal $\alpha + 1$ it is enough to observe that if $\{(A_i, B_i) : i = 1, \dots, n\}$ is essential in K_1, then $\{((\pi^\alpha_1)^{-1}(A_i), (\pi^\alpha_1)^{-1}(B_i)) : i = 1, \dots, n\}$ is essential in K_α and hence $\{((\pi^{\alpha+1}_1)^{-1}(A_i), (\pi^{\alpha+1}_1)^{-1}(B_i)) : i = 1, \dots, n\} = \{((\pi^{\alpha+1}_\alpha)^{-1}((\pi^\alpha_1)^{-1}(A_i)), (\pi^{\alpha+1}_\alpha)^{-1}((\pi^\alpha_1)^{-1}(B_i))) : i = 1, \dots, n\}$ is essential in $K_{\alpha+1}$.

Let α be a limit ordinal and that for each $\beta < \alpha$ the map $\pi^\beta_1 \colon K_\beta \to K_1$ is essential-preserving. Let $\{(A_i, B_i) : i = 1, \dots, n\}$ be an essential family in K_1 and assume that $\{(\pi^\alpha_1)^{-1}(A_i), (\pi^\alpha_1)^{-1}(B_i)) : i = 1, \dots, n\}$ is not essential in K_α. Then by Theorem 3.1.4 for each $i \leq n$ there are closed disjoint sets $C_i \supseteq (\pi^\alpha_1)^{-1}(A_i), D_i \supseteq (\pi^\alpha_1)^{-1}(B_i)$ such that

$$\bigcup_{i=1}^{n} (C_i \cup D_i) = K_\alpha.$$

By Lemma 3.1.7 for each i there is $\beta_i < \alpha$ such that $\pi^\alpha_{\beta_i}[C_i], \pi^\alpha_{\beta_i}[D_i]$ are disjoint subsets of K_{β_i}. In particular $\pi^\alpha_\beta[C_i], \pi^\alpha_\beta[D_i]$ are disjoint closed subsets of K_β, where

$\beta = \max\{\beta_i : i \leq n\}$. Since K_α is an inverse limit of surjective maps π^α_β is also surjective and so

$$\bigcup_{i=1}^{n} (\pi^\alpha_\beta[C_i] \cup \pi^\alpha_\beta[D_i]) = K_\beta.$$

Moreover, $(\pi^\beta_1)^{-1}(A_i) \subseteq \pi^\alpha_\beta[C_i]$ and $(\pi^\beta_1)^{-1}(B_i) \subseteq \pi^\alpha_\beta[D_i]$, so $\{(\pi^\beta_1)^{-1}(A_i), \pi^\alpha_\beta[C_i] : i \leq n\}$ is not essential in K_β which contradicts the inductive assumption. □

We will need some basic but important properties of the covering dimension.

Theorem 3.1.9. *[38, Theorem 3.1.3] If M is a closed subspace of a normal space X, then $\dim M \leq \dim X$.*

Theorem 3.1.10. *[38, Theorem 3.1.8] Let $n \in \omega \cup \{\infty\}$. If a normal space X is a union of countably many closed subspaces $\{F_i\}_{i\in\omega}$ with $\dim F_i \leq n$, then $\dim X \leq n$.*

Theorem 3.1.11. *[38, Theorem 3.2.13] If X, Y are non-empty compact Hausdorff spaces, then $\dim(X \times Y) \leq \dim X + \dim Y$.*

Theorem 3.1.12. *[38, Theorem 3.4.11] If K is an inverse limit of compact Hausdorff spaces of dimension at most n, then $\dim K \leq n$.*

Definition 3.1.13. [38, p. 170] Let A be a subspace of a space X. We define the relative dimension of A as

$$\mathrm{rd}_X\, A = \sup\{\dim F : F \subseteq A, F \text{ closed in } X\}.$$

Lemma 3.1.14. *Let $n \in \omega \cup \{\infty\}$. Assume that a normal space X can be represented as a union $U \cup F$ where F is finite and $\mathrm{rd}_X\, U \leq n$. Then $\dim X \leq n$.*

Proof. This is a special case of [38, Lemma 3.1.6] (which says that if $X = \bigcup_{i=0}^{\infty} F_i$ and for each $k \in \omega$ the subspace $\bigcup_{i=0}^{k} F_i$ is closed in X, and $\mathrm{rd}_X\, F_k \leq n$, then $\dim X \leq n$) where $F_0 = F, F_1 = U$ and $F_n = \varnothing$ for $n > 1$. □

Theorem 3.1.15. *Assume that compact Hausdorff spaces X and Y can be represented as $X = U \cup F, Y = V \cup E$ where U, V are open, E, F are finite, $U \cap F = V \cap E = \varnothing$ and U is homeomorphic to V. Then $\dim X = \dim Y$.*

Proof. By Theorem 3.1.9 we have $\mathrm{rd}_X\, U \leq \dim X$ and by Lemma 3.1.14 $\dim X \leq \mathrm{rd}_X\, U$, so $\dim X = \mathrm{rd}_X\, U$. By the same argument $\dim Y = \mathrm{rd}_Y\, V$. Since X, Y are compact we have

$$\mathrm{rd}_X\, U = \sup\{\dim F : F \subseteq U, F \text{ compact}\}$$

and

$$\mathrm{rd}_Y\, V = \sup\{\dim F : F \subseteq V, F \text{ compact}\}.$$

But U and V are homeomorphic, so every compact subset of U is homeomorphic to some compact subset of V and vice versa, and hence $\mathrm{rd}_X\, U = \mathrm{rd}_Y\, V$. This gives $\dim X = \mathrm{rd}_X\, U = \mathrm{rd}_Y\, V = \dim Y$. □

Theorem 3.1.16. *Suppose that K is a metrizable compact space and μ is a non-zero Radon measure on K. Then there is a compact zero-dimensional subset $Z \subseteq K$ such that $\mu(Z) \neq 0$.*

Proof. Let $\{d_n\}_{n\in\omega}$ be a countable dense subset of K. For every $n \in \omega$ pick a countable local base $\{U_i^n\}_{i\in\omega}$ at d_n such that $\overline{U}_{i+1} \subseteq U_i$ for $i \in \omega$. Then for every $n \in \omega$ there is $k_n \in \omega$ such that

$$\sum_{i=k_n}^{\infty} |\mu|(\partial U_i^n) < \frac{\|\mu\|}{2^{n+1}}.$$

In particular we have

$$|\mu|(Y) \neq 0$$

where

$$Y = K \setminus \bigcup_{n=0}^{\infty} \bigcup_{i=k_n}^{\infty} \partial U_i^n.$$

Moreover, Y is zero-dimensional, since $\{U_i^n \cap Y : n \in \omega, i \geq k_n\} = \{(U_i^n \setminus \partial U_i^n) \cap Y : n \in \omega, i \geq k_n\}$ forms a base of Y consisting of clopen sets. By regularity of μ there is a compact set $Z \subseteq Y$ with $\mu(Z) \neq 0$ which is zero-dimensional as a compact subset of zero-dimensional space Y. □

3.2 Spaces $C(K)$ with few operators

We will follow the terminology from [88]. We say that a bounded linear operator $T\colon C(K) \to C(K)$ is a weak multiplication, if it is of the form $T = gI + S$, where g is a continuous function on K, I is the identity operator and $S\colon C(K) \to C(K)$ is weakly compact. T is called a weak multiplier, if $T^* = gI + S$ for some bounded Borel map $g\colon K \to \mathbb{R}$ and weakly compact $S\colon C(K)^* \to C(K)^*$.

Definition 3.2.1. Let K be a compact Hausdorff space. We say that the Banach space $C(K)$ has few operators if every bounded linear operator $T\colon C(K) \to C(K)$ is a weak multiplication.

Lemma 3.2.2. *Suppose that K is a c.c.c. compact Hausdorff space and that $C(K) \sim C(L)$ for a compact Hausdorff space L. Then L is also c.c.c.*

Proof. By [113, Theorem 4.5(a)] a compact space M is c.c.c. if and only if $C(M)$ contains no isomorphic copy of $c_0(\omega_1)$, so in particular given property is an isomorphism invariant. □

Lemma 3.2.3. *Let K be a compact Hausdorff space. If K has a non-trivial convergent sequence, then $C(K)$ admits a complemented copy of c_0. In particular, if $C(K)$ has few operators, then K has no non-trivial convergent sequences.*

Proof. The fact that non-trivial convergent sequences give rise to complemented copies of c_0 is well-known (see [62]). The second part of the lemma follows from [84, Theorem 13 (3)]. □

Lemma 3.2.4. *Assume that K is a separable connected compact Hausdorff space such that $C(K)$ has few operators and L is a compact Hausdorff space such that $C(K) \sim C(L)$. Let J be the set of isolated points in L and $L' = L \backslash J$. Then J is a countable set and L' has no isolated points.*

Proof. Since K is separable, it is c.c.c., so by Lemma 3.2.2 L is also c.c.c. In particular J is countable.

Obviously, if J is finite, then L' has no isolated points, so we may assume that J is infinite. Suppose that $x \in L'$ is an isolated point. Then $L' \backslash \{x\}$ is a closed subspace of L, so there is an open set $V \subseteq L$ such that $x \in V$ and $\overline{V} \cap (L' \backslash \{x\}) = \varnothing$. $\overline{V} \subseteq J \cup \{x\}$, so $\overline{V}$ is an infinite countable compact space with exactly one non-isolated point i.e. it is a convergent sequence. By Lemma 3.2.3 $C(L)$ admits a complemented copy of c_0, and so $C(K)$ admits a complemented copy of c_0. However, it is impossible since by [84, Theorem 13 (a)] $C(K)$ is indecomposable. □

Definition 3.2.5. For a compact space K and a function $f \in C(K)$ we denote by M_f the operator $M_f \colon C(K) \to C(K)$ given by $M_f(g) = fg$.

In the next lemmas we will use the following characterization of weakly compact operators on Banach spaces of continuous functions from [30, p. 160].

Theorem 3.2.6. *If K is a compact Hausdorff space, then an operator T on $C(K)$ is weakly compact if and only if for every bounded sequence $(e_n)_{n\in\omega}$ of pairwise disjoint functions (i.e. $e_n \cdot e_m = 0$ for $n \neq m$) we have $\lim_{n\to\infty} \|T(e_n)\| = 0$.*

Lemma 3.2.7. *Let L be a compact Hausdorff space, J the set of isolated points in L, and $L' = L \backslash J$. Assume that $f \in C(L)$ is such that $f \restriction L' = 0$. Then M_f is weakly compact.*

Proof. Fix any bounded pairwise disjoint sequence $(e_n)_{n\in\omega}$ of elements of $C(L)$. Without loss of generality we may assume that $\|e_n\| \leq 1$ for each n. Let $\varepsilon > 0$. Since f is continuous and equal to 0 on L' there is only finitely many points x such that $|f(x)| \geq \varepsilon$. Hence for n large enough we have $\|M_f(e_n)\| = \|fe_n\| < \varepsilon$, which means that $\lim_{n\to\infty} \|M_f(e_n)\| = 0$. Now Theorem 3.2.6 implies that M_f is weakly compact. □

Lemma 3.2.8. *Assume that K has no isolated points and $f \in C(K)$ is such that M_f is weakly compact. Then $f = 0$.*

Proof. Assume that $f \neq 0$. Then there is non-empty open set $U \subset K$ such that $|f(x)| \geq \varepsilon$ for $x \in U$ and some $\varepsilon > 0$. Since there are no isolated points in K, U is infinite, so there are pairwise disjoint open subsets $U_n \subseteq U$. Let $e_n \in C(K)$ be such that $e_n(x) = 1$ for some $x \in U_n, e_n(x) = 0$ for $x \in K \backslash U_n$ and $\|e_n\| = 1$. Then for each $n \in \omega$ we have $\|M_f e_n\| \geq \varepsilon$, so by Theorem 3.2.6 M_f is not weakly compact. □

Lemma 3.2.9. *Let $f \in C(L)$ for L compact Hausdorff and assume that there is a non-isolated point $x_0 \in L$ such that $|f(x_0)| = \|f\|$. If $R \colon C(L) \to C(L)$ is a weakly compact operator, then $\|f\| \leq \|M_f + R\|$.*

Proof. Since x_0 is non-isolated there are distinct points $x_n \in L$ such that $|f(x_n)| > \|f\| - 1/n$. By passing to a subsequence we may assume that $\{x_n : n \in \omega\}$ is a relatively discrete subset of L.

Take pairwise disjoint open sets $U_n \subseteq \{x \in K : |f(x)| > \|f\| - 1/n\}, x_n \in U_n$. For each $n \in \omega$ pick $e_n \in C(L)$ such that $\|e_n\| = 1$ and $e_n \restriction (L \backslash U_n) = 0$. In particular $(e_n)_{n\in\omega}$ are pairwise disjoint functions, so by Theorem 3.2.6 $\lim_{n\to\infty} \|R(e_n)\| = 0$. Moreover, $\|M_f(e_n)\| = \|fe_n\| \geq \|f\| - 1/n$ (from the property of U_n). Hence we get that $\|M_f + R\| \geq \|(M_f + R)(e_n)\| = \|M_f(e_n) + R(e_n)\| \geq \|M_f(e_n)\| - \|R(e_n)\| \geq \|f\| - 1/n - \|R(e_n)\|$. By taking limit with $n \to \infty$ we get $\|M_f + R\| \geq \|f\|$. □

Remark 3.2.10. *If K and L are compact Hausdorff spaces, and $T\colon C(K) \to C(L)$ is an isomorphism of Banach spaces, then T induces an isomorphism of the Banach algebras $\Phi_T\colon \mathcal{B}(C(L)) \to \mathcal{B}(C(K))$ given by*

$$\Phi_T(U) = T^{-1}UT.$$

If $R \in \mathcal{B}(C(L))$ is a weakly compact operator, then $\Phi_T(R)$ is also weakly compact as a composition of a weakly compact operator with bounded operators. Similarly, if $S \in \mathcal{B}(C(K))$ is weakly compact, then $\Phi_T^{-1}(S)$ is weakly compact.

For the rest of this section we will assume that K and L are compact Hausdorff spaces, L' is the set of non-isolated points of L, $C(K)$ has few operators and $T\colon C(K) \to C(L)$ is an isomorphism of Banach spaces.

Definition 3.2.11. Let Φ_T be such as in Remark 3.2.10. We define an operator $\Psi_T\colon C(L') \to C(K)$ by putting for each $f' \in C(L')$

$$\Psi_T(f') = g,$$

for $g \in C(K)$ satisfying $\Phi_T(M_f) = M_g + R$, where R is weakly compact and $f \in C(L)$ is such that $f' = f \restriction L'$.

In other words, Ψ_T is defined in the way such that the following diagram commutes:

$$\begin{tikzcd}
C(L) \arrow[r, "M"] \arrow[d, "\mathcal{R}"'] & \mathcal{B}(C(L)) \arrow[r, leftrightarrow, "\Phi_T"] & \mathcal{B}(C(K)) \arrow[r, "\pi"] & \mathcal{B}(C(K))/\mathcal{WC}(C(K)) \arrow[d, leftrightarrow, "\mathcal{I}"] \\
C(L') \arrow[rrr, dashed, "\Psi_T"'] & & & C(K)
\end{tikzcd}$$

Here $\mathcal{R}$ stands for the restriction operator (i.e. $\mathcal{R}(f) = f \restriction L'$), $M(f) = M_f$, π is the natural surjection onto the quotient algebra $\mathcal{B}(C(K))/\mathcal{WC}(C(K))$, where $\mathcal{WC}(C(K))$ is the closed ideal in $\mathcal{B}(C(K))$ consisting of weakly compact operators and $\mathcal{I}\colon \mathcal{B}(C(K))/\mathcal{WC}(C(K)) \to C(K)$ is the isometry given by $\mathcal{I}([M_g]) = g$.

Lemma 3.2.12. *Suppose that K has no isolated points. Then the induced operator $\Psi_T\colon C(L') \to C(K)$ from Definition 3.2.11 is a well-defined bounded linear and multiplicative operator.*

Proof. Take any $f' \in C(L')$ and let $f_1, f_2 \in C(L)$ and $g_1, g_2 \in C(K)$ be such that $f_1 \restriction L' = f_2 \restriction L' = f'$ and

$$\Phi_T(M_{f_i}) = M_{g_i} + R_i \text{ for } i = 1, 2,$$

where R_1, R_2 are weakly compact. Then $(f_1 - f_2) \restriction L' = 0$, so by Lemma 3.2.7 $M_{f_1} - M_{f_2} = M_{f_1 - f_2}$ is weakly compact. This implies that

$$\begin{aligned} M_{g_1 - g_2} = M_{g_1} - M_{g_2} = R_1 - \Phi_T(M_{f_1}) - R_2 + \Phi_T(M_{f_2}) = \\ = R_1 - R_2 - \Phi_T(M_{f_1} - M_{f_2}) \end{aligned}$$

is weakly compact since $\Phi_T(M_{f_1} - M_{f_2})$ is weakly compact (by Remark 3.2.10). Since K has no isolated points, Lemma 3.2.8 implies that $g_1 - g_2 = 0$, so Ψ_T is well-defined.

For the linearity and multiplicativeness fix $f_1' = f_1 \restriction L', f_2' = f_2 \restriction L' \in C(L), a, b \in \mathbb{R}$ and put $\Psi_T(f_1') = g_1, \Psi_T(f_2') = g_2$. We have

$$\begin{aligned} \Phi_T(M_{af_1 + bf_2}) = \Phi_T(aM_{f_1} + bM_{f_2}) = a\Phi_T(M_{f_1}) + b\Phi_T(M_{f_2}) = \\ = M_{ag_1} + aR_1 + M_{bg_2} + bR_2 = M_{ag_1 + bg_2} + aR_1 + bR_2 \end{aligned}$$

and

$$\begin{aligned} \Phi_T(M_{f_1 f_2}) = \Phi_T(M_{f_1} M_{f_2}) = \Phi_T(M_{f_1})\Phi_T(M_{f_2}) = \\ = (M_{g_1} + R_1)(M_{g_2} + R_2) = M_{g_1 g_2} + R_1 M_{g_2} + M_{g_1} R_2 + R_1 R_2. \end{aligned}$$

But $aR_1 + bR_2$ and $R_1 M_{g_2} + M_{g_1} R_2 + R_1 R_2$ are weakly compact as the sums of weakly compact operators composed with bounded operators. Hence $\Psi_T(af_1' + bf_2') = ag_1 + bg_2$ and $\Psi_T(f_1' f_2') = g_1 g_2$.

Now we will show that Ψ_T is bounded. Pick any $f' \in C(L')$. By the Tietze theorem f' has an extension $f \in C(L)$ satisfying $\|f\| = \|f'\|$. From Lemma 3.2.9 we get that if $\Phi_T(M_f) = M_g + R$, then $\|g\| \leq \|M_g + R\| \leq \|\Phi_T\|\|M_f\| = \|\Phi_T\|\|f\| = \|\Phi_T\|\|f'\|$, so $\|\Psi_T\| \leq \|\Phi_T\|$. □

Lemma 3.2.13. *Suppose that K is separable and connected. Then there is $c > 0$ such that for every $f' \in C(L')$ we have $\|\Psi_T(f')\| \geq c\|f'\|$ i.e. Ψ_T is an isomorphism onto its range. In particular Ψ_T has closed range.*

Proof. Assume that $\Psi_T(f') = g$. Let $f \in C(L)$ be an extension of f' such that $\|f\| = \|f'\|$. We have $\Phi_T(M_f) = M_g + R$ for some weakly compact R, so $\Phi_T^{-1}(M_g) = M_f - \Phi_T^{-1}(R)$. $\Phi_T^{-1}(R)$ is weakly compact by Remark 3.2.10, so from Lemma 3.2.9 we get

$$\begin{aligned} \|f\| \leq \|M_f - \Phi_T^{-1}(R)\| = \|\Phi_T^{-1} \circ \Phi_T(M_f - \Phi_T^{-1}(R))\| = \|\Phi_T^{-1}(M_g + R - R)\| = \\ = \|\Phi_T^{-1}(M_g)\| \leq \|\Phi_T^{-1}\|\|M_g\| = \|\Phi_T^{-1}\|\|g\|. \end{aligned}$$

Hence it is enough to take $c = \frac{1}{\|\Phi_T^{-1}\|}$. □

Proposition 3.2.14. *Suppose that K is separable and connected. Let $S\colon C(K) \to C(K)$ be given by $S(f) = \Psi_T(T(f) \upharpoonright L')$. Then*

$$\ker(S) = T^{-1}(\{g \in C(L) : g \upharpoonright L' = 0\})$$

and it is a separable subspace of $C(K)$.

$$\begin{tikzcd} C(K) \arrow[r, "T"] \arrow[rrr, bend right, "S"'] & C(L) \arrow[r, "\mathcal{R}"] & C(L') \arrow[r, "\Psi_T"] & C(K) \end{tikzcd}$$

Proof. By Lemma 3.2.4 the set J of isolated points in L is countable, so we may write $J = \{x_n : n \in \omega\}$. Let $\chi_{\{x_n\}}$ be the characteristic function of $\{x_n\}$. Observe that $\overline{\operatorname{span}\{\chi_{\{x_n\}} : n \in \omega\}} = \{g \in C(L) : g \upharpoonright L' = 0\}$ is a separable subspace of $C(L)$, so it is enough to show that $\ker(S) = T^{-1}(\{g \in C(L) : g \upharpoonright L' = 0\})$, since T is an isomorphism.

Assume that $S(f) = 0$. Then $\Psi_T(T(f) \upharpoonright L') = 0$, so $\Phi_T(M_{T(f)}) = M_0 + R = R$ is weakly compact and hence $M_{T(f)} = T\Phi_T(M_{T(f)})T^{-1}$ is also weakly compact as a composition of a weakly compact operator with bounded operators. From Theorem 3.2.6 $\lim_{n\to\infty} \|T(f)e_n\| = 0$ for every bounded disjoint sequence $(e_n)_{n\in\omega}$. This implies that $\lim_{n\to\infty} \|(T(f) \upharpoonright L')e_n\| = 0$ for every bounded disjoint sequence $(e_n)_{n\in\omega}$. By applying Theorem 3.2.6 once again we get that $M_{T(f)\upharpoonright L'}$ is weakly compact as an operator on $C(L')$. Since L' has no isolated points (by Lemma 3.2.4) we get that $T(f) \upharpoonright L' = 0$ by Lemma 3.2.8 i.e. $f \in T^{-1}(\{g \in C(L) : g \upharpoonright L' = 0\})$, so $\ker(S) \subseteq T^{-1}(\{g \in C(L) : g \upharpoonright L' = 0\})$.

If $g \in C(L)$ is such that $g \upharpoonright L' = 0$, then by Lemma 3.2.7 M_g is weakly compact, so $S(T^{-1}(g)) = \Psi(g \upharpoonright L') = \Psi(0) = 0$ and hence $T^{-1}(g) \in \ker(S)$. □

Proposition 3.2.15. *Suppose that K is separable and connected. Let $S = \Psi_T(T(f) \upharpoonright L')$. Write S as a sum $S = M_e + W$ with W weakly compact. Then M_e is an isomorphism of $C(K)$.*

Proof. It is enough to prove that $e(x) \neq 0$ for every $x \in K$. Indeed, if it is the case, then M_g is the inverse of M_e for $g = \frac{1}{e}$.

Assume that $e(z) = 0$ for some $z \in K$ and aim for a contradiction. Then using the technique from the proof of Lemma 3.2.9 we construct pairwise disjoint non-empty open subsets $U_n \subseteq K$ such that $\|e \upharpoonright U_n\| \leq \frac{1}{n}$ for each $n \in \omega$. Let V_n be non-empty open sets such that $\overline{V}_n \subseteq U_n$.

By Lemma 3.2.3 K has no convergent sequences and hence for every $n \in \omega$ the space $\overline{V}_n$ is non-metrizable as an infinite (because V_n has no isolated points) compact set without convergent sequences. We get that points in $\overline{V}_n$ cannot be separated by countable family of continuous functions (otherwise, if $(f_n)_{n\in\omega}$ separated points of $\overline{V}_n$, $(f_1, f_2, \dots)\colon \overline{V}_n \to \mathbb{R}^n$ would be a homeomorphism onto a compact subspace of metrizable space), so since $\ker(S)$ is separable, there are points $x_n, y_n \in \overline{V}_n \subseteq U_n$ such that $d(x_n) = d(y_n)$ for all $d \in \ker(S)$. Let $f_n \in C(K)$ be such that $\|f_n\| = 1, f_n(x_n) =$

$1, f_n(y_n) = 0$ and $f_n \restriction (K \backslash U_n) = 0$. Then for all $d \in \ker(S)$

$$\begin{aligned} \|f_n - d\| \geq \max\{|f_n(x_n) - d(x_n)|, |f_n(y_n) - d(y_n)|\} = \\ = \max\{|1 - d(x_n)|, |d(x_n)|\} \geq 1/2. \end{aligned}$$

Since $f_n \restriction (K \backslash U_n) = 0$ and $\|e \restriction U_n\| \leq \frac{1}{n}$ we have $\|ef_n\| \leq \frac{1}{n}$, so $\lim_{n\to\infty} \|ef_n\| = 0$.

Ψ_T has closed range (Lemma 3.2.13) and $T, \mathcal{R}$ are surjective, so S has also closed range. By the first isomorphism theorem (see e.g. [39, Corollary 2.26]) $S[C(K)]$ is isomorphic to $C(K)/\ker(S)$, so since the distance of f_n from $\ker(S)$ is greater than $1/2$ for all $n \in \omega$, there is $c > 0$ such that $\|S(f_n)\| > c$ for all $n \in \omega$.

$$\begin{tikzcd} C(K) \arrow[rr, "S"] \arrow[dr] & & S[C(K)] \\ & C(K)/\ker(S) \arrow[ur, "\sim"'] & \end{tikzcd}$$

But on the other hand we have

$$\|S(f_n)\| = \|ef_n + W(f_n)\| \leq \|ef_n\| + \|W(f_n)\| \to 0$$

when $n \to \infty$, since we have $\lim_{n\to\infty} \|ef_n\| = 0$ and $\lim_{n\to\infty} \|W(f_n)\| = 0$ (because W is weakly compact and (f_n) are bounded and pairwise disjoint), so we get a contradiction. $\square$

Recall that an operator $R\colon X \to Y$ is called strictly singular, if for every infinite-dimensional subspace $X' \subseteq X$ the restriction $R \restriction X'$ is not isomorphism. We cite the result from [101].

Theorem 3.2.16. *Let X be a compact Hausdorff space. A bounded operator $R\colon C(X) \to C(X)$ is weakly compact if and only if it is strictly singular.*

If we apply the above theorem to [90, Proposition 2.c.10] we get the following.

Theorem 3.2.17. *Let $E\colon C(X) \to C(X)$ be an operator with a closed range for which* $\dim\ker(E) < \infty$ *and* $\dim(C(X)/E(C(X))) < \infty$. *Let $R\colon C(X) \to C(X)$ be weakly compact. Then $E + R$ also has a closed range and* $\dim((C(X))/(E+R)(C(X)) < \infty$.

Corollary 3.2.18. *Suppose that K is separable and connected. Let $S = \Psi_T(T(f) \restriction L')$.*

Then the range of S is finite-codimensional in $C(K)$. In particular the range of Ψ_T is finite-codimensional in $C(K)$.

Proof. Since M_e is an isomorphism (by Proposition 3.2.15) and W is weakly compact we may apply Theorem 3.2.17 to $S = M_e + W$. $\square$

Since $\Psi_T\colon C(L') \to C(K)$ is a bounded linear multiplicative operator (Lemma 3.2.12), there is $\varphi\colon K \to L'$ such that $\Psi_T(f) = f \circ \varphi$ for $f \in C(L')$ (see e.g. [119, Theorem 7.7.1]). From Lemma 3.2.13 and Corollary 3.2.18 we get that Ψ_T is an embedding with finite-codimensional range, so the induced map φ is surjective and has only finitely many fibers containing more than one element and each of these fibers is finite. In particular $K = U \cup F$ where F is a finite set and $\varphi \restriction U$ is a homeomorphism and we get the following theorem.

Theorem 3.2.19. *Suppose that K is a separable connected compact Hausdorff space such that $C(K)$ has few operators and L is a compact Hausdorff space such that $C(K) \sim C(L)$.*

Then K and L are homeomorphic modulo finite set i.e. there are open subsets $U \subseteq K, V \subseteq L$ and finite sets $E \subseteq K, F \subseteq L$ such that U, V are homeomorphic and $K = U \cup E, L = V \cup F$.

Corollary 3.2.20. *If $\dim(K) = n$ and K is a compact, separable and connected Hausdorff space such that $C(K)$ has few operators, then for each compact Hausdorff space L such that $C(K) \sim C(L)$ we have $\dim(L) = n$.*

Proof. Use Theorem 3.2.19 and Theorem 3.1.15. □

3.3 Extensions of compact spaces

In this section we consider the notion of strong extension from [80]. We describe the methods of controlling the dimension in constructions of compact spaces using strong extensions. We prove that strong extensions cannot lower the dimension of initial space and we show how to construct extensions that cannot rise the dimension.

Definition 3.3.1. Let K be a compact Hausdorff space and $(f_n)_{n\in\omega}$ be a sequence of pairwise disjoint continuous functions $f_n \colon K \to [0,1]$. Define

$$D((f_n)_{n\in\omega}) = \bigcup\{U : U \text{ is open and } \{n : \operatorname{supp}(f_n) \cap U \neq \varnothing\} \text{ is finite}\}.$$

We say that $L \subseteq K \times [0,1]$ is the extension of K by $(f_n)_{n\in\omega}$ if and only if L is the closure of the graph of $(\sum_{n\in\omega} f_n) \restriction D((f_n)_{n\in\omega})$. We say that this is a strong extension, if the graph of $\sum_{n\in\omega} f_n$ is a subset of L.

Lemma 3.3.2. *[80, Lemma 4.1] If $(f_n)_{n\in\omega}$ are pairwise disjoint continuous functions on K with values in $[0,1]$, then $\sum_{n\in\omega} f_n$ is well-defined and continuous in the dense open set $D((f_n)_{n\in\omega})$.*

Lemma 3.3.3. *[80, Lemma 4.4] Strong extension of a connected compact Hausdorff space is connected.*

Note that there are known examples of extensions of connected compact spaces which are not connected (see [10]), so the assumption that considered extensions are strong is necessary.

Lemma 3.3.4. *Let K be a separable compact Hausdorff space with a countable dense set $Q = \{q_n :\in \omega\}$ and let L be an extension of K with the natural projection $\pi\colon L \to K$. Assume that $Q' = \{q'_n : n \in \omega\}$ is a subset of L such that $\pi(q'_n) = q_n$ for every $n \in \omega$. Then Q' is a dense subset of L.*

Proof. Let $(f_n)_{n\in\omega}$ be a sequence of pairwise disjoint continuous functions such that L is the extension of K by $(f_n)_{n\in\omega}$. By [80, Lemma 4.3 a)] $\pi^{-1}(D((f_n)_{n\in\omega})$ is dense in L. Moreover, $\pi\restriction\pi^{-1}(D((f_n)_{n\in\omega}))$ is a homeomorphism as a projection of graph of continuous function onto its domain. Since Q is dense in K and $D((f_n)_{n\in\omega})$ is open, $Q\cap D((f_n)_{n\in\omega})$ is dense in $D((f_n)_{n\in\omega})$. Hence we get that $\pi^{-1}(Q\cap D((f_n)_{n\in\omega}))$ is dense in L. But if $q_n\in D((f_n)_{n\in\omega})$, then $\pi^{-1}(q_n)=\{q'_n\}$, so $Q'\supseteq\pi^{-1}(Q\cap D((f_n)_{n\in\omega})$ is also dense in L. □

The following lemma is a special case of [80, Lemma 4.5].

Lemma 3.3.5. *Suppose that K is a compact metric space and that for every $n\in\omega$ X_1^n, X_2^n are disjoint relatively discrete subsets of K such that $\overline{X_1^n}\cap\overline{X_2^n}\neq\varnothing$. Let $(f_n)_{n\in\omega}$ be a pairwise disjoint sequence of continuous functions from K into $[0,1]$. For an infinite subset $B\subseteq\omega$ denote by $K(B)$ the extension of K by $(f_n)_{n\in B}$. For $i=0,1$ and $n\in\omega$ put*

$$X_i^n(B)=\{(x,t): x\in X_i^n, t=\sum_{k\in B}f_k(x)\}.$$

Then there is an infinite $N\subseteq\omega$ such that for every $B\subseteq N$:

(1) $K(B)$ is a strong extension of K by $(f_n)_{n\in B}$,
(2) $\overline{X_1^n(B)}\cap\overline{X_2^n(B)}\neq\varnothing$ for every $n\in\omega$, where the closures are taken in $K(B)$.

Proposition 3.3.6. *If L is a strong extension of a compact Hausdorff space K with the natural projection $\pi\colon L\to K$, then π is essential-preserving.*

Proof. Let $(f_k)_{k\in\omega}$ be such that L is a strong extension of K by $(f_k)_{k\in\omega}$.
Let $\{(A_i,B_i): i=1,2,\dots,n\}$ be an essential family in K and assume that the family $\{(\pi^{-1}(A_i),\pi^{-1}(B_i)): i=1,2,\dots,n\}$ is not essential in L. By Theorem 3.1.4 there are closed sets $C_i\supseteq\pi^{-1}(A_i), D_i\supseteq\pi^{-1}(B_i)$ such that $C_i\cap D_i=\varnothing$ for each $i\leq n$ and

$$\bigcup_{i=1}^n(C_i\cup D_i)=L.$$

Since C_i, D_i are compact, there are sets U_i, V_i open in $K\times[0,1]$ such that $C_i\subseteq U_i$, $D_i\subseteq V_i$ and $U_i\cap V_i=\varnothing$ for every $i\leq n$.

For each $k\in\omega$ denote by L_k the graph of $\sum_{i\leq k}f_i$ and let $\pi_k\colon L_k\to K$ be the projection onto K.

Claim 1. *For every $k\in\omega$ we have*

$$L_k\setminus\bigcup_{i=1}^n(U_i\cup V_i)\neq\varnothing.$$

Proof of the claim. Assume that there is N such that

$$L_N\subseteq\bigcup_{i=1}^n(U_i\cup V_i).$$

Then for every $k \geq N$

$$L_k \backslash L_N = \operatorname{graph}(\sum_{i=N+1}^{k} f_i | \operatorname{supp}(\sum_{i=N+1}^{k} f_i)) \subseteq L \subseteq \bigcup_{i=1}^{n} (U_i \cup V_i) \tag{3.1}$$

(the first equality holds, because the supports of f_i's are pairwise disjoint), so we have

$$L_k \subseteq \bigcup_{i=1}^{n} (U_i \cup V_i).$$

Put $A_i^k = \pi_k^{-1}(A_i), B_i^k = \pi_k^{-1}(B_i)$ and observe that the family $\{(A_i^k, B_i^k) : i \leq n\}$ is essential in L_k since π_k is a homeomorphism. Hence there is $i \leq n$ such that $A_i^k \nsubseteq U_i$ or $B_i^k \nsubseteq V_i$. Indeed, otherwise $U_i \cap L_k, V_i \cap L_k$ would be disjoint open subsets of L_k with

$$\bigcup_{i=1}^{n} ((U_i \cap L_k) \cup (V_i \cap L_k)) = L_k,$$

which contradicts the fact that $\{(A_i^k, B_i^k) : i \leq n\}$ is essential (by Theorem 3.1.4). Without loss of generality there are infinitely many k such that $A_1^k \backslash U_1 \neq \varnothing$. For every $k \in \omega$ we have

$$A_1^{k+1} \backslash A_1^k = \pi_{k+1}^{-1}(A_1) \backslash \pi_k^{-1}(A_1) = \operatorname{graph}(f_{k+1} \upharpoonright (A_1 \cap \operatorname{supp}(f_{k+1})) \subseteq \pi^{-1}(A_1) \subseteq U_1.$$

In particular $(A_1^k \backslash U_1)_{k \in \omega}$ form a decreasing sequence of non-empty compact sets. Hence

$$A = \bigcap_{k=1}^{\infty} A_1^k \backslash U_1 \neq \varnothing.$$

We have $A \subseteq L$ since if $(x,t) \in A$, then $f_k(x) = 0$ for all k, so $\sum_{k \in \omega} f_k(x) = 0$ and hence $(x,t) = (x,0)$ is an element of the graph of $\sum_{k \in \omega} f_k$ which is a subset of L. Moreover $A \subseteq A_1 \times [0,1]$, so $A \subseteq (A_1 \times [0,1]) \cap L = \pi^{-1}(A_1)$ which contradicts the assumption that $\pi^{-1}(A_1) \subseteq U_1$ and completes the proof of the claim. □

To finish the proof of the proposition put

$$F_k = L_k \backslash \bigcup_{i=1}^{n} (U_i \cup V_i)$$

and observe that $(F_k)_{k \in \omega}$ is a decreasing sequence of non-empty compact sets (by (1) from the claim), so as in the case of the set A from the claim we get that

$$F = \bigcap_{k=1}^{\infty} F_k$$

is a non-empty subset of the graph of $\sum_{k \in \omega} f_k$, so $F \subseteq L$ (because the extension is strong), which is a contradiction, since F is disjoint from $\bigcup_{i \leq n} (U_i \cup V_i) \supseteq L$. □

Lemma 3.3.7. *Suppose that K is a compact metric space with $0 < \dim(K) \leq n$ and $f_k \colon K \to [0,1]$ are pairwise disjoint continuous functions such that the set*

$$Z = K \backslash D((f_k)_{k \in \omega})$$

is zero-dimensional. Assume that L is a strong extension of K by $(f_k)_{k \in \omega}$. Then $\dim L \leq n$.

Proof. Let π be the natural projection from L onto K. $\pi^{-1}(D((f_k)_{k\in\omega}))$ is an open subset of a metric space, so it is a union of countably many closed sets, each of dimension at most n since $\pi^{-1}(D((f_k)_{k\in\omega}))$ is homeomorphic to $D((f_k)_{k\in\omega})$ (see Theorem 3.1.9). The set $\pi^{-1}(Z)$ is included in $Z \times [0,1]$ so $\dim \pi^{-1}(Z) \leq 1 \leq n$ by Theorem 3.1.11. Hence $L = \pi^{-1}(D((f_k)_{k\in\omega})) \cup \pi^{-1}(Z)$ is a countable union of closed sets of dimension at most n. Now Theorem 3.1.10 gives the inequality $\dim L \leq n$. □

Corollary 3.3.8. *Let γ be an ordinal number. Suppose that $\{K_\alpha : \alpha < \gamma\}$ is an inverse system of compact Hausdorff spaces such that:*

- *for every α the map $\pi_\alpha^{\alpha+1} : K_{\alpha+1} \to K_\alpha$ is a strong extension by pairwise disjoint continuous functions $(f_n^\alpha)_{n\in\omega}$ and the set $Z_\alpha = K_\alpha \backslash D((f_n^\alpha)_{n\in\omega})$ is zero-dimensional,*
- *if α is a limit ordinal, then K_α is the inverse limit of $\{K_\beta : \beta < \alpha\}$.*

Denote by K_γ the inverse limit of $\{K_\alpha : \alpha < \gamma\}$. Then $\dim K_\gamma = \dim K_1$.

Proof. The inequality $\dim K_\gamma \geq \dim K_1$ follows from Proposition 3.3.6 and Theorem 3.1.8. The inequality $\dim K_\gamma \leq \dim K_1$ follows from Lemma 3.3.7 and Theorem 3.1.12. □

3.4 The main construction

Theorem 3.4.1. *[88, Lemma 2.4] Suppose that K is a compact Hausdorff space. If a bounded linear operator $T\colon C(K) \to C(K)$ is not a weak multiplier, then there are $\delta > 0$, a pairwise disjoint sequence $(g_n)_{n\in\omega} \subseteq C_I(K)$ and pairwise disjoint open sets $(V_n)_{n\in\omega}$ such that*

$$\mathrm{supp}(g_n) \cap V_m = \varnothing$$

for all $n, m \in \omega$ and

$$|T(g_n)|V_n| > \delta$$

for all $n \in \omega$.

In particular, if $x_n \in V_n$ and $\mu_n = T^*(\delta_{x_n})$ for $n \in \omega$, then $|\int g_n d\mu_n| = |T(g_n)(q_{l_n})| > \delta$, and so $|\mu_n|(\mathrm{supp}(g_n)) \geq |\int g_n d\mu_n| > \delta$.

The idea behind the construction is as follows. We will construct a compact space K as the inverse limit of spaces $K_\alpha \subseteq [0,1]^\alpha$ (so the final space is a subset of $[0,1]^{\mathfrak{c}}$). For each bounded sequence $(\mu_n)_{n\in\omega}$ of Radon measures on $[0,1]^{\mathfrak{c}}$ and a sequence of pairwise disjoint open sets $(V_n)_{n\in\omega}$ we want to use a strong extension in such a way that in the final space there will be no sequence $(g_n)_{n\in\omega}$ for which the properties from Theorem 3.4.1 are satisfied. However, we need to consider $2^{\mathfrak{c}}$ sequences of Radon measures on $[0,1]^{\mathfrak{c}}$, while there are only $\mathfrak{c}$ steps in the construction. In order to handle this we will use $\diamondsuit$ (cf. Lemma 3.4.4).

Proposition 3.4.2. *Let K be a compact metrizable space and $(\mu_n)_{n\in\omega}$ be a bounded sequence of Radon measures on K. Assume that $(U_n)_{n\in\omega}$ is a sequence of pairwise disjoint open sets and $\delta > 0$ is such that $|\mu_n|(U_n) > \delta$ for $n \in \omega$. Then there is an infinite set $N \subseteq \omega$, continuous pairwise disjoint functions $f_n\colon K \to [0,1]$ and $\varepsilon > 0$ such that*

(1) $\mathrm{supp}(f_n) \subseteq U_n$ *for* $n \in N$,
(2) $|\int f_n d\mu_n| > \varepsilon$ *for* $n \in N$,
(3) $\sum\{|\int f_m d\mu_n| : n \neq m, m \in N\} < \varepsilon/3$ *for* $n \in N$,
(4) $K\backslash D((f_n)_{n\in N})$ *is zero-dimensional.*

Proof. Since μ_n's are Radon measures there is $\delta' > 0$ and open sets $U_n' \subseteq U_n$ such that $|\mu_n(U_n')| > \delta'$ for $n \in \omega$. Without loss of generality we may assume that $U_n' = U_n$ and $\delta' = \delta$.

Put $\nu_n = \mu_n \restriction U_n$ for $n \in \omega$. Let N' be such that the sequence $(\nu_n)_{n\in N'}$ has the weak* limit ν. Since $|\int 1 d\nu_n| > \delta$ for every n, we have $|\int 1 d\nu| \geq \delta$, so ν is a non-zero measure. By Theorem 3.1.16 there is a compact zero-dimensional subset $Z \subseteq K$ and $\varepsilon > 0$ such that $|\nu(Z)| > 2\varepsilon$. Since Z is a closed subset of a metrizable space and ν is a regular measure, there is a decreasing sequence of open sets $(G_n)_{n\in N'}$ such that $Z = \bigcap G_n$ and $|\nu(G_n)| > 2\varepsilon$ for all $n \in N'$.

Note that if $f \in C_I(K)$ is such that $\mathrm{supp}(f) \subseteq G_n$ and $|\int f d\nu| > 2\varepsilon$, then for big enough $l \in N'$ we have $|\int f d\nu_l| > 2\varepsilon$ and so $|\nu_l|(G_n) = |\nu_l|(G_n \cap U_l) > 2\varepsilon$. Hence for each $l \in N'$ we may pick $f_l \in C_I(K)$ such that $\mathrm{supp}\, f_l \subseteq G_n \cap U_l$ and $|\int f_l d\nu_l| = |\int f_l d\mu_l| > \varepsilon$.

For each $n \in N'$ let $l_n \in N'$ be such that $\mathrm{supp}\, f_{l_n} \subseteq G_n \cap U_{l_n}$, $|\int f_{l_n} d\mu_{l_n}| > \varepsilon$ and $(l_n)_{n\in N'}$ is an increasing sequence. Let $N'' = \{l_n : n \in N'\}$. For every $M \subseteq N''$ denote $Z_M = K\backslash D((f_{l_n})_{n\in M})$. If $x \in K\backslash Z$, then there is an open neighbourhood $V \ni x$ such that for big enough $n \in M$ we have $V \cap G_n = \varnothing$ and so $V \cap \mathrm{supp}(f_{l_n}) = \varnothing$. Hence $V \subseteq D((f_{l_n})_{n\in M})$, which gives $x \notin Z_M$. This implies that $Z_M \subseteq Z$, so in particular Z_M is zero-dimensional and condition (4) is satisfied for any choice of $M \subseteq N''$. Now we use Rosenthal's lemma (see [31, p. 82] or [120]) to obtain an infinite $N \subseteq N''$ such that the 3rd condition is also satisfied. □

We will need the following strengthening of [80, Lemma 6.2].

Lemma 3.4.3. *Let K be a compact, connected metrizable space with a countable dense set $Q = \{q_n : n \in \omega\}$. Let U, V be open subsets of K such that $\overline{U} \cap \overline{V} \neq \varnothing$. Then there is a sequence $(f_n)_{n\in\omega}$ of pairwise disjoint functions $f_n \in C_I(K)$ and infinite sets $A_0, A_1, S_0, S_1 \subseteq \omega$ such that:*

(1) the sets $\{q_n : n \in S_0\} \subseteq U, \{q_n : n \in S_1\} \subseteq V$ are relatively discrete,
(2) $A_i \subseteq S_i$ and $|S_i\backslash A_i| = \omega$ for $i = 0, 1$,
(3) for every infinite $B \subseteq \omega$ in the extension $K(B)$ of K by $(f_n)_{n\in B}$ there are disjoint closed sets $F_0, F_1 \subseteq K(B)$ and distinct $x_0, x_1 \in K(B)$ such that for $i = 0, 1$

$$x_i \in \overline{\pi^{-1}(U) \cap \{q_n^B : n \in A_i\}} \cap \overline{\pi^{-1}(V) \cap \{q_n^B : n \in S_i\backslash A_i\}}$$

and

$$\{q_n^B : n \in A_i\} \subseteq F_i,$$

where $q_j^B = (q_j, t)$ *and* $t = \sum_{n\in B} f_n(q_j)$,

(4) $|K \backslash D(f_n)_{n\in B}| = 1$ *(in particular* $K\backslash D(f_n)_{n\in B}$ *is zero-dimensional).*

Proof. Fix any compatible metric d on K. Pick any $x \in \overline{U} \cap \overline{V}$. Since K is connected, x is not an isolated point. For $n \in \omega$ put $U'_n = U \cap \mathcal{B}(x, 1/n), V'_n = V \cap \mathcal{B}(x, 1/n)$ (where $\mathcal{B}(x, \varepsilon)$ is the open ball with x as the center and radius ε with respect to d) and let $U_n \subseteq U'_n, V_n \subseteq V'_n$ be non-empty open sets such that the members of the family $\{U_n, V_n : n \in \omega\}$ are pairwise disjoint. Take continuous functions $f_n \in C(K)$ and $k_n, l_n \in \omega$ such that:

- $q_{k_n} \in U_n, q_{l_n} \in V_n$,
- $f_n(q_{k_{2n}}) = f_n(q_{l_{2n}}) = 1$,
- $\mathrm{supp}(f_n) \subseteq U_{2n} \cup V_{2n}$.

Let $B \subseteq \omega$ be infinite. For (2) and (3) it is enough to take $S_0 = \{k_{2n+1}, l_{2n+1} : n \in \omega\}, A_0 = \{k_{2n+1} : n \in \omega\}, S_1 = \{k_{2n}, l_{2n} : n \in \omega\}, A_1 = \{k_{2n} : n \in \omega\}$, $x_0 = (x, 0), x_1 = (x, 1)$ and $F_0 = K(B) \cap (K \times [0, 1/3]), F_1 = K(B) \cap (K \times [2/3, 1])$. (1) is satisfied since U_n, V_m are pairwise disjoint for $n, m \in \omega$.

(4) follows from the fact that x is the only point all of whose neighborhoods intersect all but finitely many U_n's and V_n's, so we have $K\backslash D(f_n)_{n\in B} = \{x\}$.

□

Lemma 3.4.4. *Assume* $\diamondsuit$*. Then there is a sequence* $(M^\alpha, \mathcal{U}^\alpha, L^\alpha)_{\alpha<\omega_1}$ *such that:*

- $M^\alpha = (\mu_n^\alpha)_{n\in\omega}$ *is a bounded sequence of Radon measures on* $[0, 1]^\alpha$,
- $\mathcal{U}^\alpha = (U_{n,m}^\alpha)_{n,m\in\omega}$ *is a sequence of basic open sets in* $[0, 1]^\alpha$,
- $L^\alpha = (l_n^\alpha)_{n\in\omega}$ *is a sequence of distinct natural numbers,*

and for every:

- *bounded sequence* $(\mu_n)_{n\in\omega}$ *of Radon measures on* $[0, 1]^{\omega_1}$,
- *sequence* $(U_{n,m})_{n,m\in\omega}$ *of basic open sets in* $[0, 1]^{\omega_1}$,
- *increasing sequence* l_n *of natural numbers*

there is a stationary set $S \subseteq \omega_1$ *such that for* $\beta \in S$ *we have*

- $\mu_n \restriction C([0, 1]^\beta) = \mu_n^\beta$,
- $\pi_\beta[U_{n,m}] = U_{n,m}^\beta$,
- $l_n = l_n^\beta$,

where π_β *denotes the natural projection from* $[0, 1]^{\omega_1}$ *onto* $[0, 1]^\beta$.

Proof. First we will show that there is a sequence $(M_0^\alpha)_{\alpha<\omega_1}$ such that $M_0^\alpha = ((\nu_n^\alpha)_{n\in\omega})$ is a bounded sequence of Radon measures on $[0, 1]^\alpha$ and for every bounded sequence

$(\nu_n)_{n\in\omega}$ of Radon measures on $[0,1]^{\omega_1}$ there is a stationary set $S \subseteq \omega_1$ such that for $\beta \in S$ we have $\nu_n \restriction C([0,1]^\beta) = \nu_n^\beta$.

We will use the identification of Radon measures on $[0,1]^{\omega_1}$ with bounded functionals on $C([0,1]^{\omega_1})$ described in Section 1.2. For a finite set $F \in [\omega_1]^{<\omega}$ denote by w_F the product $\prod_{\alpha\in F} w_\alpha$, where $w_\alpha \in C([0,1]^{\omega_1}), w_\alpha(x) = x(\alpha)$. Observe that finite linear combinations of w_F's form a subalgebra of $C([0,1]^{\omega_1})$. If $x, y \in [0,1]^{\omega_1}$ are distinct points with $x(\alpha) \neq y(\alpha)$, then $w_\alpha(x) \neq w_\alpha(y)$, so by the Stone-Weierstrass theorem this subalgebra is dense in $C([0,1]^{\omega_1})$. Hence if ν is a Radon measure on $[0,1]^{\omega_1}$ then it is determined by the values of $\nu(w_F)$ for $F \in [\omega_1]^{<\omega}$ (note also that in the same way if $\beta < \omega_1$, then $\nu \restriction C([0,1]^\beta)$ is determined by the values of $\nu(w_F)$ for $F \in [\beta]^{<\omega}$). So we can represent each Radon measure ν on $[0,1]^{\omega_1}$ by the function

$$\varphi_\nu \colon [\omega_1]^{<\omega} \to \mathbb{R},\ \varphi(F) = \nu(w_F)$$

(and then $\nu \restriction C([0,1]^\beta)$ is represented by $\varphi_\nu \restriction [\beta]^{<\omega}$), and each countable sequence $M = (\nu_n)_{n\in\omega}$ we can represent by the function

$$\varphi_M \colon [\omega_1]^{<\omega} \times \omega \to \mathbb{R},\ \varphi_M(F, n) = \nu_n(w_F).$$

Let $\Phi_1 \colon \omega_1 \to [\omega_1]^{<\omega} \times \omega$ be a bijection such that for each limit ordinal $\gamma \in Lim \cap \omega_1$ the restriction $\Phi_1 \restriction \gamma$ is bijection onto $[\gamma]^{<\omega} \times \omega$ (to see that such a bijection exists it is enough to note that for every $\gamma \in Lim \cap \omega_1$ there is a bijection $\phi_\gamma \colon [\gamma, \gamma+\omega) \to ([\gamma+\omega]^{<\omega} \times \omega) \setminus ([\gamma]^{<\omega} \times \omega)$ and take $\Phi_1 \restriction [\gamma, \gamma+\omega) = \phi_\gamma$). We need to fix one more bijection $\Phi_2 \colon \mathbb{R} \to \omega_1$ ($\diamondsuit$ implies CH, so such a bijection exists). Put

$$\psi_M = \Phi_2 \circ \varphi_M \circ \Phi_1,\ \psi_M \colon \omega_1 \to \omega_1.$$

Since $\Phi_1 \restriction \gamma$ is a bijection onto $[\gamma]^{<\omega} \times \omega$ for all limit γ we may treat $\psi_M \restriction \gamma$ as a representation of the sequence of measures $(\nu_n \restriction C([0,1]^\gamma))_{n\in\omega}$.

We will use the following characterization of $\diamondsuit$ (see [28, Theorem 2.7]):
There exists a sequence $(f_\alpha)_{\alpha<\omega_1}, f_\alpha \colon \alpha \to \alpha$ such that for for each $f \colon \omega_1 \to \omega_1$ the set $\{\alpha : f \restriction \alpha = f_\alpha\}$ is stationary.

For $\alpha \in \omega_1$ let M_0^α be a sequence of Radon measures on $[0,1]^\alpha$ represented by f_α, if f_α is a representation for some such sequence (otherwise we pick M_0^α in any way). Let M be a bounded sequence of of measures on $[0,1]^{\omega_1}$ and let $S = \{\alpha : \psi_M \restriction \alpha = f_\alpha\}$. Since for limit $\gamma < \omega_1$ the function $\psi_M \restriction \gamma$ is a representation of some sequence of measures we get that for $\alpha \in Lim \cap S$ the function $\psi_M \restriction \alpha$ is the representation of a sequence M_0^α. Moreover the set $S \cap Lim$ is a stationary subset of ω_1, so the first part of the proof is complete.

To show the existence of a sequence $(M^\alpha, \mathcal{U}^\alpha, L^\alpha)_{\alpha<\omega_1}$ required in the Lemma, we need to observe that each triple $(M, \mathcal{U}, L)$ may be represented as a bounded countable sequence of Radon measures on $[0,1]^{\omega_1}$. Indeed, any basic open set $U \in \mathcal{U}$ may be treated as a measure λ_U on $[0,1]^{\omega_1}$, given by $\lambda_U(A) = \lambda(A \cap U)$, where λ is a product measure of ω_1 Lebesgue measures on $[0,1]$ (note that if U, V are different basic open sets, then some of their sections differ on a non-trivial interval, so we have $\lambda_U \neq \lambda_V$) and L may be represented as δ_{x_L} where $x_L = (y_l, 0, 0, \dots)$ and $y_l = g(L)$ for some fixed bijection g between the set of sequences of natural numbers and $[0,1]$. $\square$

Proposition 3.4.5. *Assume $\diamondsuit$. Then for every $k > 0, k \in \omega \cup \{\infty\}$ there is a compact Hausdorff space K satisfying the following:*

(1) $\dim K = k$*,*
(2) K is separable with a countable dense set $Q = \{q_n : n \in \omega\}$,
(3) K is connected,
(4) for every:
   - *sequence $(U_n)_{n\in\omega}$ of pairwise disjoint open sets which are countable unions of basic open sets (basic open set in K is a set of the form $W \cap K$, where W is a basic open set in $[0,1]^{\omega_1}$),*
   - *relatively discrete sequence $(q_{l_n} : n \in \omega) \subseteq Q$ with $q_{l_n} \notin U_m$ for $n, m \in \omega$,*
   - *bounded sequence $(\mu_n)_{n\in\omega}$ of Radon measures on K such that $|\mu_n|(U_n) > \delta$ for some $\delta > 0$,*

   there is $\varepsilon > 0$, continuous functions $(f_n)_{n\in\omega} \subseteq C_I(K)$ and infinite sets $B \subseteq N \subseteq \omega$ such that:

   (1) (f_n) is a sequence of pairwise disjoint functions with $\mathrm{supp}(f_n) \subseteq U_n$ *for $n \in \omega$,*
   (2) $|\int f_n d\mu_n| > \varepsilon$ for $n \in B$,
   (3) $\sum\{|\int f_m d\mu_n| : m \in B\backslash\{n\}\} < \varepsilon/3$ for $n \in N$,
   (4) $\{f_n : n \in B\}$ has supremum in the lattice $C(K)$,
   (5) $\overline{\{q_{l_n} : n \in B\}} \cap \overline{\{q_{l_n} : n \in N\backslash B\}} \neq \varnothing$,

(5) whenever U, V are open subsets of K such that $\overline{U} \cap \overline{V} \neq \varnothing$, then $\overline{U} \cap \overline{V}$ contains at least two points.

We will start with the description of the construction. Then we will prove that the constructed space satisfies the required conditions.

Construction 3.4.6. *Assume $\diamondsuit$. We will construct by induction on $\alpha < \omega_1$ an inverse system $(K_\alpha)_{\alpha<\omega_1}$ with the limit K, where $K_\alpha \subseteq [0,1]^\alpha$ and countable dense sets $Q_\alpha = \{q_n \restriction \alpha : n \in \omega\} \subseteq K_\alpha$.*

We start with $K_k = [0,1]^k$ (or $K_\omega = [0,1]^\omega$ in the case $k = \infty$) and we pick Q_k to be any countable dense subset of K_k. If α is a limit ordinal then we take as K_α the inverse limit of $(K_\beta)_{\beta<\alpha}$.

Denote by Even and Odd the sets consisting of even and odd (respectively) countable ordinals greater than k. Let $(M^\alpha, \mathcal{U}^\alpha, L^\alpha)_{\alpha<\omega_1}$ be as in Lemma 3.4.4 and fix an enumeration $(U_\alpha, V_\alpha)_{\alpha\in\mathrm{Odd}}$ of pairs of open subsets of $[0,1]^{\omega_1}$ which are countable unions of basic open sets, and require that each such a pair occurs in the sequence uncountably many times (such an enumeration exists since by CH there is $\omega_1^\omega = \omega_1$ open sets, which are countable unions of basic open sets in $[0,1]^{\omega_1}$).

First we describe the construction of $K_{\alpha+1}$ where α is an even ordinal. We assume that K_α is already constructed and for each $\beta < \alpha$ the following are satisfied:

(1) if $\beta \in$ Even then we have infinite sets $b^*_\beta \subseteq a^*_\beta \subseteq \omega$ such that $\{q_n \restriction \alpha : n \in a^*_\beta\}$ is relatively discrete and

$$\overline{\{q_n \restriction \alpha : n \in b^*_\beta\}} \cap \overline{\{q_n \restriction \alpha : n \in a^*_\beta \backslash b^*_\beta\}} \neq \varnothing.$$

(2) if $\beta \in$ Odd then we have infinite sets $b^i_\beta \subseteq a^i_\beta \subseteq \omega$ for $i = 0, 1$ such that the set $\{q_n \restriction \alpha : n \in a^i_\beta\}$ is relatively discrete and

$$\overline{\{q_n \restriction \alpha : n \in b^i_\beta\}} \cap \overline{\{q_n \restriction \alpha : n \in a^i_\beta \backslash b^i_\beta\}} \neq \varnothing$$

for $i = 0, 1$.

Put $U^\alpha_n = \bigcup_{m\in\omega} U^\alpha_{n,m}$. We will say that even step α is non-trivial if

- there is $\delta > 0$ such that $|\mu^\alpha_n|(U^\alpha_n \cap K_\alpha) > \delta$ for each $n \in \omega$,
- $(U^\alpha_n \cap K_\alpha)_{n\in\omega}$ are pairwise disjoint,
- $\{q_{l^\alpha_n} : n \in \omega\}$ is relatively discrete in K_α,
- $\{q_{l^\alpha_n} : n \in \omega\} \cap U^\alpha_m = \varnothing$ for $m \in \omega$.

Otherwise we call this step trivial and we put $K_{\alpha+1} = K_\alpha \times \{0\}$ and $q_n \restriction \alpha+1 = q_n \restriction \alpha^\frown 0$.

Assume that we are in a non-trivial case. Apply proposition 3.4.2 for $U_n = U^\alpha_n \cap K_\alpha, \mu_n = \mu^\alpha_n$ to obtain $(f^\alpha_n)_{n\in\omega} \subseteq C_I(K_\alpha)$, infinite $N \subseteq \omega$ and $\varepsilon > 0$ such that

- $\mathrm{supp}(f^\alpha_n) \subseteq U^\alpha_n \cap K_\alpha$ for $n \in N$,
- $|\int f^\alpha_n d\mu^\alpha_n| > \varepsilon$ for $n \in N$,
- $\sum\{|\int f^\alpha_m d\mu_n| : n \neq m, m \in N\} < \varepsilon/3$ for $n \in N$,
- $K_\alpha \backslash D((f^\alpha_n)_{n\in N})$ is zero-dimensional.

By Lemma 3.3.5, without loss of generality (by passing to an infinite subset of N) we may assume that for all infinite $B \subseteq N$ the extension $K_\alpha(B)$ of K_α by $(f^\alpha_n)_{n\in B}$ is strong and for each $\beta < \alpha$ and $i \in \{*, 0, 1\}$ we have

$$\overline{\{q^B_n \restriction \alpha + 1 : n \in b^i_\beta\}} \cap \overline{\{q^B_n \restriction \alpha + 1 : n \in a^i_\beta \backslash b^i_\beta\}} \neq \varnothing,$$

where

$$q^B_l \restriction \alpha + 1 = q_l \restriction \alpha^\frown t, t = \sum_{n\in B} f^\alpha_n(q_l \restriction \alpha),$$

and the closures are taken in $K_\alpha(B)$.

Let $a^*_\alpha = \{l^\alpha_n : n \in N\}$. Then

$$N = \{n \in \omega : l^\alpha_n \in a^*_\alpha\}. \tag{$*$}$$

We will show that there is infinite $b^*_\alpha \subseteq a^*_\alpha$ such that

$$\overline{\{q_n \restriction \alpha : n \in b^*_\alpha\}} \cap \overline{\{q_n \restriction \alpha : n \in a^*_\alpha \backslash b^*_\alpha\}} \neq \varnothing.$$

Suppose otherwise. Then since K_α is a compact metrizable space, for each $X \subseteq a^*_\alpha$ there are disjoint open sets U_X, V_X such that

$$\overline{\{q_n \restriction \alpha : n \in X\}} \subseteq U_X, \overline{\{q_n \restriction \alpha : n \in a^*_\alpha \backslash X\}} \subseteq V_X,$$

and U_X, V_X are finite unions of members of some fixed countable base in K_α. There are uncountably many choices of X and only countably many pairs of such open sets in K_α, so for some $X \neq Y$ we have $\{U_X, V_X\} = \{U_Y, V_Y\}$ which is a contradiction.

Let b^*_α be such that

$$\overline{\{q_n \restriction \alpha : n \in b^*_\alpha\}} \cap \overline{\{q_n \restriction \alpha : n \in a^*_\alpha \backslash b^*_\alpha\}} \neq \varnothing$$

and define

$$B = \{n \in N : l^\alpha_n \in b^*_\alpha\}. \tag{$**$}$$

To finish the construction at this step we put $K_{\alpha+1} = K_\alpha(B), q_n \restriction \alpha+1 = q^B_n \restriction \alpha+1$ and observe that (1) is satisfied for a^*_α, b^*_α, because if

$$x \in \overline{\{q_n \restriction \alpha : n \in b^*_\alpha\}} \cap \overline{\{q_n \restriction \alpha : n \in a^*_\alpha \backslash b^*_\alpha\}},$$

then

$$(x, 0) \in \overline{\{q_n \restriction \alpha+1 : n \in b^*_\alpha\}} \cap \overline{\{q_n \restriction \alpha+1 : n \in a^*_\alpha \backslash b^*_\alpha\}},$$

since $f^\alpha_n(q_k \restriction \alpha) = 0$ for all $n \in B$ and $k \in a_\alpha$.

At step $\alpha \in$ Odd we assume that we are given a^i_β, b^i_β satisfying (1) and (2) from the even step for all $\beta < \alpha$ (where $i = *$ if $\beta \in$ Odd and $i \in \{0,1\}$ if $\beta \in$ Even). We call this step non-trivial, if the closures of $\pi_\alpha[U_\alpha]$ and $\pi_\alpha[V_\alpha]$ have non-empty intersection. If the case is non-trivial we use Lemma 3.4.3 (note that Lemma 3.3.3 implies that K_α is connected) to find appropriate $(f_n)_{n\in\omega} \subseteq C_I(K_\alpha), A_i$ and S_i for $i = 0, 1$. In the same way as in the even step we find $B \subseteq \omega$ such that $K_\alpha(B)$ is a strong extension of K_α and the conditions (1) and (2) are preserved in $K_\alpha(B)$ for $\beta < \alpha$. To finish this step we define $K_{\alpha+1} = K_\alpha(B), a^i_\alpha = S_i, b^i_\alpha = A_i$ and $q_n \restriction \alpha+1 = q^B_n \restriction \alpha+1$. Lemma 3.4.3 guarantees that the condition (2) holds at the step $\alpha + 1$.

In both cases the density of $Q_{\alpha+1} = \{q_n \restriction \alpha+1 : n \in \omega\}$ in $K_{\alpha+1}$ follows from Lemma 3.3.4.

Proof of Proposition 3.4.5. We will show that the space constructed above satisfies the required conditions. (1) follows from Corollary 3.3.8 and the fact that $[0,1]^k$ is a k-dimensional space. Q is a countable dense set in K, since each Q_α is dense in K_α for $\alpha < \omega_1$. Connectedness follows from inductive argument using Lemma 3.3.3.

Let U_n, l_n, μ_n be as in (4). Let $U_n = \bigcup_{m\in\omega} U_{n,m} \cap K$ where $U_{n,m}$ are basic open sets in $[0,1]^{\omega_1}$. Every $U_{n,m}$ is determined by finitely many coordinates, so there is $\gamma < \omega_1$ such that $\pi^{-1}_\gamma(\pi_\gamma[U_{n,m}]) = U_{n,m}$ for $n \in \omega$, where π_γ is the natural projection from $[0,1]^{\omega_1}$ onto $[0,1]^\gamma$ (so $U_{n,m}$ are determined by first γ coordinates). By Lemma 3.4.4 there is $\alpha > \gamma, \alpha \in$ Even such that for $n \in \omega$

- $\mu_n \upharpoonright C(K_\alpha) = \mu_n^\alpha$,
- $\pi_\alpha[U_{n,m}] = U_{n,m}^\alpha$,
- $l_n = l_n^\alpha$.

Let $(f_n^\alpha)_{n\in B}$ be such that in the α-th step of construction. Since $(f_n^\alpha)_{n\in B}$ satisfy conditions of Proposition 3.4.2, functions $f_n = f_n^\alpha \circ \pi_\alpha$ satisfy conditions (a-c). (d) follows from [80, Lemma 4.6] and the fact that $K_{\alpha+1}$ is a strong extension of K_α by $(f_n)_{n\in B}$. By construction we have

$$\overline{\{q_n : n \in b_\alpha^*\}} \cap \overline{\{q_n : n \in a_\alpha^* \backslash b_\alpha^*\}} \neq \varnothing$$

and by $(*)$ and $(**)$

$$\{q_n : n \in b_\beta^*\} = \{q_{l_n} : n \in B\},$$

$$\{q_n : n \in a_\alpha^* \backslash b_\alpha^*\} = \{q_{l_n} : n \in N \backslash B\},$$

which gives (e).

Now we will prove (5). Fix open sets $U, V \subseteq K$ such that $\overline{U} \cap \overline{V} \neq \varnothing$. As K is separable it is c.c.c. so there are open $U' \subseteq U, V' \subseteq V$ which are countable unions of basic open sets such that $\overline{U'} = \overline{U}$ and $\overline{V'} = \overline{V}$ (namely it is enough to take as U' the union of a maximal antichain of open subsets in U, and similarly for V'). Without loss of generality we may assume that $U' = U$ and $V' = V$. Since U, V are countable unions of basic open sets, there is $\gamma < \omega_1$ such that U, V are determined by coordinates less than γ. Let $\alpha > \gamma, \alpha \in \mathrm{Odd}$ be such that $U = U_\alpha \cap K, V = V_\alpha \cap K$. Then $\overline{\pi_\alpha[U]} \cap \overline{\pi_\alpha[V]}$ is nonempty so α-th step in construction is nontrivial. By construction we have for $i = 0, 1$

$$\overline{\{q_n \upharpoonright \beta : n \in b_\alpha^i\}} \cap \overline{\{q_n \upharpoonright \beta : n \in a_\alpha^i \backslash b_\alpha^i\}} \neq \varnothing$$

for all $\beta > \alpha$, so there are $x_i \in \overline{U} \cap \overline{V}$ such that

$$x_i \in \overline{\{q_n : n \in b_\alpha^i\}} \cap \overline{\{q_n : n \in a_\alpha^i \backslash b_\alpha^i\}}.$$

To finish the proof we need only to notice that $x_0 \neq x_1$, but this follows form the fact that a_α^i, b_α^i were chosen to satisfy Lemma 3.4.3(3). $\square$

Lemma 3.4.7. *Suppose that $(U_n)_{n\in\omega}$ is a sequence of pairwise disjoint open subsets of a compact Hausdorff space K. Let $M, N \subset \omega$ be infinite sets such that $M \cap N$ is finite. Assume that $(f_m)_{m\in M}, (g_n)_{n\in N} \subseteq C_I(K)$ are such that* $\mathrm{supp}(f_m) \subseteq U_m, \mathrm{supp}(g_n) \subseteq U_n$ *for $m \in M, n \in N$ and the suprema $f_{\sup} = \sup\{f_m : m \in M\}, g_{\sup} = \sup\{g_n : n \in N\}$ exist in $C_I(K)$. Denote*

$$f = f_{\sup} - \sum_{m\in M} f_m, \; g = g_{\sup} - \sum_{n\in N} g_n.$$

Then f, g are Borel functions with disjoint supports.

Proof. f and g are Borel functions since they are pointwise sums of countably many continuous functions. Put $D = M \cap N$ and note that since D is finite the function $\sum_{m\in D} f_m$ is continuous. We will show that

$$\sup\{f_m : m \in M\backslash D\} = \sup\{f_m : m \in M\} - \sum_{m\in D} f_m. \tag{+}$$

Let $x \in K$. If $x \in \operatorname{supp}(f_n)$ for some $n \in M\backslash D$, then $\sum_{m\in D} f_m(x) = 0$, so

$$(\sup\{f_m : m \in M\} - \sum_{m\in D} f_m)(x) = \sup\{f_m : m \in M\}(x) \geq f_n(x)$$

for every $n \in M\backslash D$. If $x \notin \operatorname{supp}(f_n)$ for every $n \in M\backslash D$, then since f_n's have disjoint supports we get that

$$(\sup\{f_m : m \in M\} - \sum_{m\in D} f_m)(x) \geq 0 = f_n(x)$$

for $n \in M\backslash D$. Hence

$$\sup\{f_m : m \in M\} - \sum_{m\in D} f_m \geq f_n$$

for $n \in M\backslash D$ in the lattice $C(K)$. Let $h \in C(K)$ be such that

$$\sup\{f_m : m \in M\} - \sum_{m\in D} f_m \geq h \geq f_n$$

for $n \in M\backslash D$. Since f_n's have disjoint supports we have

$$\sup\{f_m : m \in M\} \geq h + \sum_{m\in D} f_m \geq \sum_{m\in M} f_m.$$

But

$$\sup\{f_m : m \in M\}(x) = \sum_{m\in M} f_m(x)$$

for $x \in D((f_n)_{n\in M})$, so

$$\sup\{f_m : m \in M\} - \sum_{m\in D} f_m = h,$$

because $\sup\{f_m : m \in M\} - \sum_{m\in D} f_m$ and h are continuous functions equal on the set $D((f_n)_{n\in M})$, which is dense in K (by Lemma 3.3.2). This completes the proof of the equality (+).

From (+) we get that

$$\sup\{f_m : m \in M\backslash D\} - \sum_{m\in M\backslash D} f_m = \sup\{f_m : m \in M\} - \sum_{m\in M} f_m = f.$$

In particular in the definition of f we may replace M with $M\backslash D$ and assume that $M \cap N = \varnothing$.

We will show that in this case we have $\operatorname{supp}(f_{\sup}) \cap \operatorname{supp}(g_{\sup}) = \varnothing$, which will finish the proof since $\operatorname{supp}(f) \subseteq \operatorname{supp}(f_{\sup})$ and $\operatorname{supp}(g) \subseteq \operatorname{supp}(g_{\sup})$ (the inclusions hold because $f \leq f_{\sup}, g \leq g_{\sup}$ and f, g are non-negative). First we observe that for each

$n \in N$ we have $\mathrm{supp}(f_{\sup}) \cap \mathrm{supp}(g_n) = \varnothing$. Indeed, if it is not the case, then there is $x \in U_n$ such that $f_{\sup}(x) > 0$. Then by the Tietze extension theorem we may find $h \in C_I(K)$ such that $h(x) = 0$ and $h \restriction K \backslash U_n = f_{\sup} \restriction K \backslash U_n$. But then $f_{\sup} > h \geq f_m$ for every $m \in M$, which is a contradiction with the fact that $f_{\sup}$ is the supremum of f_m's. Now, in the same way we show that if $\mathrm{supp}(f_{\sup}) \cap \mathrm{supp}(g_{\sup}) \neq \varnothing$, then there is h' such that $g_{\sup} > h' > g_n$ for $n \in N$. □

Theorem 3.4.8. *Assume $\diamondsuit$. For each $k > 0$ there is a compact Hausdorff, separable, connected space K such that $C(K)$ has few operators and* $\dim K = k$.

Proof. We will show that if K is the space with properties from Proposition 3.4.5, then $C(K)$ has few operators. K satisfies Proposition 3.4.5(5), so by [80, Theorem 2.7, Lemma 2.8] it is enough to show that all operators on $C(K)$ are weak multipliers.

Assume that there is a bounded linear operator $T : C(K) \to C(K)$, which is not a weak multiplier. By Theorem 3.4.1 there is a pairwise disjoint sequence $(g_n)_{n\in\omega} \subseteq C_I(K)$ and pairwise disjoint open sets $(V_n)_{n\in\omega}$ such that $g_n \restriction V_m = 0$ for $n, m \in \omega$ and $|T(g_n) \restriction V_n| > \delta$ for some $\delta > 0$. For $n \in \omega$ let $U_n = \mathrm{supp}(g_n)$. Let $g'_n \in C([0,1]^{\omega_1})$ be an extension of g_n and $U'_n = \mathrm{supp}(g'_n)$. By Mibu's theorem (see [94]) for every $n \in \omega$ there is $\alpha_n < \omega_1$ such that whenever $x, y \in [0,1]^{\omega_1}, x \restriction \alpha_n = y \restriction \alpha_n$, we have $g'_n(x) = g'_n(y)$. Hence U'_n is an open set of the form $W_n \times [0,1]^{\omega_1 \backslash \alpha_n}$, where W_n is an open set in $[0,1]^{\alpha_n}$. Since α_n is countable, W_n is a union of countably many basic open set in $[0,1]^{\alpha_n}$. Thus for every $n \in \omega$ the set U'_n is a union of countably many basic open sets in $[0,1]^{\omega_1}$ and $U_n = U'_n \cap K$ is a union of countably many basic open sets in K.

Let $(l_n)_{n\in\omega}$ for $n \in \omega$ be such that $q_{l_n} \in V_n$ (so in particular $\{q_{l_n} : n \in \omega\}$ is relatively discrete in K) and define $\mu_n = T^*(\delta_{q_{l_n}})$. Then $|\int g_n d\mu_n| = |T(g_n)(q_{l_n})| > \delta$. Since $\mathrm{supp}(g_n) \subseteq U_n$ and $\|g_n\| \leq 1$ we get that $|\mu_n|(U_n) \geq |\int g_n d\mu_n| > \delta$.

By Proposition 3.4.5 for every infinite subset $A \subseteq \omega$ there are infinite sets $B_A \subseteq N_A \subseteq A$, continuous functions $(f_{n,A})_{n\in A} \subseteq C_I(K)$ and ε_A such that

(a) $(f_{n,A})_{n\in A}$ is a sequence of pairwise disjoint functions with $\mathrm{supp}(f_{n,A}) \subseteq U_n$ for $n \in A$,
(b) $|\int f_{n,A} d\mu_n| > \varepsilon_A$ for $n \in B_A$,
(c) $\sum\{|\int f_{m,A} d\mu_n| : n \neq m, m \in B_A\} < \varepsilon_A/3$ for $n \in N_A$,
(d) $\{f_{n,A} : n \in B_A\}$ has its supremum in the lattice $C(K)$,
(e) $\overline{\{q_{l_n} : n \in B_A\}} \cap \overline{\{q_{l_n} : n \in N_A \backslash B_A\}} \neq \varnothing$.

Put $f_A = \sup\{f_{n,A} : n \in B_A\} - \sum_{m\in B_A} f_{m,A}$. We will show that there is an infinite set $M \subseteq \omega$ such that

$$\int f_M d\mu_n = 0. \tag{++}$$

Suppose this is not the case. Let $\{M_\xi : \xi < \omega_1\}$ be a family of infinite subsets of ω such that for $\xi \neq \xi'$ the set $M_\xi \cap M_{\xi'}$ is finite. Assume (++) does not hold for every M_ξ. Then there is $n \in \omega$ such that

$$\int f_{M_\xi} d\mu_n \neq 0$$

for uncountably many ξ's. By Lemma 3.4.7 $f_{M_\xi}, f_{M_{\xi'}}$ have disjoint supports for $\xi \neq \xi'$, so in particular there is an uncountable family of non-null (with respect to μ_n) Borel sets in K, which is a contradiction.

Put $f_n = f_{n,M}, \varepsilon = \varepsilon_M$, $B = B_M$ and $N = N_M$. Let $f = \sup\{f_n : n \in B\}$. By (b), (c), (++) and the definition of μ_n we get that for $n \in B$

$$|T(f)(q_{l_n})| = |\int f d\mu_n| = |\int f_n d\mu_n + \int \sum_{m \in B\setminus\{n\}} f_m| \geq$$
$$|\int f_n d\mu_n| - |\int \sum_{m \in B\setminus\{n\}} f_m| \geq \varepsilon - \varepsilon/3 = 2\varepsilon/3.$$

For $n \in N\setminus B$ (c) gives

$$|T(f)(q_{l_n})| = |\int \sum_{m \in B} f_m d\mu_n| < \varepsilon/3.$$

As $T(f)$ is a continuous function on K we obtain that

$$\overline{\{q_{l_n} : n \in B\}} \cap \overline{\{q_{l_n} : n \in N\setminus B\}} = \varnothing,$$

which contradicts (e). □

Theorem 3.4.9. *Assume $\diamondsuit$. Then for every $k \in \omega \cup \{\infty\}$ there is a compact Hausdorff space K such that $\dim(K) = k$ and whenever $C(K) \sim C(L)$, $\dim(L) = k$.*

Proof. For $k = 0$ every finite space K works. If $k > 0$, then the space from Theorem 3.4.8 has the required property by Corollary 3.2.20. □

3.5 Remarks and questions

The first natural question concerning our results is whether Theorem 3.4.9 is true without any additional assumption.

Question 3.5.1. *Let $k \in \omega\setminus\{0\}$. Is there (in* ZFC*) a compact Hausdorff space K such that $\dim(K) = k$ and whenever $C(K) \sim C(L)$, $\dim(L) = k$?*

In the light of Theorem 3.2.19 to show that the Question 3.5.1 has positive answer it would be enough to prove that the following question has positive answer.

Question 3.5.2. *Let $k \in \omega\setminus\{0\}$. Is there (in* ZFC*) a compact, separable, connected Hausdorff space K such that $\dim K = k$ and $C(K)$ has few operators?*

The original construction of a Banach space $C(K)$ where all the operators are weak multipliers was carried out in ZFC ([80]). In this construction we set all sequences of pairwise disjoint continuous functions on $[0,1]^{\mathfrak{c}}$ into a sequence of length $\mathfrak{c}$, and the choice of the strong extension at α-th step depends on the α-th sequence of functions. Later, in order to prove that K satisfies the required conditions, we look at any sequence $(\mu_n)_{n\in\omega}$ of Radon measures on K and show that we can find sequences of continuous functions satisfying properties (a-e) from Proposition 3.4.5. However, in this approach we may obtain an infinite-dimensional space, since used strong extensions may increase the dimension. One can try to proceed in a similar way, by applying only those extensions that preserve the dimension. The problem is that we do not know, whether the extension by the sequence of functions given at some step changes the dimension, since it depends on the earlier steps (i.e. it depends on the bookkeeping of sequences of continuous functions on $[0,1]^{\mathfrak{c}}$). Consequently, there may be a sequence of measures on the final space, for which every suitable sequence of functions appears at a step, in which using the extension would increase the dimension.

Although the main reason to use the diamond principle is the guessing of measures in Lemma 3.4.4, we also needed the continuum hypothesis to ensure that all intermediate spaces from our construction are metrizable. At that point we used the fact that for every non-zero Radon measure on metrizable compact space there is a zero-dimensional G_δ compact subset of non-zero measure (Theorem 3.1.16). In the light of this theorem the following problem seems to be interesting.

Problem 3.5.3. *Describe the class of compact Hausdorff spaces K such that for every non-zero Radon measure μ on K there is a zero-dimensional compact subset $L \subseteq K$ such that $\mu(L) \neq 0$.*

Assume that K is such that $C(K)$ has few operators. Then by [117, Proposition 4.8] there is a space L such that $C(K) \sim C(L)$, but $C(L)$ does not have few operators. However, by Theorem 3.2.19 the topology of L is very close to K, at least if we assume that K is separable and connected.

Question 3.5.4. *Suppose that K is a compact Hausdorff space such that every operator $T\colon C(K) \to C(K)$ is a weak multiplier and $C(L) \sim C(K)$ for some compact Hausdorff space. Is it true that K and L are homeomorphic modulo finitely many points in the sense of Theorem 3.2.19?*

One may also ask, what properties K should have to satisfy Theorem 3.4.9. There are known examples of "nice" spaces K such that if $C(K) \sim C(L)$, then L is not zero-dimensional. For instance Avilés and Koszmider showed that there is such a space which is quasi Radon-Nikodym ([8]) and Plebanek gave a consistent example of such a space which is a Corson copmact ([109]).

Chapter 4

Grothendieck vs Nikodym

4.1 Introduction

In 1953, Grothendieck [64, Section 4] proved that the space l_∞ of bounded sequences has the following property:

All weak-convergent sequences in the dual space l_∞^* are also weakly convergent.*

The above theorem motivated the following definition.

Definition 4.1.1. A Banach space X has the **Grothendieck property** if all weak*-convergent sequences in the dual space X^* are also weakly convergent.

Research on the Grothendieck property has long history and is still ongoing [12, 16, 29, 61–63, 66, 73, 81, 127]. If X is of the form $C(K)$ for a compact space K, then X has the Grothendieck property if and only if each weak*-convergent sequence of Radon measures on K is also weakly convergent. Recall that l_∞ is isometric to the Banach space $C(\beta\mathbb{N})$ of continuous functions on the Stone-Čech compactification of the natural numbers. Moreover, $\beta\mathbb{N}$ is the Stone space of the Boolean algebra $\mathcal{P}(\mathbb{N})$.

Schachermayer, inspired by the Grothendieck's result, introduced the notion of the Grothendieck property for Boolean algebras [116, Definition 2.3].

Definition 4.1.2. A Boolean algebra $\mathbb{A}$ has the **Grothendieck property**, if the Banach space $C(\mathrm{St}(\mathbb{A}))$ of continuous functions on the Stone space of $\mathbb{A}$ has the Grothendieck property.

Analogously, motivated by Nikodym's paper [99], Schachermayer defined the Nikodym property for Boolean algebras [116, Definition 2.4].

Definition 4.1.3. A Boolean algebra $\mathbb{A}$ has the **Nikodym property**, if every sequence (μ_n) of bounded finitely additive signed measures on $\mathbb{A}$, which is pointwise convergent to zero (i.e. for all $A \in \mathbb{A}$ we have $\lim_{n\to\infty} \mu_n(A) = 0$) is bounded in the norm (i.e. $\sup_{n\in\mathbb{N}} ||\mu_n||$ is bounded, cf. Section 1.2).

The Nikodym property is similar to the Grothendieck property in many ways. For example, if a Boolean algebra $\mathbb{A}$ has the Grothendieck or Nikodym property, then its Stone space does not contain non-trivial convergent sequences. In [64] and [7] the authors proved that complete Boolean algebras have both the Grothendieck and Nikodym properties. The completeness assumption can be relaxed to some combinatorial property (SCP) introduced by Haydon [67, Definition 1A, Proposition 1B], which is even weaker than σ-completeness [51]. Other connections between the Grothendieck property and the Nikodym property may be found in [96, 118]. Both of the properties were also considered in a recent paper by Żuchowski in the context of filters on $\mathbb{N}$ [137].

However, the Grothendieck and Nikodym properties are not equivalent. There are Boolean algebras with the Nikodym property but without the Grothendieck property, e.g. the Boolean algebra of Jordan measurable subsets of the unit interval [92],[116, Propositions 3.2, 3.3]. The question if there is a Boolean algebra with the Grothendieck property, but without the Nikodym property turned out to be much more difficult. This is the central question of the chapter.

Question 4.1.4. *Does there exist a Boolean algebra with the Grothendieck property that does not have the Nikodym property?*

So far there was only one known example of such a Boolean algebra. It was constructed by Talagrand in [128]. However, his construction uses the continuum hypothesis (CH) and so the question of the existence of such a Boolean algebra in ZFC remains open. Since Talagrand's construction there was no much progress in this subject, so it was natural to ask the following question.

Question 4.1.5. *Is it consistent with* $\neg\mathsf{CH}$ *that there is a Boolean algebra with the Grothendieck property but without the Nikodym property?*

In this chapter we answer this problem in the affirmative. Moreover, the algebra we construct has cardinality ω_1.

Theorem 4.5.15. *It is consistent with* $\neg\mathsf{CH}$ *that there is a Boolean algebra of size* ω_1 *with the Grothendieck property but without the Nikodym property.*

Very recently, there has been released a preprint by Sobota and Zdomskyy [125] with a proof that Martin's axiom (MA) implies the existence of such an algebra of cardinality $\mathfrak{c}$.

The proof of Theorem 4.5.15 strongly relies on the ideas behind Talagrand's construction. His Boolean algebra consists of Borel sets with certain symmetry property (we call such sets *balanced sets*). This ensures that the constructed Boolean algebra will not have the Nikodym property.

We define a σ-centered forcing notion that extends a given countable balanced Boolean algebra to a bigger one that is still balanced. Moreover, some sequences of measures (picked by a generic filter) which were weak*-convergent in the initial algebra are no longer weak*-convergent in the extension. Then we show that in the model obtained from the finite support iteration of length ω_1 of such forcing notions, there exists a balanced Boolean algebra with the Grothendieck property. The idea behind this forcing comes from the work of Koszmider [79] and of Fajardo [40]. In the former paper Koszmider introduced a notion of forcing that adds a Boolean algebra of cardinality ω_1, whose Stone space does not contain non-trivial convergent sequences. Fajardo adapted this method to obtain a Banach space $C(K)$ of small density and with few operators. In particular, this space has the Grothendieck property. In this chapter we show how to combine this approach with the theory of balanced algebras to obtain a Boolean algebra without the Nikodym property.

Most of the results concerning fundamental properties of balanced sets (see Section 4.3) that we use in this chapter are essentially due to Talagrand. However, our construction requires some significant changes. Since the construction is rather complicated and technical, we decided to include detailed proofs at each step. We also show how to construct a balanced Boolean algebra with the Grothendieck property under CH using our modification of Talagrand's method (see Theorem 4.4.8).

Another interesting related issue is the question about the possible sizes of Boolean algebras with the Grothendieck and Nikodym properties. There always exists such an algebra of size $\mathfrak{c}$ (e.g. $\mathcal{P}(\mathbb{N})$). It is well-known that if $\mathfrak{p} = \mathfrak{c}$, then $\mathfrak{c}$ is the only possible size of such an algebra (it follows from [68, Corollary 3F]). In particular, it happens under MA. Brech showed the consistency of the existence of a Boolean algebra with the Grothendieck property of cardinality smaller than $\mathfrak{c}$ [17]. In [100, Chapter 52, Question 10] Koszmider asked whether it is consistent that there is no Boolean algebra with the Grothendieck property of size $\mathfrak{p}$. It turned out that the answer is positive [9, Proposition 6.18]. Recently Sobota and Zdomskyy published several articles on cardinal characteristics related to Boolean algebras with the Grothendieck or Nikodym property [121–124, 126]. The Boolean algebra we construct is the first example of a Boolean algebra with the Grothendieck property and without the Nikodym property of size $\omega_1 < \mathfrak{c}$ (in particular, our model satisfies $\mathfrak{p} = \omega_1 < \mathfrak{c}$, see Corollary 4.5.14) and the first construction of such a Boolean algebra of size less than $\mathfrak{c}$. In particular, the Stone space of this algebra is another example of a Efimov space. In fact, if we want only to obtain a Boolean algebra with the Grothendieck property, then our forcing can be simplified in a natural way (by dropping some restrictions on the conditions).

It is also worth mentioning that the Grothendieck and Nikodym properties are also discussed in the non-commutative setting in the category of C*-algebras. The definition of the Grothendieck property for C*-algebras is the same as for general Banach spaces. We say that a C*-algebra A has the Nikodym property, if every sequence of bounded linear functionals on A that is convergent to 0 on projections is bounded in the norm. If $\mathbb{A}$ is a Boolean algebra, then it has the Nikodym property if and only if $C(\mathrm{St}(\mathbb{A}))$ has the Nikodym property, when considered as a C*-algebra. The Nikodym property is especially interesting in the case when given C*-algebra has many projections, e.g. when its real rank is zero. It is well-known that von Neumann algebras have both the Grothendieck and Nikodym properties (see [105, Corollary 7] and [26, Theorem 1]). The problem of the existence of C*-algebras with the Grothendieck property and without the Nikodym property is still open in ZFC even in the non-commutative case.

Question 4.1.6. *Is there a C*-algebra of real rank zero, which has the Grothendieck property, but does not have the Nikodym property?*

The structure of the chapter is the following. In Section 4.2 we introduce the property $(\mathcal{G})$ of Boolean algebras and the property of being balanced. Then we show that they imply the Grothendieck property and the negation of the Nikodym property respectively. Section 4.3 is devoted to properties of finite balanced families (this includes the behavior of balanced families under basic operations and approximating balanced families with families of clopen subsets of the Cantor set) and tools for extending countable balanced Boolean algebras. In Section 4.4 we show a method of extending countable balanced Boolean algebras to bigger ones in a way that destroys the weak*-convergence of given sequences of measures. Then we show how to apply this method to construct a Boolean algebra with the Grothendieck property and without Nikodym property assuming the continuum hypothesis. In Section 4.5 we describe a σ-centered notion of forcing that forces the existence of a Boolean algebra with the Grothendieck property and without the Nikodym property. In the last section we include final remarks and state some open questions.

4.2 Grothendieck and Nikodym properties

In this section we will reduce the problem of the existence of a Boolean algebra with the Grothendieck property and without the Nikodym property by introducing the property $(\mathcal{G})$ and the notion of a balanced Boolean algebra.

We start with the notion of semibalanced sets that describes these subsets $A \subseteq C$ for which the occurrences of 1's and -1's at r-th coordinate of elements of A are almost equally distributed for large enough r. For this we introduce measures φ_n on $\mathrm{Bor}(C)$ for $n \in \mathbb{N}$, given by the formula

$$\varphi_n(A) = \int_A \delta_n d\lambda$$

for $A \in \mathrm{Bor}(C)$, where $\delta_n \colon C \to \{-1, 1\}, \delta_n(x) = x_n$.

Definition 4.2.1. Let $A \in \mathrm{Bor}(C); m \in \mathbb{N}; \varepsilon > 0$. The set A is (m, ε)**-semibalanced** if

$$\forall r > m \ |\varphi_r(A)| < \frac{\varepsilon}{r}. \tag{3.1.1}$$

We say that A is **semibalanced**, if for every $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that A is (m, ε)-semibalanced.

Definition 4.2.2. Let $A \in \mathrm{Bor}(C); m, t \in \mathbb{N}; t \geq m; \varepsilon > 0$. We say that A is (m, t, ε)**-balanced** if for every $s \in \{-1, 1\}^m$

$$\frac{\lambda(A \cap \langle s \rangle)}{\lambda(\langle s \rangle)} < \frac{\varepsilon}{m} \text{ or } \frac{\lambda(\langle s \rangle \backslash A)}{\lambda(\langle s \rangle)} < \frac{\varepsilon}{m} \tag{3.2.1}$$

and

$$\forall r \in (m,t]\ \frac{|\varphi_r(A \cap \langle s\rangle)|}{\lambda(\langle s\rangle)} < \frac{\varepsilon}{r}. \tag{3.2.2}$$

We say that A is (m,ε)**-balanced** if for every $s \in \{-1,1\}^m$ the condition (3.2.1) is satisfied and

$$\forall r > m\ \frac{|\varphi_r(A \cap \langle s\rangle)|}{\lambda(\langle s\rangle)} < \frac{\varepsilon}{r}. \tag{3.2.3}$$

Definition 4.2.3. We say that a finite subfamily $\mathcal{A} \subseteq \mathrm{Bor}(C)$ is (m,ε)**-balanced** if every $A \in \mathcal{A}$ is (m,ε)-balanced.

We say that a family $\mathbb{B} \subseteq \mathrm{Bor}(C)$ is **balanced**, if for every finite subfamily $\mathcal{A} \subseteq \mathbb{B}$ and every $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that $\mathcal{A}$ is (m,ε)-balanced. A set $A \in \mathrm{Bor}(C)$ is **balanced**, if $\{A\}$ is balanced.

Remark 4.2.4. *If $\mathcal{A}$ is a finite balanced family and $\varepsilon > 0$, then for every $n \in \mathbb{N}$ there is $m > n$ such that $\mathcal{A}$ is (m,ε)-balanced.*

Note that if $\mathbb{B}$ is balanced, then every member of $\mathbb{B}$ is balanced, but the reverse implication does not hold in general. It may happen that A and B are balanced, while the sets $\{m \in \mathbb{N} : A \text{ is } (m,\varepsilon)\text{-balanced}\}$ and $\{m \in \mathbb{N} : B \text{ is } (m,\varepsilon)\text{-balanced}\}$ are disjoint for some $\varepsilon > 0$.

Lemma 4.2.5. *Let $A \in \mathrm{Bor}(C), \varepsilon > 0$ and $m, t \in \mathbb{N}$, where $t \geq m$. Then A is (m,ε)-balanced if and only if A is (m,t,ε)-balanced and for every $s \in \{-1,1\}^m$ the set $A \cap \langle s\rangle$ is $(t, 2^{-m}\varepsilon)$-semibalanced. In particular, if A is balanced, then it is semibalanced.*

Proof. If A is (m,ε)-balanced, then it is clearly (m,t,ε)-balanced and by (3.2.3) for any $s \in \{-1,1\}^m$ and $r > t$ we have

$$|\varphi_r(A \cap \langle s\rangle)| < \lambda(\langle s\rangle)\frac{\varepsilon}{r} = \frac{\varepsilon}{2^m r},$$

which shows that $A \cap \langle s\rangle$ is $(t, 2^{-m}\varepsilon)$-semibalanced.

The above inequality also shows that if A is $(t, 2^{-m}\varepsilon)$-semibalanced, then (3.2.3) is satisfied for $s \in \{-1,1\}^m$ and $r > t$. In particular, if A is (m,t,ε)-balanced and $(t, 2^{-m}\varepsilon)$-semibalanced, then it is (m,ε)-balanced.

To see that any balanced set is semibalanced fix $\varepsilon > 0$ and $m \in \mathbb{N}$ such that A is (m,ε)-balanced. From the first part of the lemma applied to $t = m$ we get that for $r > m$

$$|\varphi_r(A)| \leq \sum_{s\in\{-1,1\}^m} |\varphi_r(A \cap \langle s\rangle)| < \sum_{s\in\{-1,1\}^m} \frac{\varepsilon}{2^m r} = \frac{\varepsilon}{r},$$

so A is semibalanced. $\square$

We will present a few examples to illustrate the above definitions.

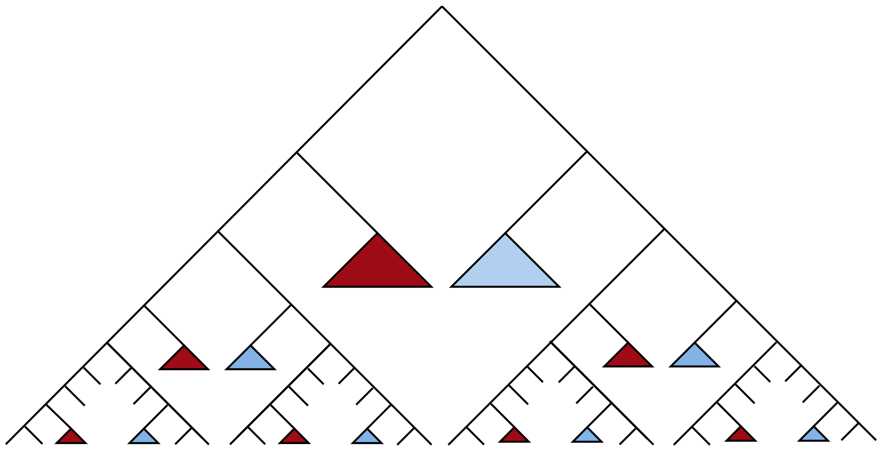
Example 4.2.6. Every clopen subset of C is (m,ε)-balanced for every $\varepsilon > 0$ and sufficiently large $m \in \mathbb{N}$.

Example 4.2.7. We will construct an open balanced set U that is not clopen. More precisely, U will be $\left(2^n, \frac{2^{n+2}}{2^{2^n}}\right)$-balanced for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ consider the set Z_n of all sequences of the form $(a_1, a_2, \ldots, a_{2^n})$ of length 2^n with values in $\{-1, 1\}$ with the following properties:

(1) if $n = 1$, then $a_1 = -a_2$,
(2) if $n \neq 1$, then $a_1 = a_2$,
(3) $\forall l < n$ $a_{2^{l-1}+1} = a_{2^{l-1}+2} = \cdots = a_{2^l}$,
(4) $a_{2^{n-1}+1} = a_{2^{n-1}+2} = \cdots = a_{2^n-1} = -a_{2^n}$,

and put $Z = \bigcup_{n\in\mathbb{N}} Z_n$. On the figure below, the red (dark) sets are of the form $\langle s\rangle$, where $s \in Z_n$ and $s_{2^n} = 1$ for some $n \in \mathbb{N}$, while blue (light) sets are of the form $\langle s\rangle$, where $s \in Z_n$, $s_{2^n} = -1$.



Let $U = \bigcup_{s\in Z}\langle s\rangle$. Consider three cases.

1. If $s \in Z$ then
$$m\frac{\lambda(\langle s\rangle \backslash U)}{\lambda(\langle s\rangle)} = m\frac{\lambda(\varnothing)}{\lambda(\langle s\rangle)} = 0.$$
2. If $s \in \{-1,1\}^{2^n}$ for $n \in \mathbb{N} \setminus \{1\}$ satisfy conditions (2), (3) and

(4') $a_{2^{n-1}+1} = a_{2^{n-1}+2} = \cdots = a_{2^n-1} = a_{2^n}$,

then there are only 4 nonempty sets of the form $U \cap \langle s^i\rangle$, for $i \in \{1,2,3,4\}$ where $s^i \in \{-1,1\}^{2^{n+1}}$ and $\langle s^i\rangle \subseteq \langle s\rangle$. Hence
$$m\frac{\lambda(U\cap\langle s\rangle)}{\lambda(\langle s\rangle)} < m\frac{\sum_{i=1}^4 \lambda(\langle s^i\rangle)}{\lambda(\langle s\rangle)} \leq 2^n\frac{4\cdot 2^{-2^{n+1}}}{2^{-2^n}} = \frac{2^{n+2}}{2^{2^n}}.$$
3. In other cases
$$m\frac{\lambda(U\cap\langle s\rangle)}{\lambda(\langle s\rangle)} = m\frac{\lambda(\varnothing)}{\lambda(\langle s\rangle)} = 0.$$

For every $r \in \mathbb{N}$ the distribution of 1's and -1's in the elements of U at the r-th coordinate is symmetric, as can be easily seen in the figure - the blue (light) sets are symmetric to the red (dark) ones (i.e. $\langle s\rangle$ is a blue set if and only if $\langle -s\rangle$ is red). Thus, for every $r \in \mathbb{N}$ we have
$$\frac{|\varphi_r(U\cap\langle s\rangle)|}{\lambda(\langle s\rangle)} = 0.$$

Proposition 4.2.8. *If a Boolean algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$ consists of semibalanced sets, then $\mathbb{B}$ does not have the Nikodym property. In particular, if $\mathbb{B}$ is balanced, then it does not have the Nikodym property.*

Proof. Consider a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathbb{B}$ given by

$$\mu_n(A) = n\varphi_n(A).$$

This sequence is pointwise convergent to 0, i.e. $\lim_{n\to\infty} |\mu_n(A)| = 0$ for all $A \in \mathbb{B}$. Indeed, for $A \in \mathbb{B}$ and for any $\varepsilon > 0$ there exist $m \in \mathbb{N}$ such that for all $n > m$ by (3.1.1) we have

$$|\mu_n(A)| = n|\varphi_n(A)| \leq n\frac{\varepsilon}{n} = \varepsilon.$$

However, $(\mu_n)_{n\in\mathbb{N}}$ is not bounded in the norm, because

$$\sup_{n\in\mathbb{N}} ||\mu_n|| = \sup_{n\in\mathbb{N}} n\|\lambda\| = \infty.$$

In particular, $\mathbb{B}$ does not have the Nikodym property.

If $\mathbb{B}$ is balanced, then by Lemma 4.2.5 it consists of semibalanced sets, so it does not have the Nikodym property by the first part of the lemma. □

Example 4.2.9. The following Boolean algebra considered by Plebanek[1] is interesting in the context of our considerations:

$$\mathbb{B}_P = \{B \in \mathrm{Bor}(C) : \lim_{n\to\infty} n\psi_n(B) = 0\},$$

where $\psi_n(B) = \min\{\lambda(B \triangle C) : C \in \mathbb{A}_n\}$.

Each element of $\mathbb{B}_P$ is semibalanced, so by Proposition 4.2.8 $\mathbb{B}_P$ does not have the Nikodym property. Indeed, one needs first to observe that if $B \in \mathbb{B}_P$ and $n < m$, then

$$\varphi_m(B) \leq \psi_n(B).$$

To show that every $B \in \mathbb{B}_P$ is semibalanced take any $B \in \mathbb{B}_P$ and $\varepsilon > 0$. Since $n\psi_n(B)$ converges to 0, we can find $m \in \mathbb{N}$ such that for every $r \geq m$ we have $r\psi_r(B) < \frac{\varepsilon}{2}$. Then for every $r > m$

$$r|\varphi_r(B)| \leq r\psi_{r-1}(B) \leq 2(r-1)\psi_{r-1}(B) < \varepsilon.$$

However, $\mathbb{B}_P$ is not balanced. To see this, take

$$B = \bigcup_{k=2}^{\infty} \langle s_k \rangle,$$

where $s_k \in \{-1, 1\}^k$ is of the form $s_k = (-1, \ldots, -1, 1, 1)$ for $k \geq 2$. Then

$$n\psi_n(B) = n\sum_{k>n} \lambda(\langle s_k \rangle) = \frac{n}{2^n}$$

[1] Personal communication

converges to 0, so $B \in \mathbb{B}_P$. But B is not balanced. Indeed, for $k \in \mathbb{N}$ consider $s'_k \in \{-1,1\}^k$ of the form $s'_k = (-1,\ldots,-1,1)$. Then

$$\frac{\lambda(B \cap \langle s'_k \rangle)}{\lambda(\langle s'_k \rangle)} = \frac{\lambda(\langle s'_k \rangle \setminus B)}{\lambda(\langle s'_k \rangle)} = \frac{1}{2}.$$

There are no non-trivial convergent sequences in $\mathrm{St}(\mathbb{B}_P)$. However, this Boolean algebra does not have the Grothendieck property. The sequence of measures on $\mathbb{B}_P$ given by

$$\vartheta_n(A) = 2^n \varphi_{2^n}(A \cap \langle s'_n \rangle),$$

(where s'_n is as above) is weak*-convergent, but it is not weakly convergent[2].

Moreover, according to Borodulin-Nadzieja[3] no semibalanced Boolean algebra containing $\mathbb{B}_P$ has the Grothendieck property.

In order to take care of the Grothendieck property it is enough to restrict the choice of sequences of measures to those with pairwise disjoint Borel supports and norms equal to 1.

Definition 4.2.10. Let $\mathbb{A}$ be a Boolean algebra. We say that a sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{A}$ is **normal**, if:

- $\forall n \in \mathbb{N}\ \|\nu_n\| = 1$,
- $(\widetilde{\nu}_n)_{n\in\mathbb{N}}$ has pairwise disjoint Borel supports.

The following definition will be important throughout the chapter.

Definition 4.2.11. We say that a Boolean algebra $\mathbb{B}$ satisfies $(\mathcal{G})$ if for every normal sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}$ there are $G \in \mathbb{B}$, an antichain $\{H_0^n, H_1^n : n \in \mathbb{N}\} \subseteq \mathbb{B}$ and strictly increasing sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ of natural numbers such that for all $n \in \mathbb{N}$

(a) $G \cap H_1^n = \varnothing$,
(b) $|\nu_{a_n}|(H_0^n) \geq 0.9$ and $|\nu_{b_n}|(H_1^n) \geq 0.9$,
(c) $|\nu_{a_n}(G \cap H_0^n)| \geq 0.3$.

Note that Schachermayer introduced the property (G) in [116] as a name for the Grothendieck property. Our property $(\mathcal{G})$ is different. It implies the Grothendieck property, but the reverse implication does not hold.

To show that the property $(\mathcal{G})$ implies the Grothendieck property we will use the following lemma known as The Kadec-Pełczyński-Rosenthal Subsequence Splitting Lemma:

Lemma 4.2.12. *[3, Lemma 5.2.7] Let K be a compact space. For every bounded sequence $(\widetilde{\nu}_n)_{n\in\mathbb{N}} \subseteq M(K)$ there exists a non-negative real r and a subsequence $(\widetilde{\nu}_{n_k})_{k\in\mathbb{N}}$, each element of which may be decomposed into a sum of two measures $\widetilde{\nu}_{n_k} = \widetilde{\mu}_k + \widetilde{\theta}_k$, where $\widetilde{\mu}_k, \widetilde{\theta}_k \in M(K)$, satisfying the following conditions:*

[2] The idea behind this sequence is due to Avilés.
[3] Personal communication.

(1) the measures $\widetilde{\mu}_k$ are supported by pairwise disjoint Borel sets,
(2) $(\widetilde{\theta}_k)_{k\in\mathbb{N}}$ is weakly convergent,
(3) $||\widetilde{\mu}_k|| = r$, for every $k \in \mathbb{N}$.

Proposition 4.2.13. *If a Boolean algebra $\mathbb{B}$ satisfies $(\mathcal{G})$, then $\mathbb{B}$ has the Grothendieck property.*

Proof. Suppose $\mathbb{B}$ has the property $(\mathcal{G})$, but does not have the Grothendieck property. That is, there is a sequence $\widetilde{\mu}_n$ of measures on $\mathrm{St}(\mathbb{B})$ weak*-convergent to a measure $\widetilde{\mu}$, which is not weakly convergent. Without loss of generality, by passing to a subsequence, we can assume that no subsequence of $(\widetilde{\mu}_n)_{n\in\mathbb{N}}$ is weakly convergent. Indeed, suppose that each subsequence of $(\widetilde{\mu}_n)_{n\in\mathbb{N}}$ contains a weakly convergent subsequence. Since the sequence is weak*-convergent to $\widetilde{\mu}$, such a subsequence must be also weakly convergent to $\widetilde{\mu}$. Then each subsequence of $(\widetilde{\mu}_n)_{n\in\mathbb{N}}$ has a subsequence weakly convergent to $\widetilde{\mu}$, so the whole sequence is weakly convergent to $\widetilde{\mu}$, which gives a contradiction.

Since $(\widetilde{\mu}_n)_{n\in\mathbb{N}}$ is weak*-convergent, it is bounded in the norm (cf. [39, Theorem 3.88]). By Lemma 4.2.12 we can find a real r and a subsequence $(\widetilde{\mu}_{n_k})_{k\in\mathbb{N}}$ each element of which may be decomposed into the sum of two measures $\widetilde{\mu}_{n_k} = \widetilde{\nu}_k + \widetilde{\theta}_k$ satisfying

(1) the measures $\widetilde{\nu}_k$ are supported by pairwise disjoint Borel sets,
(2) $(\widetilde{\theta}_k)_{k\in\mathbb{N}}$ is weakly convergent,
(3) $||\widetilde{\nu}_k|| = r$, for every $k \in \mathbb{N}$.

Note that since $(\widetilde{\theta}_n)_{n\in\mathbb{N}}$ is weakly convergent, $(\widetilde{\nu}_n)_{n\in\mathbb{N}}$ is not. In particular, $r \neq 0$. Thus, without loss of generality, by the normalization, we can assume that $r = 1$. Then the sequence $(\nu_n)_{n\in\mathbb{N}}$ is normal. The Boolean algebra $\mathbb{B}$ satisfies the property $(\mathcal{G})$, so there are $G \in \mathbb{B}$, an antichain $\{H_0^n, H_1^n : n \in \mathbb{N}\} \subseteq \mathbb{B}$ and strictly increasing sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ of natural numbers such that for all $n \in \mathbb{N}$

(a) $G \cap H_1^n = \varnothing$,
(b) $|\nu_{a_n}|(H_0^n) \geq 0.9$ and $|\nu_{b_n}|(H_1^n) \geq 0.9$,
(c) $|\nu_{a_n}(G \cap H_0^n)| \geq 0.3$.

Hence for $n \in \mathbb{N}$ we have

(1) $|\nu_{a_n}(G)| \geq |\nu_{a_n}(G \cap H_0^n)| - |\nu_{a_n}|(G \backslash H_0^n) \geq 0.3 - 0.1 = 0.2$,
(2) $|\nu_{b_n}|(G) \leq |\nu_{b_n}|(C \backslash H_1^n) \leq 0.1$.

So there is no ν such that $\nu_n(G) \to \nu(G)$. Thus, $(\widetilde{\nu}_k)_{k\in\mathbb{N}}$ is not weak*-convergent, which is a contradiction, since $(\widetilde{\nu}_k)_{k\in\mathbb{N}} = (\widetilde{\mu}_{n_k} - \widetilde{\theta}_k)_{k\in\mathbb{N}}$ is a difference of two weak*-convergent sequences. □

Now we will introduce the property $(\mathcal{G}^*)$ similar to the property $(\mathcal{G})$, which focuses on only one sequence of measures. Then we will show that having the property $(\mathcal{G}^*)$ for enough many sequences of measures we can conclude that the property $(\mathcal{G})$ holds.

Definition 4.2.14. Let $\mathbb{B}^* \subseteq \mathbb{B}$ be Boolean algebras and let $\nu = (\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures on $\mathbb{B}^*$. We say that $(\mathbb{B}^*, \mathbb{B}, \nu)$ satisfies $(\mathcal{G}^*)$, if there are: an antichain $\{H_0^n, H_1^n : n \in \mathbb{N}\} \subseteq \mathbb{B}^*$, a set $G \in \mathbb{B}$ and strictly increasing sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ of natural numbers such that for all $n \in \mathbb{N}$

(a) $G \cap H_0^n \in \mathbb{B}^*$,
(b) $G \cap H_1^n = \varnothing$,
(c) $|\nu_{a_n}|(H_0^n), |\nu_{b_n}|(H_1^n) \geq 0.9$,
(d) $|\nu_{a_n}(G \cap H_0^n)| \geq 0.3$.

The next proposition shows the relationship between the properties $(\mathcal{G})$ and $(\mathcal{G}^*)$.

Proposition 4.2.15. *Suppose that $\mathbb{B}$ is a Boolean algebra such that for every normal sequence of measures $\nu = (\nu_n)_{n\in\mathbb{N}}$ on $\mathbb{B}$ there is a subalgebra $\mathbb{B}^* \subseteq \mathbb{B}$ such that the sequence $(\nu_n \restriction \mathbb{B}^*)_{n\in\mathbb{N}}$ is normal and $(\mathbb{B}^*, \mathbb{B}, \nu \restriction \mathbb{B}^*)$ satisfies $(\mathcal{G}^*)$. Then $\mathbb{B}$ satisfies $(\mathcal{G})$. In particular, $\mathbb{B}$ has the Grothendieck property.*

Proof. Fix any normal sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}$. Pick $\mathbb{B}^* \subseteq \mathbb{B}$ such that there exist an antichain $\{H_0^n, H_1^n : n \in \mathbb{N}\} \subseteq \mathbb{B}^*$, $G \in \mathbb{B}$ and sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ such that for the sequence $(\nu_n \restriction \mathbb{B}^*)$ the conditions (a)-(d) from Definition 4.2.14 are satisfied. In particular, $G \cap H_1^n = \varnothing$, i.e. (a) of Definition 4.2.11 holds. To see that (b) of Definition 4.2.11 is satisfied, observe that

$$\begin{aligned} |\nu_{a_n}|(H_0^n) &= \sup\{|\nu_{a_n}(A)| + |\nu_{a_n}(B)| : A, B \in \mathbb{B}, A, B \subseteq H_0^n, A \cap B = \varnothing\} \geq \\ &\geq \sup\{|\nu_{a_n}(A)| + |\nu_{a_n}(B)| : A, B \in \mathbb{B}^*, A, B \subseteq H_0^n, A \cap B = \varnothing\} = \\ &= |\nu_{a_n} \restriction \mathbb{B}^*|(H_0^n) \geq 0.9 \end{aligned}$$

and similarly $|\nu_{b_n}|(H_1^n) \geq |\nu_{b_n} \restriction \mathbb{B}^*|(H_1^n) \geq 0.9$.

For Definition 4.2.11 (c) note that

$$|\nu_{a_n}(G \cap H_0^n)| = |\nu_{a_n} \restriction \mathbb{B}^*(G \cap H_0^n)| \geq 0.3.$$

Use Proposition 4.2.13 to conclude that $\mathbb{B}$ has the Grothendieck property. □

For the reader's convenience we provide a brief sketch of our constructions. We describe consecutive steps of reasoning, starting from general motivations. The parts devoted only to the case of construction under the continuum hypothesis (from Theorem 4.4.8) are tagged (CH), while the parts devoted to the forcing construction (from Theorem 4.5.15) are tagged (F).

Construction roadmap

1. General idea: we construct an increasing sequence $(\mathbb{B}_\alpha)_{\alpha<\omega_1}$ of balanced countable subalgebras of $\mathrm{Bor}(C)$.

   (CH) For every $\alpha < \omega_1$ the triple $(\mathbb{B}^*_\alpha, \mathbb{B}_{\alpha+1}, (\nu_n^\alpha)_{n\in\mathbb{N}})$ satisfies $(\mathcal{G}^*)$, where $(\nu_n^\alpha)_{n\in\mathbb{N}}$ is a sequence of measures (on some subalgebra $\mathbb{B}^*_\alpha$ of $\mathbb{B}_\alpha$) which is given (by a proper bookkeeping) in advance (see Theorem 4.4.8).

(F) We define a finite support iteration $(\mathbb{P}_\alpha)_{\alpha\le\omega_1}$ of σ-centered forcings (Definition 4.5.9). For every $\alpha<\omega_1$ the algebra $\mathbb{B}_\alpha$ belongs to the α-th intermediate model obtained from this iteration. The triples $(\mathbb{B}_\alpha, \mathbb{B}_{\alpha+1}, (\nu_n)_{n\in\mathbb{N}})$ satisfy $(\mathcal{G}^*)$ for uncountably many sequences of measures (on $\mathbb{B}_\alpha$), whose choice depends on a generic filter in $\mathbb{P}_\alpha$ (for the connection between the choice of sequences of measures and a generic filter see Lemma 4.5.6).

We finish by taking the Boolean algebra

$$\mathbb{B} = \bigcup_{\alpha<\omega_1} \mathbb{B}_\alpha,$$

which is balanced and satisfies $(\mathcal{G})$. Applying Proposition 4.2.8 and Proposition 4.2.13 we obtain Theorem 4.4.8 and Theorem 4.5.15.

2. We start with $\mathbb{B}_0 = \mathrm{Clop}(C)$. At limit steps we take unions. The only non-trivial step is the construction of $\mathbb{B}_{\alpha+1}$ from $\mathbb{B}_\alpha$. In this case we extend $\mathbb{B}_\alpha$ by a new set $G\in \mathrm{Bor}(C)$ which is a union of countably many pairwise disjoint elements of $\mathbb{B}_\alpha$:

$$G = \bigcup_{n\in\mathbb{N}} G_n$$

that satisfy the hypothesis of Lemma 4.3.7 (this ensures that $\mathbb{B}_{\alpha+1}$ is balanced). Moreover, we require that

(CH) G (together with some antichain $\{H_0^n, H_1^n\}_{n\in\mathbb{N}} \subseteq \mathbb{B}_\alpha$) is a witness for the property $(\mathcal{G}^*)$ for the triple $(\mathbb{B}^*_\alpha, \mathbb{B}_{\alpha+1}, (\nu_n^\alpha)_{n\in\mathbb{N}})$,

(F) there is an antichain $\{H_n\}_{n\in\mathbb{N}} \subseteq \mathbb{B}_\alpha$ such that for every sequence $(\nu_n)_{n\in\mathbb{N}}$ satisfying the hypothesis of Proposition 4.5.7 the set G together with some subset of $\{H_n\}_{n\in\mathbb{N}}$ witnesses the property $(\mathcal{G}^*)$ for $(\mathbb{B}_\alpha, \mathbb{B}_{\alpha+1}, (\nu_n)_{n\in\mathbb{N}})$.

3. From now on we will assume that α is fixed and we will focus on the construction of G_n's, H_n's and H_i^n's for $i=0,1$.

(CH) We define $(G_n)_{n\in\mathbb{N}}, (H_0^n)_{n\in\mathbb{N}}, (H_1^n)_{n\in\mathbb{N}}$ by induction on $n\in\mathbb{N}$ (see Lemma 4.4.6). In order to obtain the property $(\mathcal{G}^*)$ we need to ensure that for every $n\in\mathbb{N}$ the set $G_n\cap H_0^n$ is "big" in the sense of some measure from the sequence $(\nu_n^\alpha)_{n\in\mathbb{N}}$ while $G_n \cap H_1^k = \varnothing$ for $k,n\in\mathbb{N}$.

(F) The sets G_n's and H_n's appear in the forcing conditions chosen by a generic filter. Lemma 4.5.6 will imply that for an appropriate sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures, for infinitely many $n\in\mathbb{N}$ the set $G_n\cap H_n$ is "big" in the sense of some measure from this sequence, while $G_n\cap H_k=\varnothing$ for every $n\in\mathbb{N}$ and infinitely many $k\in\mathbb{N}$.

4. Given finite sequences $(G_k)_{k\le n}, (H_0^k)_{k\le n}, (H_1^k)_{k\le n}$ (or $(H_k)_{k\le n}$ in the case of forcing), we extend them using Lemma 4.4.4 and Lemma 4.4.5 (applied to $\widehat{G} = \bigcup_{k\le n} G_k$ and $\widehat{H} = \bigcup_{k\le n}(H_0^k\cup H_1^k)$ or $\widehat{H} = \bigcup_{k\le n} H_k$).

(CH) The set G_{n+1} consists of 2 parts: $G_{n+1} = L\cup M$, where $L = G_{n+1}\cap H_0^{n+1}$ is the part witnessing the property $(\mathcal{G}^*)$ and M is a very small set disjoint from H_i^k's (so it has no influence on $(\mathcal{G}^*)$).

(F) It follows that there are $l_1, l_2 > n$ such that $G_{l_1} = L \cup M$, where $L = G_{l_1} \cap H_{l_1}$ witnesses the property $(\mathcal{G}^*)$ and $M \cap H_k = \varnothing$ for $k \in \mathbb{N}$, while $G_{l_2} \cap H_{l_2} = \varnothing$ (this will ensure that $G \cap H_{l_2} = \varnothing$).

5. At the same time, to make sure that the hypothesis of Lemma 4.3.7 is satisfied (cf. item 2), we need to pick G_n's in such a way that the families $\mathcal{F}(\mathbb{B}_n, \bigcup_{i \le k} G_i)$ (where $\mathbb{B}_n$'s are some finite subalgebras of $\mathbb{B}_\alpha$) will have appropriate degrees of balance for $k, n \in \mathbb{N}$. However, we do not have enough control over the choice of L, which can affect the balance. Thus, we need to fix it with the help of M.
6. The choice of M depends on L in the way described in Proposition 4.3.8. The main idea behind this choice is to reduce the problem to finite combinatorics. We work with some finite subalgebra $\mathbb{H}$ of $\mathbb{B}$ containing the sets that have appeared in the construction so far (including L) that is sufficiently well balanced (cf. Lemma 4.3.12).
7. Lemma 4.3.5 shows that there is $n \in \mathbb{N}$ and a Boolean homomorphism $h\colon \mathbb{H} \to \mathbb{A}_n$ (recall that $\mathbb{A}_n$ is a finite Boolean algebra consisting of clopen subsets of C) such that every $A \in \mathbb{H}$ is well-approximated by $h(A)$. The choice of M and most of the crucial calculations take place in $\mathbb{A}_n$ (see Lemma 4.3.11). These include the use of techniques such as probability inequalities involving weighted Rademacher sums (Lemma 4.3.9) and analysis in finite-dimensional subspaces of the Hilbert space $\mathcal{L}_2(C)$.

4.3 Properties of balanced families

This section is devoted to the combinatorics of balanced sets and families. In a series of lemmas we will describe basic properties of balanced sets and show how to modify a given set to a balanced one.

We start with a few simple observations.

Lemma 4.3.1. *If $A, B \in \mathrm{Bor}(C)$ are disjoint and (m, ε)-balanced, then $A \cup B$ is $(m, 2\varepsilon)$-balanced.*

Proof. First we check (3.2.1). Fix any $s \in \{-1,1\}^m$. If

$$\frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)} < \frac{\varepsilon}{m} \text{ or } \frac{\lambda(\langle s\rangle \backslash B)}{\lambda(\langle s\rangle)} < \frac{\varepsilon}{m}$$

then

$$\frac{\lambda(\langle s\rangle \backslash (A \cup B))}{\lambda(\langle s\rangle)} \le \min\left\{\frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)}, \frac{\lambda(\langle s\rangle \backslash B)}{\lambda(\langle s\rangle)}\right\} < \frac{\varepsilon}{m}.$$

Otherwise, since A and B satisfy (3.2.1) we have

$$\frac{\lambda(A \cap \langle s\rangle)}{\lambda(\langle s\rangle)} < \frac{\varepsilon}{m} \text{ and } \frac{\lambda(B \cap \langle s\rangle)}{\lambda(\langle s\rangle)} < \frac{\varepsilon}{m}.$$

Summing up the inequalities we get

$$\frac{\lambda((A \cup B) \cap \langle s\rangle)}{\lambda(\langle s\rangle)} < \frac{2\varepsilon}{m},$$

so $A \cup B$ satisfies (3.2.1) for 2ε.

Now we check (3.2.3). Fix $r > m$. Since $A \cap B = \varnothing$ we have

$$|\varphi_r((A \cup B) \cap \langle s \rangle)| \leq |\varphi_r(A \cap \langle s \rangle)| + |\varphi_r(B \cap \langle s \rangle)|.$$

Thus, by summing up the inequalities (3.2.3) for A and B we get

$$\frac{|\varphi_r((A \cup B) \cap \langle s \rangle)|}{\lambda(\langle s \rangle)} \leq \frac{|\varphi_r(A \cap \langle s \rangle)|}{\lambda(\langle s \rangle)} + \frac{|\varphi_r(B \cap \langle s \rangle)|}{\lambda(\langle s \rangle)} < \frac{2\varepsilon}{r},$$

so $A \cup B$ satisfies (3.2.3) for 2ε. □

Lemma 4.3.2. *Let $A \in \mathrm{Bor}(C), t \in \mathbb{N}$ and $\varepsilon > 0$. If A is (t, ε)-semibalanced then $C \backslash A$ is also (t, ε)-semibalanced.*

Proof. Let $r > t$. We have

$$\varphi_r(A) + \varphi_r(C \backslash A) = \int_A \delta_r d\lambda + \int_{C \backslash A} \delta_r d\lambda = \int_C \delta_r d\lambda = 0.$$

By (3.1.1) for A we get

$$|\varphi(C \backslash A)| = |\varphi(A)| < \frac{\varepsilon}{r},$$

which implies (3.1.1) for $C \backslash A$ and so $C \backslash A$ is (t, ε)-semibalanced. □

Lemma 4.3.3. *Suppose that $\mathbb{F}$ is a finite Boolean subalgebra of $\mathrm{Bor}(C)$ and $t \in \mathbb{N}$ is such that $\mathbb{F}$ is (t, ε)-balanced, where*

$$\varepsilon = \frac{1}{100} \inf\{\lambda(A) : A \in \mathbb{F}, \lambda(A) > 0\}.$$

Then for every $A \in \mathbb{F}$, if $\lambda(A) > 0$, then there is $s_A \in \{-1, 1\}^t$ such that

$$\frac{\lambda(A \cap \langle s_A \rangle)}{\lambda(\langle s_A \rangle)} \geq 0.99.$$

Proof. Let $A \in \mathbb{F}$ be such that $\lambda(A) > 0$. Then $\lambda(A) > \varepsilon$, so there must be $s_A \in \{-1, 1\}^t$ such that $\lambda(A \cap \langle s_A \rangle) > \varepsilon \lambda(\langle s_A \rangle)$. Hence

$$\frac{\lambda(\langle s_A \rangle \backslash A)}{\lambda(\langle s_A \rangle)} \leq 1 - \varepsilon.$$

Since A satisfies (3.2.1) we have

$$\frac{\lambda(\langle s_A \rangle \backslash A)}{\lambda(\langle s_A \rangle)} < \frac{\varepsilon}{t} \leq \varepsilon.$$

But $\varepsilon \leq 0.01$, so

$$\frac{\lambda(A \cap \langle s_A \rangle)}{\lambda(\langle s_A \rangle)} = \frac{\lambda(\langle s_A \rangle) - \lambda(\langle s_A \rangle \backslash A)}{\lambda(\langle s_A \rangle)} > 1 - \varepsilon \geq 0.99.$$

□

Lemma 4.3.4. *Let $\mathbb{H}_0$ be an (n, ε)-balanced Boolean algebra. Then the Boolean algebra $\mathbb{H}$ generated by $\mathbb{H}_0 \cup \mathbb{A}_n$ is (n, ε)-balanced.*

Proof. Take $A \in \mathbb{H}$. Then A is of the form:

$$A = \bigcup_{\langle s\rangle \in \mathrm{at}(\mathbb{A}_n)} (\langle s\rangle \cap A_s) = \bigcup_{s\in\{-1,1\}^n} (\langle s\rangle \cap A_s),$$

where $A_s \in \mathbb{H}_0$ for $s \in \{-1,1\}^n$.

Let $s_0 \in \{-1,1\}^n$. Then

$$A \cap \langle s_0\rangle = \bigcup_{s\in\{-1,1\}^n} (\langle s\rangle \cap A_s) \cap \langle s_0\rangle = A_{s_0} \cap \langle s_0\rangle$$

and

$$\langle s_0\rangle \backslash A = \langle s_0\rangle \backslash \bigcup_{s\in\{-1,1\}^n} (\langle s\rangle \cap A_s) = \langle s_0\rangle \backslash A_{s_0}.$$

By (3.2.1) we have

$$\frac{\lambda(A_{s_0} \cap \langle s_0\rangle)}{\lambda(\langle s_0\rangle)} < \frac{\varepsilon}{n} \text{ or } \frac{\lambda(\langle s_0\rangle \backslash A_{s_0})}{\lambda(\langle s_0\rangle)} < \frac{\varepsilon}{n}.$$

If $\frac{\lambda(A_{s_0}\cap\langle s_0\rangle)}{\lambda(\langle s_0\rangle)} < \frac{\varepsilon}{n}$, then

$$\frac{\lambda(A \cap \langle s_0\rangle)}{\lambda(\langle s_0\rangle)} = \frac{\lambda(A_{s_0} \cap \langle s_0\rangle)}{\lambda(\langle s_0\rangle)} < \frac{\varepsilon}{n}.$$

If $\frac{\lambda(\langle s_0\rangle\backslash A_{s_0})}{\lambda(\langle s_0\rangle)} < \frac{\varepsilon}{n}$, then

$$\frac{\lambda(\langle s_0\rangle \backslash A)}{\lambda(\langle s_0\rangle)} = \frac{\lambda(\langle s_0\rangle \backslash A_{s_0})}{\lambda(\langle s_0\rangle)} < \frac{\varepsilon}{n},$$

so A also satisfies (3.2.1).

Let $r > n$ and let $s_0 \in \{-1,1\}^n$. Then by (3.2.3)

$$\frac{|\varphi_r(A \cap \langle s_0\rangle)|}{\lambda(\langle s_0\rangle)} = \frac{|\varphi_r(A_{s_0} \cap \langle s_0\rangle)|}{\lambda(\langle s_0\rangle)} < \frac{\varepsilon}{r}.$$

□

The next lemma shows that while dealing with finite balanced families we can approximate them with finite families of clopen subsets of C. This will allow us to reduce many problems to the combinatorics of finite Boolean algebras $\mathbb{A}_n$ for $n \in \mathbb{N}$.

Lemma 4.3.5. *Suppose that $\mathbb{H} \subseteq \mathrm{Bor}(C)$ is a finite subalgebra that is (n,ε)-balanced for some $n \in \mathbb{N}$ and $\varepsilon < 1/3$. Then the function $h_n\colon \mathbb{H} \to \mathbb{A}_n$ given by*

$$h_n(A) = \bigcup \left\{ \langle s\rangle : s \in \{-1,1\}^n, \frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)} < \varepsilon \right\}$$

is a homomorphism of Boolean algebras and for every $A \in \mathbb{H}$ we have

$$\lambda\left(A \triangle h_n(A)\right) < \varepsilon/n.$$

Proof. For the first part of the lemma we need to show that $h_n(C) = C, h_n(A \cup B) = h_n(A) \cup h_n(B)$ and $h_n(C \backslash A) = C \backslash h_n(A)$ for every $A, B \in \mathbb{H}$.

The first equality holds since for every $s \in \{-1,1\}^n$ we have

$$\frac{\lambda(\langle s\rangle \backslash C)}{\lambda(\langle s\rangle)} = \frac{\lambda(\varnothing)}{\lambda(\langle s\rangle)} = 0.$$

The second equality follows from the fact that

$$\frac{\lambda(\langle s\rangle \backslash (A \cup B))}{\lambda(\langle s\rangle)} < \varepsilon \text{ iff } \frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)} < \varepsilon \text{ or } \frac{\lambda(\langle s\rangle \backslash B)}{\lambda(\langle s\rangle)} < \varepsilon.$$

Indeed, if

$$\frac{\lambda(\langle s\rangle \backslash (A \cup B))}{\lambda(\langle s\rangle)} < \varepsilon,$$

then

$$\frac{\lambda((A \cup B) \cap \langle s\rangle)}{\lambda(\langle s\rangle)} \geq \frac{2}{3}$$

and so

$$\max\left\{\frac{\lambda(A \cap \langle s\rangle)}{\lambda(\langle s\rangle)}, \frac{\lambda(B \cap \langle s\rangle)}{\lambda(\langle s\rangle)}\right\} \geq \frac{1}{3} > \varepsilon.$$

Since A and B are (n, ε)-balanced we have

$$\max\left\{\frac{\lambda(A \cap \langle s\rangle)}{\lambda(\langle s\rangle)}, \frac{\lambda(B \cap \langle s\rangle)}{\lambda(\langle s\rangle)}\right\} > 1 - \varepsilon$$

or equivalently

$$\min\left\{\frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)}, \frac{\lambda(\langle s\rangle \backslash B)}{\lambda(\langle s\rangle)}\right\} < \varepsilon.$$

Conversely, if

$$\frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)} < \varepsilon \text{ or } \frac{\lambda(\langle s\rangle \backslash B)}{\lambda(\langle s\rangle)} < \varepsilon,$$

then

$$\frac{\lambda(\langle s\rangle \backslash (A \cup B))}{\lambda(\langle s\rangle)} \leq \min\left\{\frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)}, \frac{\lambda(\langle s\rangle \backslash B)}{\lambda(\langle s\rangle)}\right\} < \varepsilon.$$

The equality $h_n(C \backslash A) = C \backslash h_n(A)$ holds since for $s \in \{-1,1\}^n$

$$\langle s\rangle \subseteq C \backslash h_n(A) \text{ iff } \frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)} \geq \varepsilon \text{ iff } \frac{\lambda(A \cap \langle s\rangle)}{\lambda(\langle s\rangle)} \leq 1 - \varepsilon \text{ iff}$$
$$\frac{\lambda(A \cap \langle s\rangle)}{\lambda(\langle s\rangle)} < \varepsilon \text{ iff } \frac{\lambda(\langle s\rangle \backslash (C \backslash A))}{\lambda(\langle s\rangle)} < \varepsilon \text{ iff } \langle s\rangle \subseteq h_n(C \backslash A).$$

For the second part of the lemma we notice that for every $s \in \{-1,1\}^n$ if $\frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)} < \frac{\varepsilon}{n}$, then

$$\lambda(A \triangle h_n(A) \cap \langle s\rangle) = \lambda(\langle s\rangle \backslash A) < \lambda(\langle s\rangle)\frac{\varepsilon}{n}$$

and if $\frac{\lambda(\langle s\rangle \backslash A)}{\lambda(\langle s\rangle)} \geq \frac{\varepsilon}{n}$, then $\frac{\lambda(A \cap \langle s\rangle)}{\lambda(\langle s\rangle)} < \frac{\varepsilon}{n}$, and so

$$\lambda(A \triangle h_n(A) \cap \langle s\rangle) = \lambda(A \cap \langle s\rangle) < \lambda(\langle s\rangle)\frac{\varepsilon}{n}.$$

Hence

$$\lambda(A\triangle h_n(A)) = \sum_{s\in\{-1,1\}^n} \lambda(A\triangle h_n(A)\cap\langle s\rangle) \le \sum_{s\in\{-1,1\}^n} \lambda(\langle s\rangle)\frac{\varepsilon}{n} = \frac{\varepsilon}{n}.$$

□

The next lemma says that small perturbations of (m,t,ε)-balanced sets are still (m,t,ε)-balanced.

Lemma 4.3.6. *Let $m,t\in\mathbb{N}, t>m, \varepsilon>0$. Then there is $\varrho>0$ such that for every $A,B\in\mathrm{Bor}(C)$, if A is (m,t,ε)-balanced and $\lambda(B)<\varrho$, then $A\cup B$ and $A\backslash B$ are (m,t,ε)-balanced.*

Proof. Let $\varepsilon_1<\varepsilon$ be such that A is (m,t,ε_1)-balanced and let

$$\varrho = \frac{\varepsilon-\varepsilon_1}{2^m t}.$$

For every $s\in\{-1,1\}^m$ we have

$$\frac{\lambda((A\cup B)\cap\langle s\rangle)}{\lambda(\langle s\rangle)} \le \frac{\lambda(A\cap\langle s\rangle)}{\lambda(\langle s\rangle)} + \frac{\varrho}{\lambda(\langle s\rangle)} < \frac{\varepsilon_1}{m} + \frac{\varepsilon-\varepsilon_1}{t} \le \frac{\varepsilon}{m}$$

or

$$\frac{\lambda(\langle s\rangle\backslash(A\cup B))}{\lambda(\langle s\rangle)} \le \frac{\lambda(\langle s\rangle\backslash A)}{\lambda(\langle s\rangle)} < \frac{\varepsilon}{m}$$

and for every $s\in\{-1,1\}^m$, $m<r\le t$

$$\frac{|\varphi_r((A\cup B)\cap\langle s\rangle)|}{\lambda(\langle s\rangle)} \le \frac{|\varphi_r(A\cap\langle s\rangle)|}{\lambda(\langle s\rangle)} + \frac{|\varphi_r((B\backslash A)\cap\langle s\rangle)|}{\lambda(\langle s\rangle)} \le \frac{\varepsilon_1}{r} + \frac{\varepsilon-\varepsilon_1}{t} < \frac{\varepsilon}{r}.$$

Hence $A\cup B$ is (m,t,ε)-balanced. Calculations showing that $A\backslash B$ is (m,t,ε)-balanced are similar. □

In the following lemma, we provide conditions for enlarging balanced Boolean algebras to bigger ones.

Lemma 4.3.7. *Suppose that $\mathbb{B}\subseteq\mathrm{Bor}(C)$ is a balanced Boolean algebra and that*

$$\mathbb{B} = \bigcup_{n\in\mathbb{N}} \mathbb{B}_n$$

is a representation of $\mathbb{B}$ as an increasing union of finite subalgebras. Let $(m_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers and $\{G_n\}_{n\in\mathbb{N}}\subseteq\mathbb{B}$ be an antichain such that

$$\forall k\in\mathbb{N}\ \forall n\le k\ \mathcal{F}\Big(\mathbb{B}_n, \bigcup_{i\le k} G_i\Big) \text{ is } (m_n, 2^{-n})\text{-balanced},$$

where $\mathcal{F}\left(\mathbb{B}_n,\bigcup_{i\le k}G_i\right) = \left\{A\cap\bigcup_{i\le k}G_i, A\backslash\bigcup_{i\le k}G_i : A\in\mathbb{B}_n\right\}$.

Put $G=\bigcup_{n\in\mathbb{N}}G_n$. Then the Boolean algebra $\mathbb{B}'$ generated by $\mathbb{B}\cup\{G\}$ is balanced.

Proof. Let $\varepsilon > 0$ and let $\mathcal{A}'$ be a finite subfamily of $\mathbb{B}'$. Every element $A' \in \mathcal{A}'$ is of the form $(A_1 \cap G) \cup (A_2 \backslash G)$ where $A_1, A_2 \in \mathbb{B}$, so there is a finite family $\mathcal{A} \subseteq \mathbb{B}$ such that $\mathcal{A}' \subseteq \{(A_1 \cap G) \cup (A_2 \backslash G) : A_1, A_2 \in \mathcal{A}\}$. Let $n \in \mathbb{N}$ be such that $\mathcal{A} \subseteq \mathbb{B}_n$ and $1/2^{n-1} < \varepsilon$. Fix $A_1, A_2 \in \mathcal{A}$. Then for every $k > n$

$$A_1 \cap \bigcup_{i \le k} G_i \text{ and } A_2 \backslash \bigcup_{i \le k} G_i \text{ are } (m_n, 1/2^n) - \text{balanced.}$$

Hence for $s \in \{-1,1\}^{m_n}$ we have

$$\begin{aligned}\frac{\lambda(A_1 \cap G \cap \langle s \rangle)}{\lambda(\langle s \rangle)} &\le \frac{\lambda\left(A_1 \cap \bigcup_{i \le k} G_i \cap \langle s \rangle\right) + \lambda\left(\bigcup_{i > n} G_i\right)}{\lambda(\langle s \rangle)} \le \\ &\le \frac{1}{m_n 2^n} + \frac{\lambda\left(\bigcup_{i>k} G_i\right)}{\lambda(\langle s \rangle)} \xrightarrow{k \to \infty} \frac{1}{m_n 2^n}\end{aligned}$$

or

$$\frac{\lambda(\langle s \rangle \backslash (A_1 \cap G))}{\lambda(\langle s \rangle)} \le \frac{\lambda\left(\langle s \rangle \backslash \left(A_1 \cap \bigcup_{i \le k} G_i\right)\right)}{\lambda(\langle s \rangle)} \le \frac{1}{m_n 2^n}$$

and for $m > m_n$

$$\begin{aligned}\frac{|\varphi_m(A_1 \cap G \cap \langle s \rangle)|}{\lambda(\langle s \rangle)} &\le \frac{\left|\varphi_m\left(A_1 \cap \bigcup_{i \le k} G_i \cap \langle s \rangle\right)\right|}{\lambda(\langle s \rangle)} + \frac{\left|\varphi_m\left(A_1 \cap \bigcup_{i > k} G_i \cap \langle s \rangle\right)\right|}{\lambda(\langle s \rangle)} \le \\ &\le \frac{1}{m 2^n} + \frac{\lambda\left(\bigcup_{i>k} G_i\right)}{\lambda(\langle s \rangle)} \xrightarrow{k \to \infty} \frac{1}{m 2^n},\end{aligned}$$

so $A_1 \cap G$ is $(m_n, 1/2^n)$-balanced. By a similar argument $A_2 \backslash G$ is $(m_n, 1/2^n)$-balanced. By Lemma 4.3.1 the set $(A_1 \cap G) \cup (A_2 \backslash G)$ is $(m_n, 1/2^{n-1})$-balanced, and so (m_n, ε)-balanced, which completes the proof. $\square$

The next theorem is key for the construction in Lemma 4.4.5. It says how much we can modify a given finite balanced family without losing its balance.

Proposition 4.3.8. *Let $k \in \mathbb{N}$, $\eta > 0$. Let $(m_n)_{n \le k}$ be an increasing sequence of natural numbers. Let $\mathbb{B}^* \subseteq \mathbb{B} \subseteq \mathrm{Bor}(C)$ be balanced Boolean algebras and assume that $\mathrm{Clop}(C) \subseteq \mathbb{B}^*$. Let $(\mathbb{B}_n)_{n \le k} \subseteq \mathbb{B}$ be finite subalgebras. Suppose that $G, P \in \mathbb{B}^*$ and the following are satisfied:*

(A) $G \subseteq P$,
(B) $\forall n \le k$ $\mathcal{F}(\mathbb{B}_n, G)$ *is* $(m_n, 2^{-n})$*-balanced.*

Then there is $\theta > 0$ such that for every $L, Q \in \mathbb{B}^*$ satisfying

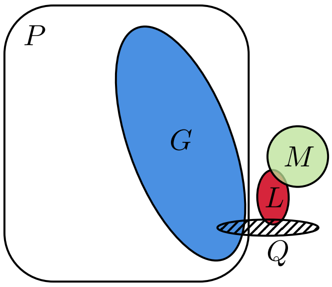
(a) $\max\{\lambda(L), \lambda(Q)\} < \theta$,
(b) $L \cap P = \varnothing$,

there is $M \in \mathbb{B}^*$ such that

(1) $M \cap (P \cup Q) = \varnothing$,

(2) $\lambda(M) < \eta$,

(3) $\forall n \leq k$ $\mathcal{F}(\mathbb{B}_n, G \cup L \cup M)$ *is* $(m_n, 2^{-n})$*-balanced.*



Before we prove the proposition, we will need a few lemmas.

The following lemma is a version of [72, Theorem 3, page 31].

Lemma 4.3.9. *Let* $n \in \mathbb{N}$. *Let* λ_n *be the standard product probability measure on the space* $\{-1,1\}^n$ $(\lambda(\{x\}) = \lambda(\{y\})$ *for every* $x, y \in \{-1,1\}^n)$. *Then for all* $(d_m)_{m\leq n} \in \mathbb{R}^n$ *and any* $\xi \in (0,1)$

$$\lambda_n\left(\left\{y \in \{-1,1\}^n : \left|\sum_{m=1}^{n} y_m d_m\right|^2 \geq \xi \sum_{m=1}^{n} |d_m|^2\right\}\right) \geq \frac{1}{3}(1-\xi)^2.$$

The above lemma will allow us to pick y from some big enough subset of $\{-1,1\}^n$ satisfying the appropriate inequality.

Lemma 4.3.10. *The sequence* $(\delta_n)_{n\in\mathbb{N}}$ *is orthonormal in the Hilbert space* $\mathcal{L}_2(C)$.

Proof. We need to show that for any $n, m \in \mathbb{N}$

$$\langle \delta_n, \delta_m \rangle = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

If $n = m$, then $\langle \delta_n, \delta_m \rangle = \|\delta_n\| = 1$.

Suppose that $n \neq m$. For $i, j \in \{-1,1\}$ let $C^i_j = \{s \in C : s_n = i, s_m = j\}$ and note that if $x \in C^i_j$, then

$$\delta_n(x)\delta_m(x) = ij.$$

Since $\lambda(C^i_j) = 1/4$ for $i, j \in \{-1,1\}$, we get

$$\langle \delta_n, \delta_m \rangle = \int_C \delta_n \delta_m d\lambda = \sum_{i,j\in\{-1,1\}} \int_{C^i_j} \delta_n \delta_m d\lambda = \sum_{i,j\in\{-1,1\}} ij\lambda(C^i_j) = 0.$$

□

Lemma 4.3.11. *Let* $t \in \mathbb{N}$. *Let* $\eta \in (0, 1/2^{t+10})$. *Let* n_0 *be large enough so that for all* $n > n_0$ *there exists* $k \in \mathbb{N}$ *that*

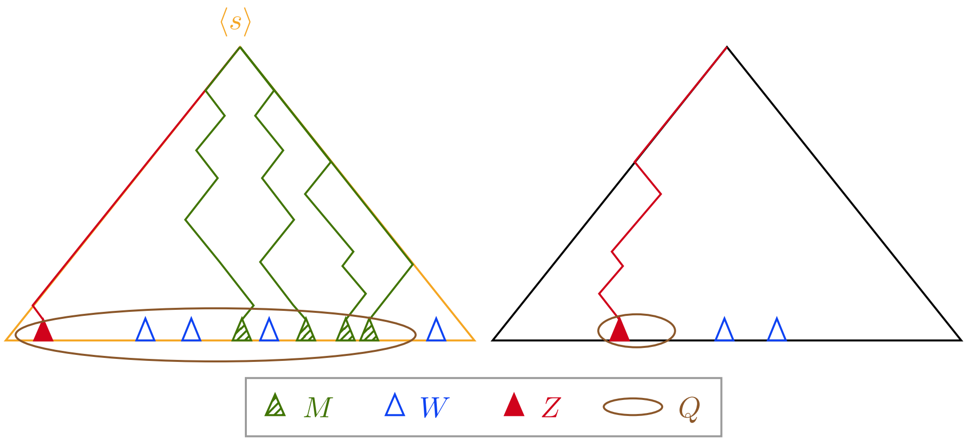
$$\frac{\eta}{2} < \frac{k}{2^n} < \eta \text{ and } \frac{n^3}{2^{n-1}} \leq \eta.$$

Let $n > n_0$ *and* k *satisfy the above inequality. Then for all*

- *$Q \in \mathbb{A}_n$ such that $\lambda(Q) < \eta$,*
- *$F \in \mathbb{A}_n$ for which there exists $s \in \{-1,1\}^t$ such that $\frac{\lambda(F\cap\langle s\rangle)}{\lambda(\langle s\rangle)} \geq 0.95$,*
- *$Z \in \mathbb{A}_n$ such that $\lambda(Z) < \frac{\eta^2}{64}$ and $Z \subseteq F$,*

there exists $M \subseteq F \setminus (Q \cup Z)$ such that

(a) $M \in \mathbb{A}_n$,
(b) $\lambda(M) = \frac{k}{2^n}$,
(c) $M \cup Z$ is (t,η)-semibalanced.



In the figure above, the lines indicating the position of the triangles forming Z and M at each level go to the left as many times as to the right, which means that for every $r \in \mathbb{N}$ we have $\varphi_r(M \cup Z) = 0$ (and so $M \cup Z$ is (t,η)-semibalanced).

Proof. Let $F' = F \setminus (Q \cup Z)$ and fix $s \in \{-1,1\}^t$ such that

$$\frac{\lambda(F \cap \langle s\rangle)}{\lambda(\langle s\rangle)} \geq 0.95.$$

In particular,

$$\frac{\lambda(F' \cap \langle s\rangle)}{\lambda(\langle s\rangle)} \geq 0.9.$$

Put $\mathcal{M} = \left\{M' \in \mathbb{A}_n : M' \subseteq F', \lambda(M') = \frac{k}{2^n}\right\}$. Since for $M' \in \mathcal{M}$ we have $M' \cap Z = \varnothing$. It follows that $\varphi_m(M' \cup Z) = \varphi_m(M') + \varphi_m(Z)$.

Define

$$S(M') = \sum_{m=t+1}^{n} (\varphi_m(M' \cup Z))^2.$$

Choose a set $M \in \mathcal{M}$ such that

$$S(M) = \min\{S(M') : M' \in \mathcal{M}\}.$$

We will show, that $M \cup Z$ is (t,η)-semibalanced. Namely, we will show, that $S(M) < \frac{\eta^2}{n^2}$. This implies that for all $n \geq m > t$ we have

$$|\varphi_m(M \cup Z)| \leq \sqrt{S(M)} < \frac{\eta}{n} \leq \frac{\eta}{m},$$

while for $m > n$ we have $|\varphi_m(M \cup Z)| = 0$ (because $Z, M \in \mathbb{A}_n$). Thus, $M \cup Z$ is (t, η)-semibalanced.
We need to show that

$$S(M) < \frac{k}{2^n} \frac{n}{2^{n-1}}. \tag{4.1}$$

Indeed, if the above inequality holds, then

$$S(M) < \frac{k}{2^n} \frac{n}{2^{n-1}} < \eta \frac{n}{2^{n-1}} = \frac{\eta}{n^2} \frac{n^3}{2^{n-1}} \leq \frac{\eta^2}{n^2}.$$

The inequality (4.1) may be written as:

$$\begin{aligned} S(M) &= \sum_{m=t+1}^{n} \varphi_m(Z)\varphi_m(M \cup Z) + \sum_{m=t+1}^{n} \varphi_m(M)\varphi_m(M \cup Z) < \\ &< \frac{k}{2^n} \frac{\sqrt{S(M)}}{4} + \frac{k}{2^n} \left(\frac{n}{2^{n-1}} - \frac{\sqrt{S(M)}}{4} \right). \end{aligned}$$

To prove it we will split it into two inequalities:

$$\sum_{m=t+1}^{n} \varphi_m(M)\varphi_m(M \cup Z) \leq \frac{k}{2^n} \left(\frac{n}{2^{n-1}} - \frac{\sqrt{S(M)}}{4} \right) \tag{4.2}$$

and

$$\sum_{m=t+1}^{n} \varphi_m(Z)\varphi_m(M \cup Z) < \frac{k}{2^n} \frac{\sqrt{S(M)}}{4}. \tag{4.3}$$

We will prove the inequalities (4.2) and (4.3) with the help of four claims. The first three are necessary to show the inequality (4.2) while the last one will prove inequality (4.3).

In Claim 2 we make use of the minimality of $S(M)$ analyzing the situation when we change M by one atom of $\mathbb{A}_m$.

Claim 2. *For any $x, y \in \{-1, 1\}^n$ such that $\langle x \rangle \subseteq M$ and $\langle y \rangle \subseteq F' \setminus M$*

$$\sum_{m=t+1}^{n} x_m \varphi_m(M \cup Z) \leq \frac{n}{2^{n-1}} + \sum_{m=t+1}^{n} y_m \varphi_m(M \cup Z), \tag{4.4}$$

where x_m and y_m are the m-th terms of the sequences x and y respectively.

Proof of the claim. Let $M' = (M \setminus \langle x \rangle) \cup \langle y \rangle$. We have

$$\begin{aligned} \varphi_m(M') &= \int_{M'} \delta_m d\lambda \quad = \int_M \delta_m d\lambda - \int_{\langle x \rangle} \delta_m d\lambda + \int_{\langle y \rangle} \delta_m d\lambda \\ &= \varphi_m(M) + \frac{1}{2^n}(y_m - x_m). \end{aligned}$$

Since M minimizes S, we have $S(M') - S(M) \geq 0$. Then

$$\begin{aligned}
S(M') - S(M) &= \sum_{m=t+1}^{n} \Big((\varphi_m(Z) + \varphi_m(M'))^2 - (\varphi_m(Z) + \varphi_m(M))^2 \Big) \\
&= \sum_{m=t+1}^{n} \Big(\varphi_m(M')^2 - \varphi_m(M)^2 + 2\varphi_m(Z)(\varphi_m(M') - \varphi_m(M)) \Big) \\
&= \sum_{m=t+1}^{n} (\varphi_m(M') - \varphi_m(M))(\varphi_m(M') + \varphi_m(M) + 2\varphi_m(Z)) \\
&= \sum_{m=t+1}^{n} \frac{1}{2^n}(y_m - x_m) \left(\frac{1}{2^n}(y_m - x_m) + 2\varphi_m(M) + 2\varphi_m(Z) \right).
\end{aligned}$$

Multiplying the above by 2^{n-1} and using the fact that $(y_m - x_m)^2 \in \{0, 4\}$ we get

$$\begin{aligned}
0 &\leq \sum_{m=t+1}^{n} (y_m - x_m) \left(\frac{1}{2^{n+1}}(y_m - x_m) + \varphi_m(M) + \varphi_m(Z) \right) \\
&\leq \sum_{m=t+1}^{n} \frac{4}{2^{n+1}} + (y_m - x_m)(\varphi_m(M) + \varphi_m(Z)) \\
&\leq \frac{n}{2^{n-1}} + \sum_{m=t+1}^{n} (y_m - x_m)\varphi_m(M \cup Z).
\end{aligned}$$

□

Let $T \colon C \to C$ be a function that swaps the sign of the coordinates from $t+1$ to n given by the formula:

$$T(y)(i) = \begin{cases} -y(i) & \text{if } i \in (t, n], \\ y(i) & \text{if } i \notin (t, n]. \end{cases}$$

Note that for $A \in \mathrm{Bor}(C)$ we have $\varphi_m(A \cup T[A]) = 0$, where $m \in \{t+1, \ldots, n\}$.

Claim 3.

$$\frac{\lambda(((F' \cap \langle s \rangle) \cap T[F' \cap \langle s \rangle]) \setminus (M \cup T[M]))}{\lambda(\langle s \rangle)} \geq 0.75.$$

Proof of the claim. Since

$$\lambda(M) = \frac{k}{2^n} < \eta < \frac{0.05}{2^{t+1}},$$

we have

$$\lambda(M \cup T[M]) \leq \frac{0.05}{2^t}.$$

Since

$$\frac{\lambda(F' \cap \langle s \rangle)}{\lambda(\langle s \rangle)} \geq 0.9,$$

we have

$$\frac{\lambda((F' \cap \langle s \rangle) \cap T[F' \cap \langle s \rangle])}{\lambda(\langle s \rangle)} \geq 0.8.$$

Therefore

$$\frac{\lambda(((F' \cap \langle s \rangle) \cap T[F' \cap \langle s \rangle]) \setminus (M \cup T[M]))}{\lambda(\langle s \rangle)} \quad \geq \quad 0.8 - 0.05 = 0.75.$$

□

It is clear that if $y \in \langle s\rangle$ then $T(y) \in \langle s\rangle$. In the proof of the next claim we will use an obvious observation that if $\frac{\lambda(A\cap\langle s\rangle)}{\lambda(\langle s\rangle)} \geq 0.75$ and $\frac{\lambda(B\cap\langle s\rangle)}{\lambda(\langle s\rangle)} > 0.25$ then there exists $y \in A \cap B \cap \langle s\rangle$.

Claim 4. *There exists $y^M \in \{-1,1\}^n$ such that $\langle y^M\rangle \subseteq (F' \setminus M) \cap T[F' \setminus M]$ and*

$$\left|\sum_{m=t+1}^{n} y^M_m \varphi_m(M \cup Z)\right| \geq \frac{\sqrt{S(M)}}{4}.$$

Proof of the claim. Recall that $S(M) = \sum_{m=t+1}^{n} \varphi_m(M \cup Z)^2$. By Lemma 4.3.9 for $\xi = 1/16$ and

$$d_m = \begin{cases} \varphi_m(M \cup Z) & \text{if } m \in (t,n], \\ 0 & \text{if } m \in [0,t], \end{cases}$$

we have

$$\lambda_n\left(\left\{y \in \{-1,1\}^n : \left|\sum_{m=t+1}^{n} y_m\varphi_m(M \cup Z)\right|^2 \geq \frac{1}{16}S(M)\right\}\right) \geq \frac{1}{3}\left(\frac{15}{16}\right)^2.$$

Hence

$$\lambda_n\left(\left\{y \in \{-1,1\}^n : \left|\sum_{m=t+1}^{n} y_m\varphi_m(M \cup Z)\right| \geq \frac{\sqrt{S(M)}}{4}\right\}\right) > 0.25.$$

Now Claim 2 implies the existence of the desired y^M. □

Note that $(F'\setminus M)\cap T[F'\setminus M] = T[(F'\setminus M)\cap T[F'\setminus M]]$. So since $\langle y^M\rangle \subseteq (F'\setminus M)\cap T[F'\setminus M]$ we also have $T[\langle y^M\rangle] \subseteq (F'\setminus M)\cap T[F'\setminus M]$. Moreover $\sum_{m=t+1}^{n} y^M_m\varphi_m(M\cup Z) = -\sum_{m=t+1}^{n} T(y^M_m)\varphi_m(M \cup Z)$. Thus, by replacing y^M with $T(y^M)$ if needed Claim 3 implies that

$$\sum_{m=t+1}^{n} y^M_m\varphi_m(M \cup Z) < -\frac{\sqrt{S(M)}}{4}. \tag{4.5}$$

From inequalities (4.4) and (4.5) for x as in Claim 1 we get

$$\sum_{m=t+1}^{n} x_m\varphi_m(M \cup Z) \leq \frac{n}{2^{n-1}} + \sum_{m=t+1}^{n} y^M_m\varphi_m(M \cup Z) \leq \frac{n}{2^{n-1}} - \frac{\sqrt{S(M)}}{4}. \tag{4.6}$$

Let $\{x^{(i)} : i \in \{1,\dots k\}\}$ be an enumeration of all $x \in \{-1,1\}^n$ such that $\langle x\rangle \subseteq M$.

$$M = \bigcup_{\substack{x\in\{-1,1\}^n \\ \langle x\rangle\subseteq M}} \langle x\rangle = \bigcup_{i=1}^{k}\langle x^{(i)}\rangle$$

Since for all $x \in \{-1,1\}^n$ and $m \leq n$ we have $x_m = 2^n \int_{\langle x\rangle} \delta_m d\lambda$ we conclude that

$$\sum_{i=1}^{k} x^{(i)}_m = 2^n \int_M \delta_m d\lambda = 2^n\varphi_m(M). \tag{4.7}$$

By (4.7) and (4.6) we obtain

$$\begin{aligned}\sum_{m=t+1}^{n} 2^n \varphi_m(M)\varphi_m(M\cup Z) &= \sum_{m=t+1}^{n}\sum_{i=1}^{k} x_m^{(i)}\varphi_m(M\cup Z) = \\ &= \sum_{i=1}^{k}\sum_{m=t+1}^{n} x_m^{(i)}\varphi_m(M\cup Z) \le k\left(\frac{n}{2^{n-1}} - \frac{\sqrt{S(M)}}{4}\right).\end{aligned} \tag{4.8}$$

Multiplying both sides of the inequality (4.8) by 2^{-n} we get

$$\sum_{m=t+1}^{n} \varphi_m(M)\varphi_m(M\cup Z) \le \frac{k}{2^n}\left(\frac{n}{2^{n-1}} - \frac{\sqrt{S(M)}}{4}\right),$$

that is the inequality (4.2).

Now we will prove (4.3).

Claim 5.

$$\sum_{m=t+1}^{n} \varphi_m(Z)\varphi_m(M\cup Z) < \frac{k}{2^n}\frac{\sqrt{S(M)}}{4}.$$

Proof of the claim. By the Cauchy-Schwarz inequality we have

$$\begin{aligned}\sum_{m=t+1}^{n} \varphi_m(Z)\varphi_m(M\cup Z) &\le \sqrt{\sum_{m=t+1}^{n}(\varphi_m(Z))^2}\sqrt{\sum_{m=t+1}^{n}\varphi_m(M\cup Z)^2} = \\ &= \sqrt{\sum_{m=t+1}^{n}(\varphi_m(Z))^2}\sqrt{S(M)}.\end{aligned}$$

By Lemma 4.3.10 the sequence $(\delta_n)_{n\in\mathbb{N}}$ is orthonormal in the Hilbert space $\mathcal{L}_2(C)$, so by the Bessel inequality we get

$$\begin{aligned}\sum_{m=t+1}^{n}(\varphi_m(Z))^2 &= \sum_{m=t+1}^{n}\left(\int_Z \delta_m d\lambda\right)^2 = \sum_{m=t+1}^{n}\left(\int_C \chi_Z\delta_m d\lambda\right)^2 = \\ &= \sum_{m=t+1}^{n}(\langle\chi_Z,\delta_m\rangle)^2 \le \|\chi_Z\|_2^2 = \lambda(Z).\end{aligned}$$

Since $\lambda(Z) < \eta^2/64$, we have $\sqrt{\lambda(Z)} < \eta/8 < k/2^{n+2}$ and so

$$\sum_{m=t+1}^{n} \varphi_m(Z)\varphi_m(M\cup Z) \le \sqrt{\lambda(Z)}\sqrt{S(M)} < \frac{k}{2^n}\frac{\sqrt{S(M)}}{4}.$$

□

Adding inequalities (4.2) and (4.3) side by side we get the following estimate

$$\sum_{m=t+1}^{n}\varphi_m(M\cup Z)^2 < \frac{k}{2^n}\frac{\sqrt{S(M)}}{4} + \frac{k}{2^n}\left(\frac{n}{2^{n-1}} - \frac{\sqrt{S(M)}}{4}\right) = \frac{k}{2^n}\frac{n}{2^{n-1}},$$

which shows (4.1) and finishes the proof. □

Lemma 4.3.12. *Let $\mathbb{B}^* \subseteq \mathbb{B}$ be balanced Boolean subalgebras of* $\mathrm{Bor}(C)$ *containing* $\mathrm{Clop}(C)$ *and let $\mathbb{F}$ be a finite subalgebra of $\mathbb{B}$. Let $P \in \mathbb{F} \cap \mathbb{B}^*$. Let $t \in \mathbb{N}, \delta > 0$. Suppose that for every $A \in \mathbb{F}$ such that $\lambda(A) > 0$ there is $s_A \in \{-1,1\}^t$ such that*

$$\frac{\lambda(A \cap \langle s_A \rangle)}{\lambda(\langle s_A \rangle)} \geq 0.99.$$

Then there is $\theta > 0$ such that for any $L, Q \in \mathbb{B}^*$, if $\max\{\lambda(L), \lambda(Q)\} < \theta$ *and $L \cap P = \varnothing$, then there is $M \in \mathbb{B}^*$ such that*

(1) $M \cap (P \cup Q) = \varnothing$,
(2) $\lambda(M) < \delta$,
(3) $\forall F \in \mathbb{F}\ (M \cup L) \cap F$ *is (t, δ)-semibalanced,*
(4) $\forall F \in \mathbb{F}\ F \backslash (M \cup L)$ *is (t, δ)-semibalanced.*

Proof. Let $\eta < \min\{\delta/(4|\mathbb{F}|), 1/2^{t+10}\}, \theta < \eta^2/64$ and let n_0 be large enough so that for every $n > n_0$

(5) there is $k \in \mathbb{N}$ such that $\frac{\eta}{2} < \frac{k}{2^n} < \eta$,
(6) $\frac{n^3}{2^{n-1}} \leq \eta$,
(7) $\frac{\eta}{n} < \frac{\eta^2}{64} - \theta$.

Fix Q and L satisfying the hypothesis of the lemma. Denote by $\mathbb{H}_0$ the Boolean algebra generated by $\mathbb{F} \cup \{Q, L\}$. Since $\mathbb{B}$ is balanced, there is $n > n_0$ such that $\mathbb{H}_0$ is (n, η)-balanced. Let $\mathbb{H}$ be the Boolean algebra generated by $\mathbb{H}_0 \cup \mathbb{A}_n$. By Lemma 4.3.4

$$\mathbb{H} \text{ is } (n, \eta)\text{-balanced.}$$

Let h_n be defined as in Lemma 4.3.5. By the same lemma, for $F \in \mathbb{F}$ we have

$$\frac{\lambda(h_n(F) \cap \langle s_F \rangle)}{\lambda(\langle s_F \rangle)} \geq \frac{\lambda(F \cap \langle s_F \rangle)}{\lambda(\langle s_F \rangle)} - \frac{\lambda(F \triangle h_n(F))}{\lambda(\langle s_F \rangle)} \geq 0.99 - \frac{2^t \eta}{n} \geq 0.95.$$

By Lemma 4.3.5 and (7)

$$\lambda(h_n(L) \cap h_n(F)) \leq \lambda(h_n(L)) \leq \lambda(L) + \frac{\eta}{n} < \theta + \frac{\eta^2}{64} - \theta = \frac{\eta^2}{64}.$$

By Lemma 4.3.11 (applied to $Z = h_n(L) \cap h_n(E)$) for every $E \in \mathrm{at}(\mathbb{F})$ there is $M_E \in \mathbb{A}_n$ such that

(a) $M_E \subseteq h_n(E) \backslash h_n(Q)$,
(b) $\lambda(M_E) < \eta < \delta/|\mathbb{F}|$,
(c) $M_E \cup (h_n(L) \cap h_n(E))$ is $(t, \delta/(4|\mathbb{F}|))$-semibalanced.

Put

$$M_0 = \bigcup_{E \in \mathrm{at}(\mathbb{F})} M_E \backslash Q, M = M_0 \backslash P.$$

Then $M \cap (P \cup Q) = \varnothing$ and $\lambda(M) < \delta$. To show (3) and (4) fix $F \in \mathbb{F}$ and $r > t$. Consider 2 cases.

Case 1. $r \leq n$.

For every $E \in \mathrm{at}(\mathbb{F})$ we have $M_E \in \mathbb{A}_n$, so $h_n(M_E) = M_E$. From (a) we get

$$\begin{aligned}
h_n(M_0 \cap h_n(F)) &= h_n(M_0 \cap F) = h_n\left(\bigcup_{E \in \mathrm{at}(\mathbb{F})} (M_E \backslash Q) \cap F\right) = \\
&= \bigcup_{\substack{E \in \mathrm{at}(\mathbb{F}) \\ E \subseteq F}} (h_n(M_E) \cap h_n(E)) \backslash h_n(Q) = \\
&= \bigcup_{\substack{E \in \mathrm{at}(\mathbb{F}) \\ E \subseteq F}} ((M_E \cap h_n(E)) \backslash h_n(Q) = \\
&= \bigcup_{\substack{E \in \mathrm{at}(\mathbb{F}) \\ E \subseteq F}} (M_E \cap h_n(E)) \backslash h_n(Q) = \bigcup_{\substack{E \in \mathrm{at}(\mathbb{F}) \\ E \subseteq F}} M_E.
\end{aligned}$$

By Lemma 4.3.5 for any $A \in \mathbb{H}$ we have

$$|\varphi_r(A)| \leq |\varphi_r(h_n(A))| + \lambda(A \triangle h_n(A)) < |\varphi_r(h_n(A))| + \frac{\eta}{n}.$$

Putting $A = (M_0 \cup L) \cap F$ and using (c) we get

$$\begin{aligned}
|\varphi_r((M_0 \cup L) \cap F)| &\leq |\varphi_r(h_n((M_0 \cup L) \cap F))| + \frac{\eta}{n} < \\
&< |\varphi_r((h_n(M_0) \cup h_n(L)) \cap h_n(F))| + \frac{\delta}{4n} = \\
&= \left|\varphi_r\left(\bigcup_{\substack{E \in \mathrm{at}(\mathbb{F}) \\ E \subseteq F}} M_E \cup (h_n(L) \cap h_n(E))\right)\right| + \frac{\delta}{4n} \leq \\
&\leq \sum_{\substack{E \in \mathrm{at}(\mathbb{F}) \\ E \subseteq F}} |\varphi_r(M_E \cup (h_n(L) \cap h_n(E)))| + \frac{\delta}{4n} \leq \\
&\leq |\mathbb{F}| \frac{\delta}{4r|\mathbb{F}|} + \frac{\delta}{4n} \leq \frac{\delta}{2r}.
\end{aligned}$$

Case 2. $r > n$.

In this case, since $\mathbb{H}$ is $(n, \delta/2)$-balanced, we have

$$|\varphi_r((M_0 \cup L) \cap F)| \leq \sum_{s \in \{-1,1\}^n} |\varphi_r((M_0 \cup L) \cap F \cap \langle s \rangle)| < 2^n \lambda(\langle s \rangle) \frac{\delta}{2r} = \frac{\delta}{2r}.$$

Hence

$$(M_0 \cup L) \cap F \text{ is } \left(t, \frac{\delta}{2}\right)\text{-semibalanced for } F \in \mathbb{F}. \tag{8}$$

Since $L \cap P = \varnothing$ and $M = M_0 \backslash P$, we get

$$(M \cup L) \cap F = (M_0 \cup L) \cap (F \backslash P).$$

Since $P \in \mathbb{F}$ we get that if $F \in \mathbb{F}$, then $F \backslash P \in \mathbb{F}$, so by (8) applied to $F \backslash P$ we get that

$$(M \cup L) \cap F \text{ is } \left(t, \frac{\delta}{2}\right)\text{-semibalanced for } F \in \mathbb{F}, \tag{9}$$

which implies (3).

For (4) note that $C\backslash(F\backslash(M \cup L)) = (C\backslash F) \cup ((M \cup L) \cap F)$. Since $C\backslash F$ and $(M \cup L) \cap F$ are disjoint we have

$$|\varphi_r(C\backslash(F\backslash(M \cup L)))| \leq |\varphi_r(C\backslash F)| + |\varphi_r((M \cup L) \cap F)|.$$

Since $\mathbb{H}$ is $(t, \delta/2)$-balanced and $C\backslash F \in \mathbb{F} \subseteq \mathbb{H}$ we have $|\varphi_r(C\backslash F)| < \delta/(2r)$ and from (9) we get that $|\varphi_r((M \cup L) \cap F)| < \delta/(2r)$. Hence

$$|\varphi_r(C\backslash(F\backslash(M \cup L)))| < \frac{\delta}{r},$$

so $C\backslash(F\backslash(M \cup L))$ is (t, δ)-semibalanced. By Lemma 4.3.2 the set $F\backslash(M \cup L)$ is (t, δ)-semibalanced, which shows (4) and finishes the proof. □

Proof of Proposition 4.3.8. Denote by $\mathbb{F}$ the subalgebra of $\mathbb{B}$ generated by

$$\{G, P\} \cup \bigcup_{n \leq k} \mathbb{B}_n \cup \mathbb{A}_{m_k}.$$

Put

$$\varepsilon = \min\left\{\frac{1}{100}\inf\{\lambda(A) : A \in \mathbb{F}, \lambda(A) > 0\}, 2^{-m_k-k-1}\right\}. \tag{4}$$

Since $\mathbb{B}$ is balanced, there is $t \in \mathbb{N}, t > m_k$ such that

$$\mathbb{F} \text{ is } (t, \varepsilon)\text{-balanced}. \tag{5}$$

By Lemma 4.3.3 for every $A \in \mathbb{F}$ there is $s_A \in \{-1, 1\}^t$ such that

$$\frac{\lambda(A \cap \langle s_A \rangle)}{\lambda(\langle s_A \rangle)} \geq 0.99.$$

By Lemma 4.3.6 and the assumption (B) of the proposition there is $\varrho > 0$ such that

$$\begin{gathered} \forall n \leq k \; \forall A \in \mathcal{F}(\mathbb{B}_n, G) \; \forall B \in \mathrm{Bor}(C) \\ \lambda(B) < \varrho \implies A \cup B, A\backslash B \text{ are } (m_n, t, 2^{-n})\text{-balanced}. \end{gathered} \tag{6}$$

By Lemma 4.3.12 (applied to $\delta = \min\left\{\eta, \varrho/2, 2^{-m_k-k-1}\right\}$), there is $0 < \theta < \varrho/2$ such that for any $L, Q \in \mathbb{B}^*$, if $\max\{\lambda(L), \lambda(Q)\} < \theta$ and $L \cap P = \varnothing$, then there is $M \in \mathbb{B}^*$ satisfying

(1') $M \cap (P \cup Q) = \varnothing$,
(2') $\lambda(M) < \min\{\eta, \varrho/2\}$,
(3') $\forall F \in \mathbb{F}$ $(M \cup L) \cap F$ is $(t, 2^{-m_k-k-1})$-semibalanced,
(4') $\forall F \in \mathbb{F}$ $F\backslash(M \cup L)$ is $(t, 2^{-m_k-k-1})$-semibalanced.

In particular, the conditions (1) and (2) of the proposition are satisfied.

In order to show that such M satisfies (3), fix $L, Q \in \mathbb{B}^*$ such that $L \cap P = \varnothing$ and $\max\{\lambda(L), \lambda(Q)\} < \theta$. Then we have

$$\lambda(L \cup M) \leq \lambda(L) + \lambda(M) < \varrho/2 + \varrho/2 = \varrho,$$

so by (6) for every $n \leq k$ the family

$$\mathcal{F}(\mathbb{B}_n, G \cup L \cup M) \text{ is } (m_n, t, 2^{-n})\text{-balanced.} \tag{7}$$

Since $\mathbb{A}_{m_n}, \mathbb{B}_n \subseteq \mathbb{F}$ for $n \leq k$ and $G \in \mathbb{F}$, for every $A \in \mathbb{B}_n$ and $s \in \{-1, 1\}^{m_n}$ we have $\langle s \rangle \cap A \cap G, A \backslash G \in \mathbb{F}$. Hence by (4), (5) and Lemma 4.2.5

$$\langle s \rangle \cap A \cap G, \langle s \rangle \cap A \backslash G \text{ are } (t, 2^{-m_n-n-1})\text{-semibalanced.} \tag{8}$$

Since $\langle s \rangle \cap A \in \mathbb{F}$, by (3') and (4')

$$\langle s \rangle \cap A \cap (L \cup M), \langle s \rangle \cap A \backslash (L \cup M) \text{ are } (t, 2^{-m_n-n-1})\text{-semibalanced.} \tag{9}$$

Since $L \cap P = M \cap P = \varnothing$ and $G \subseteq P$, the sets $L \cup M$ and G are disjoint, so by (8) and (9) for $A \in \mathbb{B}_n, s \in \{-1, 1\}^n$

$$\langle s \rangle \cap A \cap (G \cup L \cup M), \langle s \rangle \cap A \backslash (G \cup L \cup M) \text{ are } (t, 2^{-m_n-n})\text{-semibalanced.}$$

By the above, (7) and Lemma 4.2.5

$$\mathcal{F}(\mathbb{B}_n, G \cup L \cup M) \text{ is } (m_n, 2^{-n})\text{-balanced,}$$

which shows (3) and completes the proof. □

4.4 Extensions of countable balanced Boolean algebras

In this section, we will show how to enlarge a given countable balanced Boolean algebra $\mathbb{B}$ to a balanced Boolean algebra $\mathbb{B}^*$, so that $(\mathbb{B}, \mathbb{B}^*, \nu)$ satisfies the property $(\mathcal{G}^*)$, where ν is a normal sequence of measures on $\mathbb{B}$. We will also show how to deal with finitely many measures simultaneously, which will be important in the forcing construction.

Lemma 4.4.1. (Folklore) *Suppose K is a compact Hausdorff space with an open basis $\mathcal{B}$ that is closed under finite unions. Let $\widetilde{\nu} \in M(K)$ and let $\mathcal{M} \subseteq M(K)$ be a finite set of measures such that $\widetilde{\nu} \perp \widetilde{\mu}$ for every $\widetilde{\mu} \in \mathcal{M}$. Then for every $\varepsilon > 0$ there is $X \in \mathcal{B}$ such that $|\widetilde{\nu}|(X) < \varepsilon$ and for every $\widetilde{\mu} \in \mathcal{M}$ we have $|\widetilde{\mu}|(K \backslash X) < \varepsilon$.*

Proof. First, we will show that the lemma holds when $\mathcal{M} = \{\widetilde{\mu}\}$. Since $\widetilde{\nu} \perp \widetilde{\mu}$ we have $|\widetilde{\nu}| \perp |\widetilde{\mu}|$, so there exists a Borel support A of $|\widetilde{\mu}|$, such that $K \setminus A$ is a Borel support of $|\widetilde{\nu}|$. By the regularity there is closed $B \subseteq A$ such that $|\widetilde{\mu}|(B) > |\widetilde{\mu}|(K) - \varepsilon$. We can find a closed set $D \subseteq K \setminus B$ such that $|\widetilde{\nu}|(D) > |\widetilde{\nu}|(K) - \varepsilon$. Note that $|\widetilde{\mu}|(D) \leq \varepsilon$. Since $U = K \setminus D$ is an open superset of B, there are finitely many sets $B_1, \ldots B_k \in \mathcal{B}, B_i \subseteq U$

for $i \leq k$, which cover B. Let $X = \bigcup_{i \leq k} B_i$. Note that $|\widetilde{\mu}|(X) \geq |\widetilde{\mu}|(K) - \varepsilon$, and $X \subseteq K \setminus D$, so $|\widetilde{\nu}|(X) < \varepsilon$ and $|\widetilde{\mu}|(K \setminus X) < \varepsilon$.

When $\mathcal{M} = \{\widetilde{\mu}_1, \ldots \widetilde{\mu}_n\}$, by the first part of the proof for each pair $(\widetilde{\nu}, \widetilde{\mu}_i)$ where $i \leq n$ we can find $X_i \in \mathcal{B}$ such that $|\widetilde{\nu}|(X_i) < \frac{\varepsilon}{n}$ and $|\widetilde{\mu}_i|(K \setminus X_i) < \frac{\varepsilon}{n}$. Then for $X = \bigcup_{i \in 1, \ldots n} X_i$ we have $|\widetilde{\nu}|(X) < \varepsilon$ and $|\widetilde{\mu}_i|(K \setminus X) < \frac{\varepsilon}{n} < \varepsilon$. □

Lemma 4.4.2. (Pełczyński)*[74, Lemma 5.3] Let $(\widetilde{\nu}_n)_{n \in \mathbb{N}}$ be a bounded sequence of Radon measures on a compact space K. Suppose there are pairwise disjoint Borel sets $(E_n)_{n \in \mathbb{N}}$ and $c > 0$ such that $\widetilde{\nu}_n(E_n) \geq c$ for every $n \in \mathbb{N}$. Then for every $\delta > 0$ there is a subsequence $(\widetilde{\nu}_{n_k})_{n \in \mathbb{N}}$ and a sequence of pairwise disjoint open sets $(U_k)_{k \in \mathbb{N}}$ such that $\widetilde{\nu}_{n_k}(U_k) \geq c - \delta$ for every $k \in \mathbb{N}$.*

We will use the following application of the above lemma, in which K is the Stone space of a Boolean algebra.

Corollary 4.4.3. *Let $(\nu_n)_{n \in \mathbb{N}}$ be a sequence of measures on $\mathbb{B}$ and $(E_n)_{n \in \mathbb{N}}$ be a sequence of disjoint Borel sets in $\mathrm{St}(\mathbb{B})$. Let $P \in \mathbb{B}$ be such that $E_n \cap [P] = \varnothing$ for every $n \in \mathbb{N}$. Let $c, \delta > 0$. If $|\widetilde{\nu}_n|(E_n) \geq c$ for every $n \in \mathbb{N}$, then there exist a subsequence $(\nu_{n_k})_{k \in \mathbb{N}}$ and a sequence $(V_k)_{k \in \mathbb{N}}$ of pairwise disjoint elements of $\mathbb{B}$ such that $V_k \cap P = \varnothing$ and $|\nu_{n_k}|(V_k) \geq c - \delta$ for all $k \in \mathbb{N}$.*

The next lemma will let us build an antichain needed to satisfy the property $(\mathcal{G})$.

Lemma 4.4.4. *Let $\nu = (\nu_n)_{n \in \mathbb{N}}$ be a normal sequence of measures on a Boolean algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$. Let $\mathcal{M}$ be a finite set of positive measures on $\mathbb{B}$ and assume that $(|\nu_n|)_{n \in \mathbb{N}}$ has a subsequence pointwise convergent to a measure $\nu_\infty \in \mathcal{M}$. Let $d \in \mathbb{N}$ and $\varepsilon > 0$. Let $P \in \mathbb{B}$ be such that*

$$\nu_\infty(P) < 0.1.$$

Then there are $H_0, H_1 \in \mathbb{B}$ and $a, b > d$ such that

(1) H_0, H_1, P are pairwise disjoint,
(2) $\forall \mu \in \mathcal{M}\ \mu(H_0 \cup H_1) < \varepsilon$,
(3) $\lambda(H_0 \cup H_1) < \varepsilon$,
(4) $|\nu_a|(H_0), |\nu_b|(H_1) \geq 0.9$.

Proof. We may assume that $\lambda \restriction \mathbb{B} \in \mathcal{M}$, so it is enough to show (1), (2) and (4).

Since ν_∞ is the pointwise limit of a sequence of probability measures, we have $\nu_\infty(C) = 1$, and so $\nu_\infty(C \setminus P) > 0.9$. Hence there is $\delta > 0$ such that for infinitely many $n \in \mathbb{N}$ we have

$$|\nu_n|(C \setminus P) > 0.9 + \delta.$$

Since $(\widetilde{\nu}_n)_{n \in \mathbb{N}}$ has pairwise disjoint Borel supports, there are pairwise disjoint sets $(E_n)_{n \in \mathbb{N}} \subseteq \mathrm{Bor}(C)$ such that

$$|\widetilde{\nu}_n|(E_n \setminus [P]) > 0.9 + \delta$$

for infinitely many $n \in \mathbb{N}$. By Corollary 4.4.3 there is an antichain $(V_l)_{l\in\mathbb{N}} \subseteq \mathbb{B}$ and a subsequence $(\nu_{n_l})_{l\in\mathbb{N}}$ such that $V_l \cap P = \varnothing$ and $|\nu_{n_l}|(V_l) \geq 0.9$ for every $l \in \mathbb{N}$. Since $(V_l)_{l\in\mathbb{N}}$ is an antichain, we have for every $\mu \in \mathcal{M}$

$$\lim_{l\to\infty} \mu(V_l) = 0.$$

In particular, if l_1, l_2 are big enough, then we have $\mu(V_{l_1} \cup V_{l_2}) < \varepsilon$ for every $\mu \in \mathcal{M}$, so it is enough to put $a = n_{l_1}, b = n_{l_2}, H_0 = V_{l_1}, H_1 = V_{l_2}$ where $l_1 \neq l_2$ are so big that $\min\{n_{l_1}, n_{l_2}\} > \max\{d, n\}$.

□

In the next two lemmas we describe how to pick a sequence of sets, whose union will be a witness for the property $(\mathcal{G}^*)$ for a given sequence of measures.

For the purpose of the construction under CH, in the following lemma it is enough to take $\mathcal{M} = \{\nu_\infty\}$. The case when $\mathcal{M}$ consists of more than one measure will be used in Section 4.5.

Lemma 4.4.5. *Suppose we are given:*

(A) *natural numbers* $k, d \in \mathbb{N}$,

(B) *subalgebras* $\mathbb{B}^*, \mathbb{B} \subseteq \mathrm{Bor}(C)$ *and finite subalgebras* $\mathbb{B}_n \subseteq \mathbb{B}$ *for* $n \leq k+1$ *such that*

- $\mathbb{B}$ *is balanced,*
- $\mathrm{Clop}(C) \subseteq \mathbb{B}^* \subseteq \mathbb{B}$,

(C) *a normal sequence of measures* $(\nu_n)_{n\in\mathbb{N}}$ *on* $\mathbb{B}^*$ *and a finite set* $\mathcal{M}$ *of probability measures on* $\mathbb{B}^*$ *such that* $(|\nu_n|)_{n\in\mathbb{N}}$ *has a subsequence pointwise convergent to a measure* $\nu_\infty \in \mathcal{M}$,

(D) *a strictly increasing sequence of natural numbers* $(m_n)_{n\leq k}$ *and sets* $\widehat{G}, \widehat{H} \in \mathbb{B}^*$ *such that*

- $\forall n \leq k\ \mathcal{F}(\mathbb{B}_n, \widehat{G})$ *is* $(m_n, 2^{-n})$*-balanced,*
- $\forall \mu \in \mathcal{M}\ \mu(\widehat{G} \cup \widehat{H}) < 0.1$.

Then there are $a, b > d; m_{k+1} \in \mathbb{N}; G', H_0, H_1 \in \mathbb{B}^*$ *such that:*

(1) $m_{k+1} > m_k$,

(2) $\forall n \leq k+1\ \mathcal{F}(\mathbb{B}_n, \widehat{G} \cup G')$ *is* $(m_n, 2^{-n})$*-balanced,*

(3) $\forall \mu \in \mathcal{M}\ \mu(\widehat{G} \cup G' \cup \widehat{H} \cup H_0 \cup H_1) < 0.1$,

(4) $\widehat{H}, H_0, H_1$ *are pairwise disjoint,*

(5) $G' \cap (\widehat{G} \cup \widehat{H} \cup H_1) = \varnothing$,

(6) $\widehat{G} \cap (H_0 \cup H_1) = \varnothing$,

(7) $|\nu_a|(H_0), |\nu_b|(H_1) \geq 0.9$,

(8) $|\nu_a(G' \cap H_0)| \geq 0.3$.

Proof. Let $\mathbb{E}$ be the subalgebra of $\mathbb{B}$ generated by $\mathbb{B}_{k+1} \cup \{\widehat{G}, \widehat{H}\}$. Since $\mathbb{B}$ is balanced, by Remark 4.2.4 there is $m_{k+1} \in \mathbb{N}, m_{k+1} > m_k$ such that

$$\mathbb{E} \text{ is } \left(m_{k+1}, \frac{1}{2^{k+1}}\right)\text{-balanced.}$$

In particular,

$$\mathcal{F}(\mathbb{B}_{k+1}, \widehat{G}) \text{ is } \left(m_{k+1}, \frac{1}{2^{k+1}}\right)\text{-balanced.} \tag{9}$$

Put

$$\xi = 0.1 - \max_{\mu \in \mathcal{M}} \mu(\widehat{G} \cup \widehat{H}). \tag{10}$$

For every $\mu \in \mathcal{M}$ consider its Lebesgue decomposition (see [114, Theorem 6.10])

$$\mu = \mu_1 + \mu_2, \text{ where } \mu_1 \ll \lambda \text{ and } \mu_2 \perp \lambda. \tag{11}$$

In particular (by [114, Theorem 6.11]) there is $\eta > 0$ such that

$$\forall \mu \in \mathcal{M}\ \forall A \in \mathbb{B}^*\ \lambda(A) < \eta \implies \mu_1(A) < \xi/4. \tag{12}$$

By (9), the first part of (D) and Proposition 4.3.8 (applied to $P = \widehat{G} \cup \widehat{H}$, $G = \widehat{G}$) there is $\theta > 0$ such that whenever $L, Q \in \mathbb{B}^*$ and $\max\{\lambda(L), \lambda(Q)\} < \theta$, there is $M_{(L,Q)} \in \mathbb{B}^*$ such that:

(13) $M_{(L,Q)} \cap (\widehat{G} \cup \widehat{H} \cup Q) = \varnothing$,
(14) $\lambda\left(M_{(L,Q)}\right) < \eta$,
(15) $\forall n \leq k+1\ \mathcal{F}\left(\mathbb{B}_n, \widehat{G} \cup L \cup M_{(L,Q)}\right)$ is $(m_n, 2^{-n})$-balanced.

By (11) and Lemma 4.4.1 there is $X \in \mathbb{B}^*$ such that for every $\mu \in \mathcal{M}$

$$\lambda(X) < \theta, \mu_2(C \backslash X) < \xi/4. \tag{16}$$

By Lemma 4.4.4 (applied to $\varepsilon = \min\{\theta - \lambda(X), \xi/4\}$, $P = \widehat{G} \cup \widehat{H}$) there are $a, b > d$ and $H_0, H_1 \in \mathbb{B}^*$ such that

- $\lambda(H_0 \cup H_1 \cup X) < \theta$,
- $\forall \mu \in \mathcal{M}\ \mu(H_0 \cup H_1) < \xi/4$,
- $H_0 \cap H_1 = (H_0 \cup H_1) \cap (\widehat{G} \cup \widehat{H}) = \varnothing$,
- $|\nu_a|(H_0), |\nu_b|(H_1) \geq 0.9$.

Let $L \in \mathbb{B}^*$ be such that

$$L \subseteq H_0 \text{ and } |\nu_a(L)| \geq 0.3. \tag{17}$$

Let $M = M_{(L, H_0 \cup H_1 \cup X)}$. Then, in particular,

$$M \cap (X \cup \widehat{H} \cup \widehat{G} \cup H_0 \cup H_1) = \varnothing. \tag{18}$$

We put $G' = L \cup M$.

We need to verify that these definitions satisfy conditions (1)-(8).

(1) follows directly from the choice of m_{k+1}. (2) follows from ((15)).

For (3) fix any $\mu \in \mathcal{M}$. By (12) and ((14)) we have

$$\mu_1(M) < \xi/4.$$

By (16) and (18) we have

$$\mu_2(M) < \xi/4.$$

By (17)

$$\mu(L) < \xi/4.$$

Hence

$$\mu(G') = \mu(M) + \mu(L) = \mu_1(M) + \mu_2(M) + \mu(L) < 3\xi/4.$$

Finally, from (10) we get

$$\mu(\widehat{G} \cup G' \cup \widehat{H} \cup H_0 \cup H_1) = \mu(\widehat{G} \cup \widehat{H}) + \mu(G') + \mu(H_0 \cup H_1) < 0.1 - \xi + 3\xi/4 + \xi/4 = 0.1.$$

Conditions (4)-(7) follow directly from the choice of a, b, H_0, H_1, G'.

(8) follows from (17) and the fact that $G' \cap H_0 = L$. $\square$

Lemma 4.4.6. *Let $\mathbb{B}^* \subseteq \mathbb{B} \subseteq \mathrm{Bor}(C)$ be balanced countable Boolean algebras and suppose that $(\nu_n)_{n\in\mathbb{N}}$ is a normal sequence of measures on $\mathbb{B}^*$.*

Then there exists a balanced countable Boolean algebra $\mathbb{B}' \subseteq \mathrm{Bor}(C)$ such that $\mathbb{B} \subseteq \mathbb{B}'$ and $(\mathbb{B}^*, \mathbb{B}', (\nu_n)_{n\in\mathbb{N}})$ satisfies $(\mathcal{G}^*)$.

Proof. Since $\mathbb{B}^*$ is countable, the dual ball in $C(\mathrm{St}(\mathbb{B}^*))$ is metrizable and by the Banach-Alaoglu theorem it is compact in the weak* topology. Hence there is a subsequence $(|\widetilde{\nu}_{n_k}|)_{n\in\mathbb{N}}$ of $(|\widetilde{\nu}_n|)_{n\in\mathbb{N}}$ that converges to a measure $\widetilde{\nu}_\infty$ in the weak* topology. In particular, $(|\nu_{n_k}|)_{n\in\mathbb{N}}$ is pointwise convergent to ν_∞.

Let us represent $\mathbb{B}$ as an increasing union of finite subalgebras

$$\mathbb{B} = \bigcup_{n\in\mathbb{N}} \mathbb{B}_n.$$

Using Lemma 4.4.5 we construct by induction on $k \in \mathbb{N}$ sequences $(m_k)_{k\in\mathbb{N}}$, $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and $(G_k)_{k\in\mathbb{N}}, (H_0^k)_{k\in\mathbb{N}}$, $(H_1^k)_{k\in\mathbb{N}} \subseteq \mathbb{B}^*$ such that

(1) $(m_k)_{k\in\mathbb{N}}, (a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}}$ are strictly increasing,
(2) $\forall k \in \mathbb{N}\ \forall n \le k\ \mathcal{F}\left(\mathbb{B}_n, \bigcup_{i\le k} G_i\right)$ is $(m_n, 2^{-n})$-balanced,
(3) $\forall k \in \mathbb{N}\ \nu_\infty\left(\bigcup_{n\le k}(G_n \cup H_0^n \cup H_1^n)\right) < 0.1$,
(4) $\{H_0^k, H_1^k\}_{k\in\mathbb{N}}$ are pairwise disjoint,
(5) $\{G_k\}_{k\in\mathbb{N}}$ are pairwise disjoint,
(6) $G_k \cap H_i^n \neq \varnothing$ if and only if $i = 0$ and $n = k$,
(7) $\forall k \in \mathbb{N}\ |\nu_{a_k}|(H_0^k), |\nu_{b_k}|(H_1^k) \ge 0.9$,
(8) $\forall k \in \mathbb{N}\ |\nu_{a_k}(G_k \cap H_0^k)| \ge 0.3$.

Let $k \in \mathbb{N} \cup \{0\}$ and suppose we have constructed $(m_n)_{n\le k}, (a_n)_{n\le k}, (b_n)_{n\le k}$, $(G_n)_{n\le k}$, $(H_0^n)_{n\le k}$, $(H_1^n)_{n\le k}$ (if $k = 0$, then we assume that all of these sequences are empty). We apply Lemma 4.4.5 to $\mathcal{M} = \{\nu_\infty\}, \widehat{G} = \bigcup_{i\le k} G_i$, $\widehat{H} = \bigcup_{i\le k}(H_0^i \cup H_1^i)$ and $d = \max\{a_k, b_k\}$ (or $\widehat{G} = \widehat{H} = \varnothing$ and $d = 1$, if $k = 0$) to obtain m_{k+1}, $a_{k+1} = a$, $b_{k+1} = b$, $G_{k+1} = G'$, $H_0^{k+1} = H_0$, $H_1^{k+1} = H_1$ satisfying (1)-(8).

Let
$$G = \bigcup_{n\in\mathbb{N}} G_n$$
and let $\mathbb{B}'$ be the Boolean algebra generated by $\mathbb{B} \cup \{G\}$. By (2) and Lemma 4.3.7 $\mathbb{B}'$ is balanced. To see that $(\mathbb{B}^*, \mathbb{B}', (\nu_n)_{n\in\mathbb{N}})$ satisfies $(\mathcal{G}^*)$ we notice that by (6) for every $n \in \mathbb{N}$ we have $G \cap H_i^n = G_n \cap H_i^n$ for $i = 0, 1$, and so

- $|\nu_{a_n}(G \cap H_0^n)| = |\nu_{a_n}(G_n \cap H_0^n)| \geq 0.3$,
- $G \cap H_1^n = \varnothing$,

which together with (7) and (8) shows that the conditions (a)-(d) of Definition 4.2.14 are satisfied and it completes the proof. □

Properties of measures on a Boolean algebra $\mathbb{A}$ such as norm or disjointness of Borel supports depend only on countably many elements of $\mathbb{A}$. In particular, the following lemma holds.

Lemma 4.4.7. *Let $\mathbb{B} = \bigcup_{\alpha<\omega_1} \mathbb{B}_\alpha$, where $(\mathbb{B}_\alpha)_{\alpha<\omega_1}$ is an increasing sequence of countable Boolean algebras. Let $(\nu_n)_{n\in\mathbb{N}}$ be a normal sequence of measures on $\mathbb{B}$. Then there exists $\alpha < \omega_1$ such that for every $\beta > \alpha$ the sequence $(\nu_n \restriction \mathbb{B}_\beta)_{n\in\mathbb{N}}$ is normal.*

The next theorem shows how to construct a Boolean algebra with the Grothendieck property and without the Nikodym property under CH.

Theorem 4.4.8. (Talagrand, [128]) *Assume* CH. *There exists a Boolean algebra with the Grothendieck property, but without the Nikodym property.*

Proof. By Proposition 4.2.8 and Proposition 4.2.13 it is enough to construct a balanced Boolean algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$ satisfying $(\mathcal{G})$. We will define $\mathbb{B}$ as a union of a sequence of countable subalgebras $(\mathbb{B}_\alpha)_{\alpha<\omega_1}$ of $\mathrm{Bor}(C)$, which is constructed by induction.

First, using CH we fix an enumeration
$$(\nu^\alpha, \mathbb{B}^*_\alpha)_{\alpha<\omega_1}$$
of all pairs such that each $\mathbb{B}^*_\alpha$ is a countable subalgebra of $\mathrm{Bor}(C)$ and $\nu^\alpha = (\nu^\alpha_n)_{n\in\mathbb{N}}$ is a normal sequence of measures on $\mathbb{B}^*_\alpha$. We also require each such pair to appear cofinaly often in the sequence $(\nu^\alpha, \mathbb{B}^*_\alpha)_{\alpha<\omega_1}$.

Successor stage: Suppose we have constructed $\mathbb{B}_\alpha$. If $\mathbb{B}^*_\alpha$ is not a subalgebra of $\mathbb{B}_\alpha$, then we put $\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha$. If $\mathbb{B}^*_\alpha \subseteq \mathbb{B}_\alpha$, then by Lemma 4.4.6 there is a balanced Boolean algebra $\mathbb{B}' \supseteq \mathbb{B}_\alpha$ such that $(\mathbb{B}^*_\alpha, \mathbb{B}', \nu^\alpha)$ satisfies $(\mathcal{G}^*)$. We put $\mathbb{B}_{\alpha+1} = \mathbb{B}'$.

If γ is a limit ordinal then we put $\mathbb{B}_\gamma = \bigcup_{\alpha<\gamma} \mathbb{B}_\alpha$.

We will prove that $\mathbb{B} = \bigcup_{\alpha<\omega_1} \mathbb{B}_\alpha$ satisfies $(\mathcal{G})$. By Proposition 4.2.15 it is enough to show that for every normal sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}$ there is $\beta < \omega_1$ such that the sequence $(\nu_n \restriction \mathbb{B}_\beta)_{n\in\mathbb{N}}$ is normal and $(\mathbb{B}_\beta, \mathbb{B}, (\nu_n \restriction \mathbb{B}_\beta)_{n\in\mathbb{N}})$ satisfies $(\mathcal{G}^*)$. By Lemma 4.4.7 there is $\alpha < \omega_1$ such that for every $\beta \geq \alpha$ the sequence $(\nu_n \restriction \mathbb{B}_\beta)_{n\in\mathbb{N}}$ is normal. To finish the proof, pick $\beta > \alpha$ such that $\mathbb{B}^*_\beta = \mathbb{B}_\alpha \subseteq \mathbb{B}_\beta$ and $\nu^\beta = \nu \restriction \mathbb{B}_\alpha$ and notice that by the construction $(\mathbb{B}^*_\beta, \mathbb{B}_{\beta+1}, (\nu_n \restriction \mathbb{B}^*_\beta)_{n\in\mathbb{N}})$ satisfies $(\mathcal{G}^*)$, which implies that $(\mathbb{B}_\beta, \mathbb{B}, (\nu_n \restriction \mathbb{B}_\beta)_{n\in\mathbb{N}})$ satisfies $(\mathcal{G}^*)$. □

4.5 Forcing

For the purpose of this section we identify Borel subsets of C with their codes (with respect to some absolute coding, see [69, Section 25]) i.e. whenever we say about the same Borel sets in different models of ZFC we mean Borel sets coded by the same code.

If $\mathfrak{c} > \omega_1$, then the method from the previous section does not work, since it requires extending a given Boolean algebra $\mathfrak{c}$ many times, while this method does not allow us to enlarge uncountable Boolean algebras keeping them balanced. Instead, we define a notion of forcing that adds to a given balanced algebra $\mathbb{B}$ a witness for $(\mathcal{G}^*)$ for many sequences of measures (chosen by a generic filter) on $\mathbb{B}$ simultaneously. However, it is not possible to pick one extension that is suitable for every sequence, so we will iterate ω_1 such forcings and the final Boolean algebra will have cardinality $\omega_1 < \mathfrak{c}$.

Definition 4.5.1. Let $\mathbb{B} \subseteq \mathrm{Bor}(C)$ be a balanced countable Boolean algebra containing $\mathrm{Clop}(C)$ and fix a representation $\mathbb{B} = \bigcup_{n\in\mathbb{N}} \mathbb{B}_n$, where $(\mathbb{B}_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite subalgebras of $\mathbb{B}$. We define a forcing notion $\mathbb{P}_\mathbb{B}$ consisting of conditions of the form

$$p = (k^p, (m^p_n)_{n\le k^p}, (G^p_n)_{n\le k^p}, (H^p_n)_{n\le k^p}, \mathcal{M}^p),$$

where

1. $k^p \in \mathbb{N}$,
2. $(m^p_n)_{n\le k^p}$ is a strictly increasing sequence of natural numbers,
3. $\mathcal{M}^p$ is a finite set of probability measures on $\mathbb{B}$ such that $\lambda \restriction \mathbb{B} \in \mathcal{M}^p$,
4. $(G^p_n)_{n\le k^p}$ and $(H^p_n)_{n\le k^p}$ are sequences of elements of $\mathbb{B}$ such that
   1. $G^p_n \cap G^p_l = H^p_n \cap H^p_l = G^p_n \cap H^p_l = \varnothing$ for $n \neq l$,
   2. $\mu\left(\bigcup_{n\le k^p}(G^p_n \cup H^p_n)\right) < 0.1$ for all $\mu \in \mathcal{M}^p$,
   3. $\mathcal{F}\left(\mathbb{B}_n, \bigcup_{i\le k^p} G^p_i\right)$ is $(m_n, 2^{-n})-$balanced for $n \le k^p$.

We put $q \le p$, if

- $k^q \ge k^p$,
- $m^q_n = m^p_n$ for $n \le k^p$,
- $G^q_n = G^p_n$ for $n \le k^p$,
- $H^q_n = H^p_n$ for $n \le k^p$,
- $\mathcal{M}^q \supseteq \mathcal{M}^p$.

Lemma 4.5.2. *$\mathbb{P}_\mathbb{B}$ is σ-centered. In particular, $\mathbb{P}_\mathbb{B}$ satisfies c.c.c.*

Proof. For $p \in \mathbb{P}_\mathbb{B}$ define

$$f(p) = (k^p, (m^p_n)_{n\le k^p}, (G^p_n)_{n\le k^p}, (H^p_n)_{n\le k^p}).$$

If $f(p) = f(q)$, then $r \le p, q$ and $f(r) = f(p) = f(q)$, where

$$r = (k^p, (m^p_n)_{n\le k^p}, (G^p_n)_{n\le k^p}, (H^p_n)_{n\le k^p}, \mathcal{M}^p \cup \mathcal{M}^q) \in \mathbb{P}_\mathbb{B}.$$

In particular, for every $x \in f[\mathbb{P}_{\mathbb{B}}]$ the set $f^{-1}(x)$ is directed. Since $\mathbb{B}$ is countable, $f[\mathbb{P}_{\mathbb{B}}]$ is also countable, so $\mathbb{P}_{\mathbb{B}}$ is a union of countably many directed sets. Hence $\mathbb{P}_{\mathbb{B}}$ is σ-centered. □

The next few lemmas concern the basic properties of $\mathbb{P}_{\mathbb{B}}$.

Lemma 4.5.3. *Let $p \in \mathbb{P}_{\mathbb{B}}$ and $k > k^p$. Then there is $q \in \mathbb{P}_{\mathbb{B}}, q \leq p$ such that $k^q = k$.*

Proof. It is enough to show that there is such q for $k = k^p + 1$ and apply an inductive argument. For this put $\widehat{G} = \bigcup_{n \leq k^p} G^p_n, \widehat{H} = \bigcup_{n \leq k^p} H^p_n$. Since $\lambda \restriction \mathbb{B}$, by the condition (4c) of Definition 4.5.1 we have $\lambda(C \backslash (\widehat{G} \cup \widehat{H})) > 0$. Hence there is a normal sequence $(\nu_n)_{n \in \mathbb{N}}$ of measures on $\mathbb{B}$, whose supports are included in $C \backslash (\widehat{G} \cup \widehat{H})$. We may also assume that $(|\nu_n|)_{n \in \mathbb{N}}$ is pointwise convergent to a probability measure ν_∞. In particular, we have

$$\nu_\infty(\widehat{G} \cup \widehat{H}) = \lim_{n \to \infty} |\nu_n|(\widehat{G} \cup \widehat{H}) = 0.$$

Let $\mathcal{M} = \mathcal{M}^p \cup \{\nu_\infty\}$. By Lemma 4.4.5 there are $m > m_{k^p}$ and $G', H_0 \in \mathbb{B}$ such that

$$q = (k, ((m^p_n)_{n \leq k^p}, m), ((G^p_n)_{n \leq k^p}, G'), ((H^p_n)_{n \leq k^p}, H_0), \mathcal{M}^p) \in \mathbb{P}_{\mathbb{B}}.$$

We have $q \leq p$. □

Definition 4.5.4. Let us introduce the following notation for a balanced Boolean algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$ containing $\mathrm{Clop}(C)$:

- $\dot{\mathbb{G}}$ denotes the canonical name for a generic filter in $\mathbb{P}_{\mathbb{B}}$,
- $\dot{G}$ is a $\mathbb{P}_{\mathbb{B}}$-name such that

$$\mathbb{P}_{\mathbb{B}} \Vdash \dot{G} = \bigcup_{p \in \dot{\mathbb{G}}, n \in \mathbb{N}} \check{G}^p_n,$$

- $\dot{\mathbb{B}}'$ denotes a $\mathbb{P}_{\mathbb{B}}$-name such that

$$\mathbb{P}_{\mathbb{B}} \Vdash \dot{\mathbb{B}}' \text{ is the subalgebra of } \mathrm{Bor}(C) \text{ generated by } \check{\mathbb{B}} \cup \{\dot{G}\},$$

- $\dot{H}, \dot{H}_n$ for $n \in \mathbb{N}$ denote $\mathbb{P}_{\mathbb{B}}$-names such that

$$\mathbb{P}_{\mathbb{B}} \Vdash \dot{H} = (\dot{H}_n)_{n \in \mathbb{N}} \text{ and } \forall n \in \mathbb{N}\ \exists p \in \dot{\mathbb{G}}\ \dot{H}_n = \check{H}^p_n.$$

Lemma 4.5.5. *Let $n \in \mathbb{N}$ and $p \in \mathbb{P}_{\mathbb{B}}$ be such that $k^p \geq n$. Then*

$$\mathbb{P}_{\mathbb{B}} \Vdash \dot{G} \cap \dot{H}_n \in \check{\mathbb{B}}$$

and

$$p \Vdash \dot{G} \cap \dot{H}_n = \check{G}^p_n \cap \check{H}^p_n.$$

Proof. By the definitions of $\dot{G}, \dot{H}_n$ and by (4a) from Definition 4.5.1 we have

$$\mathbb{P}_{\mathbb{B}} \Vdash \dot{G} \cap \dot{H}_n = \bigcup_{\substack{q \in \dot{\mathbb{G}}, l \in \mathbb{N} \\ k^q \geq n, l}} \check{G}^q_l \cap \check{H}^q_n = \bigcup_{\substack{q \in \dot{\mathbb{G}} \\ k^q \geq n}} \check{G}^q_n \cap \check{H}^q_n.$$

If q and r are compatible and $k^q, k^r \geq n$, then $G^q_n = G^r_n$ and $H^q_n = H^r_n$, so the last union above is in fact a union of one-element family consisting of a set from $\mathbb{B}$. In particular,

$$\mathbb{P}_{\mathbb{B}} \Vdash \dot{G} \cap \dot{H}_n \in \check{\mathbb{B}}$$

and

$$\mathbb{P}_{\mathbb{B}} \Vdash \check{p} \in \dot{\mathbb{G}} \implies \dot{G} \cap \dot{H}_n = \check{G}^p_n \cap \check{H}^p_n$$

or equivalently

$$p \Vdash \dot{G} \cap \dot{H}_n = \check{G}^p_n \cap \check{H}^p_n.$$

□

In the following lemma we will make use of Lemma 4.4.5 in the case when $\mathcal{M}$ consists of many probability measures. This is where the differences between our approach and that of work [128] are crucial.

Lemma 4.5.6. *Let* $\mathbb{B} \subseteq \mathrm{Bor}(C)$ *be a balanced Boolean algebra containing* $\mathrm{Clop}(C)$*. Let* $p \in \mathbb{P}_{\mathbb{B}}$ *and* $\nu = (\nu_n)_{n\in\mathbb{N}}$ *be a normal sequence of measures on* $\mathbb{B}$ *such that the sequence* $(|\nu_n|)_{n\in\mathbb{N}}$ *is pointwise convergent on* $\mathbb{B}$ *to a measure* $\nu_\infty \in \mathcal{M}^p$*. Then*

(1) $p \Vdash \forall k \in \mathbb{N}\ \exists n, l > k\ |\check{\nu}_n|(\dot{H}_l) \geq 0.9$ *and* $|\check{\nu}_n(\dot{G} \cap \dot{H}_l)| \geq 0.3$*,*
(2) $p \Vdash \forall k \in \mathbb{N}\ \exists n, l > k\ |\check{\nu}_n|(\dot{H}_l) \geq 0.9$ *and* $\dot{G} \cap \dot{H}_l = \varnothing$*.*

Proof. In the light of Lemma 4.5.5 it is enough to show that the sets

$$\mathbb{D}_k = \{q \in \mathbb{P}_{\mathbb{B}} : k^q > k, \exists n > k\ |\nu_n|(H^q_{k^q}) \geq 0.9 \text{ and } |\nu_n(G^q_{k^q} \cap H^q_{k^q})| \geq 0.3\}$$

and

$$\mathbb{E}_k = \{q \in \mathbb{P}_{\mathbb{B}} : k^q > k, \exists n > k\ |\nu_n|(H^q_{k^q}) \geq 0.9 \text{ and } G^q_{k^q} \cap H^q_{k^q} = \varnothing\}$$

are dense below p for every $k \in \mathbb{N}$.

First, we will show that $\mathbb{D}_k$ is dense below p. Pick any $r \leq p$. By Lemma 4.5.3 we may assume that $k^r \geq k$.

We apply Lemma 4.4.5 to $\mathcal{M} = \mathcal{M}^r, \widehat{G} = \bigcup_{i\leq k^r} G^r_i, \widehat{H} = \bigcup_{i\leq k^r} H^r_i$ and $d = k$ to obtain $m = m_{k^r+1}$, $a > k$, G', H_0 so that the conditions (1)-(8) from Lemma 4.4.5 are satisfied. In particular:

(a) $m > m_{k^r}$,
(b) $G' \cap G^r_i = H_0 \cap H^r_i = G' \cap H^r_i = \varnothing$ for $i \leq k^r$,
(c) $\mu\left(\bigcup_{i\leq k^r}(G^r_i \cup H^r_i) \cup G' \cup H_0\right) < 0.1$ for all $\mu \in \mathcal{M}^r$,
(d) $\mathcal{F}\left(\mathbb{B}_n, \bigcup_{i\leq k^r} G^r_i \cup G'\right)$ is $(m_n, 2^{-n})$–balanced for $n \leq k^r + 1$,
(e) $|\nu_a|(H_0) \geq 0.9$,
(f) $|\nu_a(G' \cap H_0)| \geq 0.3$.

It follows from (a)-(d) that

$$q = (k^r + 1, ((m^r_i)_{i\leq k^r}, m), ((G^r_i)_{i\leq k^r}, G'), ((H^r_i)_{i\leq k^r}, H_0), \mathcal{M}^r) \in \mathbb{P}_{\mathbb{B}}$$

and from (e), (f) that $q \in \mathbb{D}_k$.

We show the density of $\mathbb{E}_k$ in a similar way: the difference is that instead of H_0 we pick H_1 such that

- $|\nu_a|(H_1) \geq 0.9$,
- $G' \cap H_1 = \varnothing$.

□

Directly from Lemma 4.5.5, Lemma 4.5.6 and Definition 4.2.14 we obtain

Proposition 4.5.7. *Let* $\mathbb{B} \subseteq \mathrm{Bor}(C)$ *be a balanced Boolean algebra that contains* $\mathrm{Clop}(C)$. *Let* $p \in \mathbb{P}_{\mathbb{B}}$ *and* $\nu = (\nu_n)_{n\in\mathbb{N}}$ *be a normal sequence of measures on* $\mathbb{B}$ *such that the sequence* $(|\nu_n|)_{n\in\mathbb{N}}$ *is pointwise convergent to a measure* ν_∞. *Suppose that* $\nu_\infty \in \mathcal{M}^p$. *Then*

$$p \Vdash (\check{\mathbb{B}}, \dot{\mathbb{B}}', \check{\nu}) \text{ satisfies } (\mathcal{G}^*).$$

Proposition 4.5.8. *Suppose that* $\mathbb{B} \subseteq \mathrm{Bor}(C)$ *is a balanced countable Boolean algebra containing* $\mathrm{Clop}(C)$. *Then*

$$\mathbb{P}_{\mathbb{B}} \Vdash \dot{\mathbb{B}}' \text{ is balanced.}$$

Proof. Since the property of being balanced is absolute between transitive models of ZFC

$$V^{\mathbb{P}_{\mathbb{B}}} \models \mathbb{B} \text{ is balanced.}$$

In $V^{\mathbb{P}_{\mathbb{B}}}$ for every $n \in \mathbb{N}$ we define $G_n = G_n^p$ for some $p \in \mathbb{G}$ such that $k^p \geq n$. Then $\mathbb{B}'$ is the Boolean algebra generated by $\mathbb{B} \cup \{G\}$, where $G = \bigcup_{n\in\mathbb{N}} G_n$. By Definition 4.5.1(4a,4c) the hypothesis of Lemma 4.3.7 is satisfied and so $\mathbb{B}'$ is balanced. □

Definition 4.5.9. We define an iteration $(\mathbb{P}_\alpha)_{\alpha\leq\omega_1}$ with finite supports and $\mathbb{P}_\alpha$-names $\dot{\mathbb{B}}_\alpha$ for every $\alpha \leq \omega_1$ by induction in the following way:

- $\mathbb{P}_0$ is the trivial forcing and $\mathbb{B}_0 = \mathrm{Clop}(C)$,
- having constructed $\mathbb{P}_\alpha$ and $\dot{\mathbb{B}}_\alpha$ we define

  $$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{P}_{\dot{\mathbb{B}}_\alpha}$$

  and we pick a $\mathbb{P}_{\alpha+1}$-name $\dot{\mathbb{B}}_{\alpha+1}$ such that

  $$\mathbb{P}_{\alpha+1} \Vdash \dot{\mathbb{B}}_{\alpha+1} = \dot{\mathbb{B}}'_\alpha,$$

- if γ is a limit ordinal, then we define $\mathbb{P}_\gamma$ as the iteration of $(\mathbb{P}_\alpha)_{\alpha<\gamma}$ with finite supports and we pick $\dot{\mathbb{B}}_\gamma$ so that

  $$\mathbb{P}_\gamma \Vdash \dot{\mathbb{B}}_\gamma = \bigcup_{\alpha<\gamma} \dot{\mathbb{B}}_\alpha$$

We will identify each $\mathbb{P}_\alpha$ with the subset of $\mathbb{P}_{\omega_1}$ consisting of those $p \in \mathbb{P}_{\omega_1}$, for which $p(\beta) = \mathbb{1}_\beta$ for all $\beta \geq \alpha$, where $\mathbb{1}_\beta$ denotes the maximal element of $\mathbb{P}_{\dot{\mathbb{B}}_\beta}$.

Lemma 4.5.10. *$\mathbb{P}_{\omega_1}$ is σ-centered. In particular, $\mathbb{P}_{\omega_1}$ satisfies c.c.c.*

Proof. This follows from Lemma 4.5.2 and the fact that a finite support iteration of length ω_1 of σ-centered forcings is σ-centered [129, proof of Lemma 2]. $\square$

Since $\mathbb{P}_{\omega_1}$ satisfies c.c.c. the standard closure argument shows the following (cf. [40, Lemma 5.3])

Lemma 4.5.11. *Let $\dot{\nu} = (\dot{\nu}_n)_{n\in\mathbb{N}}$ be a sequence such that*

$$\mathbb{P}_{\omega_1} \Vdash (\dot{\nu}_n)_{n\in\mathbb{N}} \textit{ is a sequence of measures on } \dot{\mathbb{B}}_{\omega_1}.$$

Let

$$C_{\dot{\nu}} = \{\alpha < \omega_1 : \mathbb{P}_{\omega_1} \Vdash \dot{\nu} \restriction \dot{\mathbb{B}}_\alpha \in V^{\mathbb{P}_\alpha}\}.$$

Then $C_{\dot{\nu}}$ is a closed and unbounded subset of ω_1.

Proposition 4.5.12. *$\mathbb{P}_{\omega_1} \Vdash \dot{\mathbb{B}}_{\omega_1}$ is balanced and satisfies $(\mathcal{G})$.*

Proof. The fact that $\mathbb{P}_{\omega_1} \Vdash \dot{\mathbb{B}}_{\omega_1}$ is balanced follows directly from Proposition 4.5.8 and the fact that increasing unions of balanced Boolean algebras are balanced.

To prove the second part of the proposition, by Proposition 4.2.15 it is enough to show that for every sequence $(\dot{\nu}_n)_{n\in\mathbb{N}}$ such that

$$\mathbb{P}_{\omega_1} \Vdash (\dot{\nu}_n)_{n\in\mathbb{N}} \text{ is a normal sequence of measures on } \dot{\mathbb{B}}_{\omega_1}$$

we have

$$\mathbb{P}_{\omega_1} \Vdash \exists_{\alpha<\omega_1}\ (\dot{\mathbb{B}}_\alpha, \dot{\mathbb{B}}_{\omega_1}, \dot{\nu} \restriction \dot{\mathbb{B}}_\alpha) \text{ satisfies } (\mathcal{G}^*).$$

Pick any $p \in \mathbb{P}_{\omega_1}$. By Lemma 4.4.7 there is $\alpha_1 < \omega_1$ and $p_1 \leq p$ such that for every $\beta \geq \alpha_1$

$$p_1 \Vdash (\dot{\nu}_n \restriction \dot{\mathbb{B}}_\beta)_{n\in\mathbb{N}} \text{ is normal.}$$

By Lemma 4.5.11 there are: $\beta \in C_{\dot{\nu}}$ such that $\alpha_1 < \beta$, $p_1 \in V^{\mathbb{P}_\beta}$ and $\mathbb{P}_\beta$-names $\dot{\nu}^\beta, \dot{\nu}^\beta_n$ for $n \in \mathbb{N}$ such that

$$\mathbb{P}_\beta \Vdash \dot{\nu}^\beta = (\dot{\nu}^\beta_n)_{n\in\mathbb{N}} = (\dot{\nu}_n \restriction \dot{\mathbb{B}}_\beta)_{n\in\mathbb{N}}.$$

Without loss of generality by passing to a subsequence we may assume that there is $\dot{\nu}_\infty$ such that

$$\mathbb{P}_\beta \Vdash (|\dot{\nu}^\beta_n|)_{n\in\mathbb{N}} \text{ is pointwise convergent to a measure } \dot{\nu}_\infty.$$

Since $p_1(\beta) = \mathbb{1}_\beta$, there is $p_2 \leq p_1$ such that

$$p_2 \restriction \beta \Vdash \dot{\nu}_\infty \in \dot{\mathcal{M}}^{\check{p}_2(\beta)}.$$

By Proposition 4.5.7 we have

$$p_2 \Vdash (\dot{\mathbb{B}}_\beta, \dot{\mathbb{B}}_{\beta+1}, \dot{\nu}^\beta) \text{ satisfies } (\mathcal{G}^*)$$

and hence

$$p_2 \Vdash \exists_{\alpha<\omega_1}\ (\dot{\mathbb{B}}_\alpha, \dot{\mathbb{B}}_{\omega_1}, \dot{\nu} \restriction \dot{\mathbb{B}}_\alpha) \text{ satisfies } (\mathcal{G}^*)$$

which completes the proof. $\square$

In particular, by Proposition 4.2.8 and Proposition 4.2.13 we obtain

Corollary 4.5.13. $\mathbb{P}_{\omega_1} \Vdash$ *there is a Boolean algebra of size* ω_1 *with the Grothendieck property but without the Nikodym property.*

The existence of a Boolean algebra with the Grothendieck property of small cardinality has influence on certain cardinal characteristics of the continuum. Below $\mathfrak{p}$ denotes the pseudointersection number, $\mathfrak{s}$ is the splitting number and $\mathrm{cov}(\mathcal{M})$ is the covering number of the ideal of meager sets in $\mathbb{R}$.

Corollary 4.5.14. $\mathbb{P}_{\omega_1} \Vdash \mathfrak{p} = \mathfrak{s} = \mathrm{cov}(\mathcal{M}) = \omega_1$.

Proof. Apply [122, Corollary 4.3]. ☐

Theorem 4.5.15. *It is consistent with* $\neg\mathsf{CH}$ *that there is a Boolean algebra of size* ω_1 *with the Grothendieck property but without the Nikodym property.*

Proof. Start with a model V of ZFC satisfying $\neg\mathsf{CH}$. Since $\mathbb{P}_{\omega_1}$ is σ-centered, it preserves cardinals and the value of the continuum, so we have

$$V^{\mathbb{P}_{\omega_1}} \models \mathfrak{c} = \mathfrak{c}^V > \omega_1^V = \omega_1.$$

By Corollary 4.5.13 in $V^{\mathbb{P}_{\omega_1}}$ there is a Boolean algebra with the Grothendieck property, but without the Nikodym property. ☐

4.6 Final remarks

Let us start with a comment concerning differences between the original Talagrand's contruction and our approach. To obtain the Grothendieck property, Talagrand uses CH to enumerate (in a sequence of length ω_1) all normalized sequences $(\nu_n)_{n\in\mathbb{N}}$ of measures on countable subalgebras of Borel subsets of the Cantor set, for which there exists an antichain $(H_n)_{n\in\mathbb{N}}$ such that $|\nu_n|(H_n) \geq 0.95$. Then for each such sequence he constructs another antichain $(G_n)_{n\in\mathbb{N}}$ satisfying the hypothesis of Lemma 4.3.7, such that for $G = \bigcup_{n\in\mathbb{N}} G_n$ we have

- $|\nu_n(G \cap H_n)| \geq 0.4$ for infinitely many $n \in \mathbb{N}$,
- $|\nu_n(G \cap H_n)| < 0.1$ for infinitely many $n \in \mathbb{N}$.

It follows that extending a given Boolean algebra with G keeps it balanced, and in the extension the sequence $(\nu_n)_{n\in\mathbb{N}}$ satisfies a property similar to $(\mathcal{G}^*)$ from Definition 4.2.14. Thus, the final algebra has the Grothendieck property and does not have the Nikodym property. The same approach was used in [125].

However, this technique applies only when we work with one sequence of measures at a time. In our method, we construct a suitable antichain $(H_n)_{n\in\mathbb{N}}$ along with $(G_n)_{n\in\mathbb{N}}$, which allows us to pick both the antichains in a generic way, making them working for uncountably many sequences of measures simultaneously.

The method of construction we have described relies strongly on the fact that the Boolean algebras we extend are countable.

Question 4.6.1. *Let $\mathbb{B} \subseteq \mathrm{Bor}(C)$ be a balanced Boolean algebra of cardinality $< \mathfrak{c}$ and ν be a normal sequence of measures on $\mathbb{B}$.*

Does there exist a balanced Boolean algebra $\mathbb{B} \subseteq \mathbb{B}' \subseteq \mathrm{Bor}(C)$ such that $(\mathbb{B}, \mathbb{B}', \nu)$ satisfies $(\mathcal{G}^*)$?

Positive answer for the above question would allow us to construct (by induction of length $\mathfrak{c}$) a balanced Boolean algebra of size $\mathfrak{c}$. Thus, it would imply the positive answer for the following question:

Question 4.6.2. *Is there (in* ZFC*) a balanced Boolean algebra with the Grothendieck property?*

One may look for candidates for Boolean algebras with the Grothendieck and without Nikodym property among maximal balanced Boolean algebras.

Question 4.6.3. *Let $\mathbb{B} \subseteq \mathrm{Bor}(C)$ be a maximal balanced Boolean algebra. Does $\mathbb{B}$ have the Grothendieck property?*

Chapter 5

The Calkin algebra in the Cohen model

5.1 Introduction

By the classical Parovičenko theorem CH implies that every compact space of cardinality at most $\mathfrak{c}$ is a continuous image of the remainder $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ of the Stone-Čech compactification of the natural numbers. It follows that every space of type $C(K)$ of density at most $\mathfrak{c}$ embeds into $\ell_\infty/c_0 \equiv C(\mathbb{N}^*)$ as a C*-algebra. It is also well-known that under CH the Banach space ℓ_∞/c_0 is isometrically universal in the class of Banach spaces of density at most $\mathfrak{c}$. On the other hand, due to a result of Brech and Koszmider [20], none of the above statements holds in the Cohen model[1].

For a separable Hilbert space H we denote by $\mathcal{B}(H)$ the C*-algebra of all bounded operators on H and by $\mathcal{K}(H)$ the ideal of $\mathcal{B}(H)$ consisting of compact operators. The quotient $\mathcal{Q}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is called the Calkin algebra and is considered to be the non-commutative analogue of $\mathbb{N}^*$ and ℓ_∞/c_0 (see [42, 43, 136]). In this chapter we will focus on possible embeddings of C*-algebras of density $\mathfrak{c}$ into $\mathcal{Q}(\ell_2)$.

It is well-known that $\mathcal{Q}(\ell_2)$ contains a *-isomorphic copy of every separable C*-algebra. Under MA every C*-algebra of density strictly smaller than $\mathfrak{c}$ embeds into $\mathcal{Q}(\ell_2)$ [47, Corollary C]. In [46] the authors proved that assuming CH the C*-algebra $\mathcal{Q}(\ell_2)$ is $\mathfrak{c}$-universal (i.e. the density of $\mathcal{Q}(\ell_2)$ equals $\mathfrak{c}$ and every C*-algebra of density at most $\mathfrak{c}$ embeds into $\mathcal{Q}(\ell_2)$). Vaccaro showed that if we assume OCA, then the class of C*-algebras that embed into $\mathcal{Q}(\ell_2)$ is not closed under tensor products [133, Theorem 1.2], which implies that $\mathcal{Q}(\ell_2)$ is not $\mathfrak{c}$-universal. In the Cohen model, by another result of Vaccaro [131, Corollary 2.5.5], in $\mathcal{Q}(\ell_2)$ there is no well-ordered strictly increasing sequence of projections of length ω_2. Hence the abelian algebra generated by such a sequence of projections does not embed into $\mathcal{Q}(\ell_2)$, though its density equals $\omega_2 \leq \mathfrak{c}$. It follows, that $\mathcal{Q}(\ell_2)$ is not $\mathfrak{c}$-universal in the Cohen model. Among other results, it is worth mentioning that PFA implies that there exists a $\mathfrak{c}$-universal C*-algebra, while

[1] By the Cohen model we mean a model obtained from a model of CH by adding ω_2 Cohen reals.

$\mathcal{Q}(\ell_2)$ is not $\mathfrak{c}$-universal, and it is consistent that there is no $\mathfrak{c}$-universal C*-algebra [46, Corollary 3.2].

We will focus on the ℓ_∞-sum of infinitely many copies of the Calkin algebra i.e. the algebra $\ell_\infty(\mathcal{Q}(\ell_2))$ consisting of bounded (in norm) sequences of elements of $\mathcal{Q}(\ell_2)$ with pointwise multiplication and the norm given by

$$\|(x_n)_{n\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}} \|x_n\|.$$

Our main result consists of a proof that in the Cohen model $\ell_\infty(\mathcal{Q}(\ell_2))$ is not isomorphic to a *-subalgebra of $\mathcal{Q}(\ell_2)$ (which in particular gives another proof that $\mathcal{Q}(\ell_2)$ is not $\mathfrak{c}$-universal in the Cohen model). Note that this is not covered by the mentioned result of Vaccaro, since every strictly increasing well-ordered sequence of length ω_2 of projections in $\ell_\infty(\mathcal{Q}(\ell_2))$ induces a strictly increasing sequence of projections in $\mathcal{Q}(\ell_2)$ of length ω_2 (namely, if $(p^\alpha)_{\alpha<\omega_2}$ is a strictly increasing sequence of projections in $\ell_\infty(\mathcal{Q}(\ell_2))$, then $p^\alpha = (p^\alpha_n)_{n\in\mathbb{N}}$, where $(p^\alpha_n)_{\alpha<\omega_2}$ is an increasing sequence of projections in $\mathcal{Q}(\ell_2)$ for $n \in \mathbb{N}$, and there is $n \in \mathbb{N}$ such that $(p^\alpha_n)_{\alpha<\omega_2}$ has ω_2 distinct values).

Our result may be seen as a non-commutative version of a theorem of [20], which says that in the Cohen model $\ell_\infty(\ell_\infty/c_0)$ cannot be included isomorphically into ℓ_∞/c_0 as a Banach space. On the other hand, CH implies that the Banach spaces $\ell_\infty(\ell_\infty/c_0)$ and ℓ_∞/c_0 are isomorphic (this fact was proved by Drewnowski and Roberts and used to obtain the primariness of the space ℓ_∞/c_0 under CH [37, Thoeorem 3.3]). The construction of this isomorphism strongly relies on the result of Negrepontis that assuming CH the closure of non-empty open F_σ subset of $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ is a retract of $\mathbb{N}^*$ [97, Corollary 3.2]. It is not clear whether a non-commutative version of the Negrepontis theorem holds under CH.

In [44] Farah compares the Calkin algebra with some coronas of the form $\mathcal{Q}(\mathcal{A}\otimes\mathcal{K}(\ell_2))$. In particular, he shows that $\mathcal{Q}(\ell_2)$ is not isomorphic to the corona of the stabilization of the Cuntz algebra i.e. $\mathcal{Q}(\mathcal{O}_\infty \otimes \mathcal{K}(\ell_2))$, though these algebras are not distinguishable from the K-theoretical point of view. The problem of the existence of such isomorphisms is important, since they may induce K-theory reversing automorphisms of $\mathcal{Q}(\ell_2)$ (such automorphisms are not known to exist consistently). We apply the result about the non-existence of an embedding of $\ell_\infty(\mathcal{Q}(\ell_2))$ into $\mathcal{Q}(\ell_2)$ to show that in the Cohen model $\mathcal{Q}(\ell_2)$ is not isomorphic to $\mathcal{Q}(\mathcal{Q}(\ell_2)\otimes\mathcal{K}(\ell_2))$. This should be compared with the fact that the Banach spaces $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$ and ℓ_∞/c_0 are not isomorphic in the Cohen model [20], while they are isomorphic, if we assume CH [37, Proposition 4.4]. It is not clear, whether this result does not follow from other known facts concerning rigidity of $\mathcal{Q}(\ell_2)$ (cf. [45, 132, 135]).

5.2 Preliminaries

5.2.1 Operators on separable Hilbert spaces

Throughout the chapter by ℓ_2 we mean the separable complex Hilbert space consisting of square-summable sequences $(a_n)_{n\in\mathbb{N}}$ of complex numbers with the standard inner product $\langle (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\rangle = \sum_{n=1}^{\infty} a_n\overline{b}_n$.

Recall that an element p of a C*-algebra $\mathcal{A}$ is a projection, if $p = p^2 = p^*$. If $P \in \mathcal{B}(\ell_2)$, then P is a projection if and only if it is an orthogonal projection onto a subspace of ℓ_2. On the set of all projections $\mathrm{Proj}(\mathcal{A})$ on $\mathcal{A}$ we introduce the ordering given by $p \le q$ if and only if $pq = p$. For $P, R \in \mathrm{Proj}(\mathcal{B}(\ell_2))$ we denote $P \le^{\mathcal{K}} Q$, if $PQ - P \in \mathcal{K}(\ell_2)$. Note that $P \le^{\mathcal{K}} Q$ if and only if $\pi(P) \le \pi(Q)$, where $\pi\colon \mathcal{B}(\ell_2) \to \mathcal{Q}(\ell_2)$ is the canonical quotient map.

Definition 5.2.1. Given an orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ in ℓ_2 we denote by $E^{(n)}$ the projection onto the closed subspace of ℓ_2 spanned by $(e_i)_{i\ge n}$.

The following lemma follows from the proof of [23, Theorem II.4.4].

Lemma 5.2.2. *Let $K \in \mathcal{K}(\ell_2)$. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of ℓ_2. Then*

$$\lim_{n\to\infty} \|KE^{(n)}\| = \lim_{n\to\infty} \|E^{(n)}K\| = 0.$$

Lemma 5.2.3. *Let $K \in \mathcal{K}(\ell_2)$ and $\varepsilon > 0$. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of ℓ_2. Suppose $v_n \in \mathcal{B}(\ell_2)$ for $n \in \mathbb{N}$ are such that*

$$\|v_n\| \in [1, 1+\varepsilon) \text{ and } \|v_n - E^{(n)}(v_n)\| < \frac{1}{n}.$$

Then $\|K(v_n)\| < \varepsilon$ for large enough $n \in \mathbb{N}$.

Proof. By Lemma 5.2.2 we have $\lim_{n\to\infty} \|KE^{(n)}n\| = 0$. Hence

$$\|K(v_n)\| = \|K(v_n - E^{(n)}(v_n)) + KE^{(n)}(v_n)\| \le \frac{1}{n}\|K\| + (1+\varepsilon)\|KE^{(n)}\| \xrightarrow{n\to\infty} 0.$$

□

5.2.2 Embedding $c_0(\mathfrak{c})$ and $\ell_\infty(c_0(\mathfrak{c}))$

For a set A by $c_0(A)$ we mean the algebra of (complex) sequences on A converging to 0 (i.e. $(a_\alpha)_{\alpha\in A} \in c_0(A)$, if for every $\varepsilon > 0$ the set $\{\alpha \in A : |a_\alpha| < \varepsilon\}$ is finite) with pointwise multiplication and the supremum norm. We put $(a_\alpha)^*_{\alpha\in A} = (\overline{a}_\alpha)_{\alpha\in A}$. If $\mathcal{A}$ is a C*-algebra, then $\ell_\infty(\mathcal{A})$ denotes the C*-algebra of bounded (in norm) sequences of elements of $\mathcal{A}$ and $c_0(\mathcal{A})$ denotes the subalgebra of $\ell_\infty(\mathcal{A})$ of sequences converging to 0 in norm.

We will need two lemmas on embedding $c_0(\mathfrak{c})$ and $\ell_\infty(c_0(\mathfrak{c}))$. Suitable embeddings are described in [20, proofs of implications $(b) \to (c)$ and $(c) \to (e)$] (those embeddings are considered in the category of Banach spaces, but the same arguments give *-embeddings).

Lemma 5.2.4. *There is a *-embedding of $c_0(\mathfrak{c})$ into ℓ_∞/c_0.*

Lemma 5.2.5. *There is a *-embedding of $\ell_\infty(c_0(\mathfrak{c}))$ into $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$.*

5.2.3 Set theory

In this chapter V will denote a universe of sets satisfying CH and $\mathbb{P}$ will denote the Cohen forcing adding ω_2 reals i.e.

$$\mathbb{P} = \{p\colon \mathrm{dom}(p) \to \{0,1\} : \mathrm{dom}(p) \in [\omega_2]^{<\omega}\}$$

with the ordering given by $q \leq p$ if and only if $q \supseteq p$.

Definition 5.2.6. Let $\sigma\colon \omega_2 \to \omega_2$ be a permutation. We define the **lifting** of σ as the automorphism $\sigma\colon \mathbb{P} \to \mathbb{P}$ given by $\sigma(p)(\sigma(x)) = p(x)$ for $p \in \mathbb{P}, x \in \mathrm{dom}(p)$. If $\dot{x} = \{(\dot{y}_i, p_i) : i \in I\}$ is a $\mathbb{P}$-name, then we denote $\sigma(\dot{x}) = \{(\sigma(\dot{y}_i), \sigma(p_i)) : i \in I\}$ (cf. [69, p. 221]).

Note that if $\check{x}$ is the canonical name for $x \in V$, then for any permutation $\sigma\colon \omega_2 \to \omega_2$ we have $\sigma(\check{x}) = \check{x}$.

Definition 5.2.7. Let $\sigma\colon \omega_2 \to \omega_2$ be a permutation such that $\sigma[S_1] = S_2$, where $S_1, S_2 \subseteq \omega_2$. We say that $p \in \mathbb{P}$ is (σ, S_1, S_2)**-symmetric**, if $\sigma(p \restriction S_1) = p \restriction S_2$.

Lemma 5.2.8. *Let $S_1, S_2 \subseteq \omega_2$ and suppose $\sigma\colon \omega_2 \to \omega_2$ is a permutation such that $\sigma[S_1] = S_2$ and $\sigma \restriction S_1 \cap S_2 = Id$. Suppose $p \in \mathbb{P}$ is (σ, S_1, S_2)-symmetric and $q \leq p$ is such that $(dom(q) \backslash dom(p)) \cap S_2 \subseteq S_1$. Let*

$$r = q \cup \sigma(q \restriction S_1).$$

Then $r \in \mathbb{P}$ is a (σ, S_1, S_2)-symmetric condition.

Proof. First, note that $r \in \mathbb{P}$ since if $\alpha \in \mathrm{dom}(q) \cap \mathrm{dom}(\sigma(q \restriction S_1))$, then either $\alpha \in S_1 \cap S_2$ (and $r(\alpha)$ is well-defined by the hypothesis that $\sigma \restriction S_1 \cap S_2 = Id$) or $\alpha \in \mathrm{dom}(p) \cap S_2$, and so $r(\alpha)$ is well-defined by the symmetry of p.

Now we will show that r is (σ, S_1, S_2)-symmetric. It is clear from the definition that $\mathrm{supp}(\sigma(r \restriction S_1)) = \mathrm{supp}(r \restriction S_2)$. Fix $\alpha \in \mathrm{dom}(r) \cap S_1$. If $\alpha \in \mathrm{dom}(p)$, then $\sigma(r)(\alpha) = r(\sigma(\alpha))$ by the symmetry of p. If $\alpha \in \mathrm{dom}(q) \backslash \mathrm{dom}(p)$, then we have $\sigma(r)(\alpha) = r(\sigma(\alpha))$ from the equality defining r. $\square$

Let us recall the definition of a nice name.

Definition 5.2.9. A $\mathbb{P}$-name $\dot{X}$ is a **nice name** for a subset of $M \in V$, if it is of the form $\dot{X} = \bigcup_{m \in M} \{\check{m}\} \times A_m$, where each A_m is an antichain in $\mathbb{P}$. The set $\bigcup_{m \in M} \bigcup_{p \in A_m} \mathrm{dom}(p)$ is called the **support** of $\dot{X}$ and is denoted by $\mathrm{supp}(\dot{X})$.

Note that since $\mathbb{P}$ satisfies c.c.c. every nice name for a subset of $\mathbb{N}$ has a countable support.

Remark 5.2.10. *If $\dot{h}$ is a nice name such that*

$$\mathbb{P} \Vdash \dot{h}\colon \mathbb{N} \to \mathbb{N} \text{ is a function},$$

then for every $p \in \mathbb{P}$ *and* $n \in \mathbb{N}$ *there is* $q \leq p$ *and* $m \in \mathbb{N}$ *such that*

$$q \Vdash \dot{h}(n) = m$$

and $dom(q) \setminus dom(p) \subseteq \operatorname{supp}(\dot{h})$.

We will need some basic facts concerning Δ-systems.

Definition 5.2.11. We say that a family of sets $\mathcal{A}$ is a **Δ-system** with **root** Δ, if for every $A, B \in \mathcal{A}$ we have $A \cap B = \Delta$ whenever $A \neq B$.

Lemma 5.2.12. *[69, Theorem 9.19] Assume* CH. *Then for every family of countable sets* $\mathcal{A}$ *of cardinality* ω_2 *there is a* Δ*-system* $\mathcal{B} \subseteq \mathcal{A}$ *with* $|\mathcal{B}| = \omega_2$.

Lemma 5.2.13. *Assume* CH. *Suppose* $(S_\alpha)_{\alpha<\omega_2}$ *is a sequence of pairwise disjoint countable subsets of* ω_2 *and* $(R_\alpha)_{\alpha<\omega_2}$ *is a sequence of countable subsets of* ω_2. *Then there are* $\xi, \eta < \omega_2$ *such that* $\xi \neq \eta$ *and*

$$S_\xi \cap R_\eta = S_\eta \cap R_\xi = \varnothing.$$

Proof. By Lemma 5.2.12 there is $A \in [\omega_2]^{\omega_2}$ such that $(R_\xi)_{\xi \in A}$ is a Δ-system with root Δ. The set Δ is countable, so there is $B \in [A]^{\omega_2}$ such that $S_\xi \cap \Delta = \varnothing$ for $\xi \in B$. Pick any $\xi \in B$. Since R_ξ is countable, there is $C \in [B]^{\omega_2}$ such that $S_\eta \cap R_\xi = \varnothing$ for $\eta \in C$. Since S_ξ is countable and the sets $(R_\eta \setminus \Delta)_{\eta \in C}$ are pairwise disjoint, we can pick $\eta \in C \setminus \{\xi\}$ such that $S_\xi \cap (R_\eta \setminus \Delta) = \varnothing$. It follows that $S_\xi \cap R_\eta = \varnothing$. □

Lemma 5.2.14. *Suppose* $A_n \in [\omega_2]^{\omega_2}, S_{n,\alpha} \in [\omega_2]^\omega$ *for* $n \in \mathbb{N}, \alpha < \omega_2$ *are such that for every* $n \in \mathbb{N}$ *the family* $(S_{n,\alpha})_{\alpha \in A_n}$ *is a* Δ*-system with root* Δ_n. *Assume that for each* $\alpha \in A_n$ *we have* $\Delta \cap S_{n,\alpha} = \Delta_n$, *where* $\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n$. *Then for every* $\xi < \omega_2$ *there is* $\gamma_\xi \in \omega_2^{\mathbb{N}}$ *such that*

(a) $\gamma_\xi(n) \in A_n$ *for* $n \in \mathbb{N}$,

(b) for distinct $(\xi, n), (\eta, m) \in \omega_2 \times \mathbb{N}$ *we have*

$$(S_{n,\gamma_\xi(n)} \setminus \Delta_n) \cap (S_{m,\gamma_\eta(m)} \setminus \Delta_m) = \varnothing,$$

(c) $\gamma_\xi \cap \gamma_\eta = \varnothing$ *for* $\xi, \eta < \omega_2, \xi \neq \eta$.

Proof. We construct $(\gamma_\xi)_{\xi<\omega_2}$ by induction on $\xi < \omega_2$ and $n \in \mathbb{N}$. Fix $\xi < \omega_2$ and $n \in \mathbb{N}$. Suppose we have constructed γ_η for $\eta < \xi$ and $\gamma_\xi(m)$ for $m < n$. Put $\delta = \sup_{\eta<\xi, n \in \mathbb{N}} \gamma_\eta(n)$ and observe that $|A_n \setminus (\delta + 1)| = \omega_2$. The set

$$B = \bigcup_{(\eta,m) <_{lex} (\xi,n)} S_{m,\gamma_\eta(m)} \setminus \Delta$$

has cardinality at most ω_1 and the family $(S_{n,\alpha} \setminus \Delta)_{\alpha<\omega_2}$ consists of non-empty pairwise disjoint sets, so there is $\beta \in A_n \setminus (\delta + 1)$ such that $S_{n,\beta} \cap B = \varnothing$. We put $\gamma_\xi(n) = \beta$. It follows directly from the construction that the conditions (a)-(c) are satisfied. □

5.3 Embedding ℓ_∞-sums into the Calkin algebra

Every sequence of vectors in ℓ_2 is a sequence of sequences of complex numbers, and every complex number is a pair of real numbers. Thus, by the standard argument we may choose bijections defined in an absolute way, that identify the sets $\ell_2^{\mathbb{N}} \sim (\mathbb{C}^{\mathbb{N}})^{\mathbb{N}} \sim \mathbb{R}^{\mathbb{N}} \sim \mathbb{N}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$. We will use such identification in the context of names for sequences of vectors - in particular, under such identification, every $v \in (\ell_2)^{\mathbb{N}}$ in $V^{\mathbb{P}}$ is named by a nice $\mathbb{P}$-name $\dot{v}$ in V. Since $\mathbb{P}$ is c.c.c. the support $\operatorname{supp}(\dot{v})$ is countable.

Lemma 5.3.1. *Assume* CH. *Suppose $A \in [\omega_2]^{\omega_2}$, $\{S_\alpha\}_{\alpha\in A}$ is a Δ-system of distinct countable sets with root Δ and for every $\alpha < \omega_2$ we are given a nice $\mathbb{P}$-name $\dot{v}_\alpha$ for a sequence of vectors in ℓ_2 such that $\operatorname{supp}(\dot{v}_\alpha) \subseteq S_\alpha$. Then there is $C \in [A]^{\omega_2}$ and permutations $\sigma_{\alpha,\beta}\colon \omega_2 \to \omega_2$ such that*

- $\sigma_{\alpha,\beta} = \sigma^{-1}_{\beta,\alpha}$
- $\sigma_{\alpha,\beta}[S_\alpha] = S_\beta$,
- $\sigma_{\alpha,\beta} \restriction \Delta = Id$,
- $\sigma_{\alpha,\beta}(\dot{v}_\alpha) = \dot{v}_\beta$

for all $\alpha, \beta \in C, \alpha \neq \beta$.

Proof. Without loss of generality we may assume that each S_α is infinite. For each $\alpha \in A$ choose a permutation $\sigma_\alpha\colon \omega_2 \to \omega_2$ such that $\sigma_\alpha[S_\alpha] = \mathbb{N}$ and $\sigma_\alpha \restriction \Delta = \sigma_\beta \restriction \Delta$ for all $\alpha, \beta \in A$. Then for every $\alpha \in \mathbb{N}$ the name $\sigma_\alpha(\dot{v}_\alpha)$ is a nice $\mathbb{P}$-name with the support included in $\mathbb{N}$. By CH there is only ω_1 such names, so there is $C \in [A]^{\omega_2}$ such that $\sigma_\alpha(\dot{v}_\alpha) = \sigma_\beta(\dot{v}_\beta)$ for $\alpha, \beta \in C$. For $\alpha, \beta \in C, \alpha < \beta$ let $\sigma_{\alpha,\beta} = \sigma_\beta^{-1}\sigma_\alpha$. Then $\sigma_{\alpha,\beta}(\dot{v}_\alpha) = \sigma_\beta^{-1}\sigma_\alpha(\dot{v}_\alpha) = \sigma_\beta^{-1}\sigma_\beta(\dot{v}_\beta) = \dot{v}_\beta$ for $\alpha, \beta \in C$. Since $\sigma_\alpha \restriction \Delta = \sigma_\beta \restriction \Delta$ we have $\sigma_{\alpha,\beta} \restriction \Delta = Id$. If $\alpha > \beta$, then we define $\sigma_{\alpha,\beta} = \sigma^{-1}_{\beta,\alpha}$. ☐

Proposition 5.3.2. *Suppose in the Cohen model $V^{\mathbb{P}}$ we are given non-compact projections $(E_{n,\alpha} : (n,\alpha) \in \mathbb{N}\times\omega_2)$ and $(B_\gamma : \gamma \in \omega_2^{\mathbb{N}})$ in $\mathcal{B}(\ell_2)$ such that if $\gamma(n) = \alpha$, then $E_{n,\alpha} \leq^{\mathcal{K}} B_\gamma$. Then there are disjoint $\gamma_1, \gamma_2 \in \omega_2^{\mathbb{N}}$ such that $B_{\gamma_1}B_{\gamma_2}$ is non-compact.*

Proof. For simplicity of the notation put $\varepsilon = 1/100$. Denote by F the subset of ℓ_2 consisting of all sequences of the form $(a_n + ib_n)_{n\in\mathbb{N}}$, where $a_n, b_n \in \mathbb{Q}$ are non-zero only for finitely many $n \in \mathbb{N}$. Observe that F is a countable dense subset of ℓ_2.

In V, for $(n,\alpha) \in \mathbb{N}\times\omega_2$ and $\gamma \in \omega_2^{\mathbb{N}}$ let $\dot{E}_{n,\alpha}, \dot{B}_\gamma$ be $\mathbb{P}$-names such that

- $\mathbb{P} \Vdash \dot{E}_{n,\alpha}$ and $\dot{B}_\gamma$ are non-compact projections in $\mathcal{B}(\ell_2)$,
- if $\gamma(n) = \alpha$, then $\mathbb{P} \Vdash \dot{E}_{n,\alpha} \leq^{\mathcal{K}} \dot{B}_\gamma$.

For $(n,\alpha) \in \mathbb{N}\times\omega_2$ let $\dot{e}_{n,\alpha}$ be a nice $\mathbb{P}$-name for a countable sequence of vectors from ℓ_2 and for $l \in \mathbb{N}$ let $\dot{E}^{(l)}_{n,\alpha}$ be $\mathbb{P}$-names such that

- $\mathbb{P} \Vdash \dot{e}_{n,\alpha}$ is an orthonormal basis of $\dot{E}_{n,\alpha}[\ell_2]$,
- $\mathbb{P} \Vdash \dot{E}^{(l)}_{n,\alpha}$ is the projection onto the closed subspace spanned by $(\dot{e}_{n,\alpha}(i))_{i\geq l}$.

In particular, $\mathbb{P} \Vdash \dot{E}_{n,\alpha} = \dot{E}^{(1)}_{n,\alpha}$.

Since F is dense in ℓ_2, for each $(n,\alpha) \in \mathbb{N} \times \omega_2$ there is a nice $\mathbb{P}$-name $\dot{v}_{n,\alpha}$ for a countable sequence of vectors such that for every $l \in \mathbb{N}$

$$\mathbb{P} \Vdash \dot{v}_{n,\alpha}(l) \in \check{F}, \|\dot{v}_{n,\alpha}(l)\| \in [1, 1+\varepsilon), \|\dot{v}_{n,\alpha}(l) - \dot{e}_{n,\alpha}(l)\| < \frac{1}{2l}. \tag{1}$$

Hence

$$\mathbb{P} \Vdash \|\dot{v}_{n,\alpha}(l) - \dot{E}^{(l)}_{n,\alpha}(\dot{v}_{n,\alpha}(l))\| \leq \|\dot{v}_{n,\alpha}(l) - \dot{e}_{n,\alpha}(l)\| + \|\dot{E}^{(l)}_{n,\alpha}(\dot{v}_{n,\alpha}(l) - \dot{e}_{n,\alpha}(l))\| \leq \frac{1}{l}. \tag{2}$$

For $n \in \mathbb{N}, \alpha \in \omega_2$ put

$$S_{n,\alpha} = \operatorname{supp}(\dot{e}_{n,\alpha}) \cup \operatorname{supp}(\dot{v}_{n,\alpha}).$$

Consider partition $\omega_2 = \bigcup_{n\in\omega} B_n$, where $|B_n| = \omega_2$ and $(B_n)_{n\in\omega}$ are pairwise disjoint. Each $S_{n,\alpha}$ is countable, so by Lemma 5.2.12, for each $n \in \omega$ there is a set $A_n \in [B_n]^{\omega_2}$ such that $(S_{n,\alpha})_{\alpha\in A_n}$ is a Δ-system with root Δ_n. The set $\Delta = \bigcup_{n\in\omega} \Delta_n$ is countable, so by a further thinning out of each A_n we may assume that for every $\alpha \in A_n$ we have $\Delta \cap S_{n,\alpha} = \Delta_n$.

By Lemma 5.3.1 we may also assume that for each $n \in \mathbb{N}$ and $\alpha, \beta \in A_n$ there is a permutation $\sigma_{n,\alpha,\beta} \colon \omega_2 \to \omega_2$ such that

- $\sigma_{n,\alpha,\beta} = \sigma^{-1}_{n,\beta,\alpha}$,
- $\sigma_n[S_{n,\alpha}] = S_{n,\beta}$,
- $\sigma_{n,\alpha,\beta} \restriction \Delta_n = Id$,
- $\sigma_{n,\alpha,\beta}(\dot{e}_{n,\alpha}) = \dot{e}_{n,\beta}$.

By Lemma 5.2.14 for every $\xi < \omega_2$ there is $\gamma_\xi \in \omega_2^{\mathbb{N}}$ such that

(a) $\gamma_\xi(n) \in A_n$ for $n \in \mathbb{N}$,
(b) for distinct $(\xi, n), (\eta, m) \in \omega_2 \times \mathbb{N}$ we have

$$(S_{n,\gamma_\xi(n)} \backslash \Delta_n) \cap (S_{m,\gamma_\eta(m)} \backslash \Delta_m) = \varnothing,$$

(c) $\gamma_\xi \cap \gamma_\eta = \varnothing$ for $\xi, \eta < \omega_2, \xi \neq \eta$.

For $n \in \mathbb{N}, \xi < \omega_2$ let $\dot{K}_{n,\xi}, \dot{L}_{n,\xi}$ be such $\mathbb{P}$-names that

$$\mathbb{P} \Vdash \dot{K}_{n,\xi} = \dot{E}_{n,\gamma_\xi(n)} - \dot{E}_{n,\gamma_\xi(n)} \dot{B}_{\gamma_\xi}, \dot{L}_{n,\xi} = \dot{E}_{n,\gamma_\xi(n)} - \dot{B}_{\gamma_\xi} \dot{E}_{n,\gamma_\xi(n)}.$$

Since $\mathbb{P} \Vdash \dot{E}_{n,\gamma_\xi(n)} \leq^{\mathcal{K}} \dot{B}_{\gamma_\xi}$ we have

$$\mathbb{P} \Vdash \dot{K}_{n,\xi} \text{ is compact}$$

and since $\mathbb{P} \Vdash \dot{L}_{n,\xi} = (\dot{K}_{n,\xi})^*$, we get that

$$\mathbb{P} \Vdash \dot{L}_{n,\xi} \text{ is compact.}$$

By Lemma 5.2.2 there is a nice $\mathbb{P}$-name $\dot{h}_\xi$ such that

$$\begin{aligned}\mathbb{P} \Vdash \dot{h}_\xi \colon \mathbb{N} \to \mathbb{N} \text{ and } \forall (n,\alpha) \in \check{\gamma}_\xi \ \forall l \geq \dot{h}_\xi(n) \\ \|\dot{E}^{(l)}_{n,\gamma_\xi(n)} \dot{K}_{n,\xi}\| < \varepsilon, \|\dot{L}_{n,\xi} \dot{E}^{(l)}_{n,\gamma_\xi(n)}\| < \varepsilon.\end{aligned} \tag{3}$$

Let $R_\xi = \operatorname{supp}(\dot{h}_\xi)$ and $S_\xi = \bigcup_{n\in\omega}(S_{n,\gamma_\xi(n)} \setminus \Delta_n)$. By (b) the sequence $(S_\xi)_{\xi<\omega_2}$ consists of pairwise disjoint subsets of ω_2. Since each $S_{n,\gamma_\xi(n)}$ is countable, S_ξ is also countable for $\xi < \omega_2$.

By Lemma 5.2.13 there are $\xi, \eta < \omega_2$ such that $\xi \neq \eta$ and

$$S_\xi \cap R_\eta = S_\eta \cap R_\xi = \varnothing.$$

We will show that

$$\mathbb{P} \Vdash \dot{B}_{\gamma_\xi} \dot{B}_{\gamma_\eta} \text{ is non-compact.}$$

Suppose this is not the case. Then by Lemma 5.2.3, (1) and (2) there is $p \in \mathbb{P}$ and $m \in \mathbb{N}$ such that

$$p \Vdash \forall l \geq m \ \|\dot{B}_{\gamma_\xi} \dot{B}_{\gamma_\eta}(\dot{v}_{n,\gamma_\xi(n)}(l))\| < \varepsilon. \tag{4}$$

Fix $n \in \mathbb{N}$ such that $\operatorname{dom}(p) \cap (S_{n,\gamma_\xi(n)} \cup S_{n,\gamma_\eta(n)}) \subseteq \Delta_n$ (such n exists, since by (b) the sequence $((S_{n,\gamma_\xi(n)} \cup S_{n,\gamma_\eta(n)}) \setminus \Delta_n)_{n\in\mathbb{N}}$ consists of pairwise disjoint sets and $\operatorname{dom}(p)$ is finite) and notice that p is $(\sigma_{n,\gamma_\xi(n),\gamma_\eta(n)}, S_{n,\gamma_\xi(n)}, S_{n,\gamma_\eta(n)})$-symmetric. Since $\operatorname{supp}(\dot{h}_\xi) = R_\xi$, by Remark 5.2.10 there is $q_\xi \leq p$ and $k_\xi \in \mathbb{N}$ such that

$$\operatorname{dom}(q_\xi) \setminus \operatorname{dom}(p) \subseteq R_\xi \text{ and } q_\xi \Vdash \dot{h}_\xi(n) = k_\xi.$$

Put

$$p_\xi = q_\xi \cup \sigma_{n,\gamma_\xi(n),\gamma_\eta(n)}(q_\xi \restriction S_{n,\gamma_\xi(n)}).$$

By Lemma 5.2.8 we have $p_\xi \in \mathbb{P}$ and p_ξ is $(\sigma_{n,\gamma_\xi(n),\gamma_\eta(n)}, S_{n,\gamma_\xi(n)}, S_{n,\gamma_\eta(n)})$-symmetric (it is also $(\sigma_{n,\gamma_\eta(n),\gamma_\xi(n)}, S_{n,\gamma_\eta(n)}, S_{n,\gamma_\xi(n)})$-symmetric, since $\sigma_{n,\gamma_\xi(n),\gamma_\eta(n)} = \sigma^{-1}_{n,\gamma_\eta(n),\gamma_\xi(n)}$). By the same argument there is $q_\eta \leq p_\xi$ and $k_\eta \in \mathbb{N}$ such that

$$\operatorname{dom}(q_\eta) \setminus \operatorname{dom}(p_\xi) \subseteq R_\eta \text{ and } q_\eta \Vdash \dot{h}_\eta(n) = k_\eta$$

and

$$p_\eta = q_\eta \cup \sigma_{n,\gamma_\eta(n),\gamma_\xi(n)}(q_\eta \restriction S_{n,\gamma_\eta(n)})$$

is an $(\sigma_{n,\gamma_\eta(n),\gamma_\xi(n)}, S_{n,\gamma_\eta(n)}, S_{n,\gamma_\xi(n)})$-symmetric element of $\mathbb{P}$.

Pick $l > \max\{k_\xi, k_\eta, m, \frac{1}{\varepsilon}\}$. By (2) we have

$$\begin{aligned}\mathbb{P} \Vdash \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}(\dot{v}_{n,\gamma_\xi(n)}(l))\| \geq \|\dot{v}_{n,\gamma_\xi(n)}(l)\| + \\ - \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}(\dot{v}_{n,\gamma_\xi(n)}(l)) - \dot{v}_{n,\gamma_\xi(n)}(l)\| \geq 1 - \varepsilon.\end{aligned} \tag{5}$$

Since $\operatorname{supp}(\dot{v}_{n,\gamma_\xi(n)}) \subseteq S_{n,\gamma_\xi(n)}$, there is $r \leq p_\eta$ and $v \in F$ such that

$$\operatorname{dom}(r) \setminus \operatorname{dom}(p_\eta) \subseteq S_{n,\gamma_\xi(n)} \text{ and } r \Vdash \check{v} = \dot{v}_{n,\gamma_\xi(n)}(l).$$

Then

$$s = r \cup \sigma_{n,\gamma_\xi(n),\gamma_\eta(n)}(r \upharpoonright S_{n,\gamma_\xi(n)})$$

is an $(\sigma_{n,\gamma_\xi(n),\gamma_\eta(n)}, S_{n,\gamma_\xi(n)}, S_{n,\gamma_\eta(n)})$-symmetric element of $\mathbb{P}$. From (5) we get

$$r \Vdash \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}(\check{v})\| \geq 1 - \varepsilon. \tag{6}$$

Since $\check{v}$ is a canonical name for an element of V we have $\sigma_{n,\gamma_\xi(n),\gamma_\eta(n)}(\check{v}) = \check{v}$ and since $s \leq \sigma_{n,\gamma_\xi(n),\gamma_\eta(n)}(r)$ we get

$$s \Vdash \|\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| \geq 1 - \varepsilon. \tag{7}$$

By the inequality

$$\mathbb{P} \Vdash \|\check{v} - \dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\|^2 + \|\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\|^2 = \|\check{v}\|^2 \leq (1+\varepsilon)^2$$

and (7) we get that

$$s \Vdash \|\check{v} - \dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\|^2 \leq (1+\varepsilon)^2 - (1-\varepsilon)^2 = 4\varepsilon \tag{8}$$

and hence

$$\begin{aligned} s \Vdash \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| &= \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}(\check{v}) - \dot{E}^{(l)}_{n,\gamma_\xi(n)}(\check{v} - \dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v}))\| \geq \\ &\geq \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}(\check{v})\| - \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}(\check{v} - \dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v}))\| \geq \\ &\geq 1 - \varepsilon - \|\check{v} - \dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| \geq 1 - \varepsilon - 2\sqrt{\varepsilon} > 1/2. \end{aligned} \tag{9}$$

On the other hand we have

$$\begin{aligned} s \Vdash \|\dot{B}_{\gamma_\xi}\dot{B}_{\gamma_\eta}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| &= \|\dot{B}_{\gamma_\xi}\dot{B}_{\gamma_\eta}(\check{v}) - \dot{B}_{\gamma_\xi}\dot{B}_{\gamma_\eta}(\check{v} - \dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v}))\| \leq \\ &\leq \|\dot{B}_{\gamma_\xi}\dot{B}_{\gamma_\eta}(\check{v})\| + \|\check{v} - \dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| \leq \varepsilon + 2\sqrt{\varepsilon}, \end{aligned} \tag{10}$$

where the last inequality follows from (4) and (8). Hence

$$\begin{aligned} s \Vdash \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| &= \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}\dot{E}_{n,\gamma_\xi(n)}\dot{E}_{n,\gamma_\eta(n)}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| = \\ &= \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}(\dot{K}_{n,\xi} + \dot{E}_{n,\gamma_\xi(n)}\dot{B}_{\gamma_\xi})(\dot{L}_{n,\eta} + \dot{B}_{\gamma_\xi}\dot{E}_{n,\gamma_\eta(n)})\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| \leq \\ &\leq \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}\dot{B}_{\gamma_\xi}\dot{B}_{\gamma_\eta}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| + \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}\dot{K}_{n,\xi}\dot{L}_{n,\eta}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| + \\ &+ \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}\dot{B}_{\gamma_\xi}\dot{L}_{n,\eta}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| + \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}\dot{K}_{n,\xi}\dot{B}_{\gamma_\eta}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| \leq \\ &\leq \|\dot{B}_{\gamma_\xi}\dot{B}_{\gamma_\eta}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| + 2(1+\varepsilon)\|\dot{L}_{n,\eta}\dot{E}^{(l)}_{n,\gamma_\eta(n)}\| + (1+\varepsilon)\|\dot{E}^{(l)}_{n,\gamma_\xi(n)}\dot{K}_{n,\xi}\|. \end{aligned}$$

By (3) and the fact that $l > \max\{k_\xi, k_\eta\}$ (which holds since $s \Vdash \max\{k_\xi, k_\eta\} = \max\{\dot{h}_\xi(n), \dot{h}_\eta(n)\}$) the last two terms of the above sum may be estimated by $3\varepsilon(1+\varepsilon)$. From this and (10) we get

$$s \Vdash \|\dot{E}^{(l)}_{n,\gamma_\xi(n)}\dot{E}^{(l)}_{n,\gamma_\eta(n)}(\check{v})\| \leq \varepsilon + 2\sqrt{\varepsilon} + 3\varepsilon(1+\varepsilon) < 1/2,$$

which is a contradiction with (9). $\square$

Theorem 5.3.3. *In the Cohen model $V^{\mathbb{P}}$ there is no *-embedding of $\ell_\infty(c_0(\omega_2))$ into $\mathcal{Q}(\ell_2)$. In particular, there is no *-embedding of $\ell_\infty(\mathcal{Q}(\ell_2))$ into $\mathcal{Q}(\ell_2)$.*

Proof. Assume that $T\colon \ell_\infty(c_0(\omega_2)) \to \mathcal{Q}(\ell_2)$ is a *-embedding. Let $\chi_{n,\alpha}, \chi_\gamma \in \ell_\infty(c_0(\omega_2))$ for $n \in \mathbb{N}, \alpha < \omega_2, \gamma \in \omega_2^{\mathbb{N}}$ be given by

$$\chi_{n,\alpha}(m)(\beta) = \begin{cases} 1, & \text{if } (n,\alpha) = (m,\beta) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\chi_\gamma(m)(\beta) = \begin{cases} 1, & \text{if } \gamma(m) = \beta \\ 0, & \text{otherwise} \end{cases}$$

Clearly $\chi_{n,\alpha}, \chi_\gamma$ are projections and if $\gamma(n) = \alpha$, then $\chi_{n,\alpha} \leq \chi_\gamma$.

Since *-embeddings preserve posets of projections, $T(\chi_{n,\alpha})$ and $T(\chi_\gamma)$ are projections in $\mathcal{Q}(\ell_2)$ for $n \in \mathbb{N}, \alpha \in \omega_2, \gamma \in \omega_2^{\mathbb{N}}$ and $T(\chi_{n,\alpha}) \leq T(\chi_\gamma)$ whenever $\gamma(n) = \alpha$. By [43, Lemma 3.1.13] there are projections $E_{n,\alpha}, B_\gamma$ in $\mathcal{B}(\ell_2)$ such that $\pi(E_{n,\alpha}) = T(\chi_{n,\alpha}), \pi(B_\gamma) = T(\chi_\gamma)$ for $n \in \mathbb{N}, \alpha \in \omega_2, \gamma \in \omega_2^{\mathbb{N}}$ (here $\pi\colon \mathcal{B}(\ell_2) \to \mathcal{Q}(\ell_2) = \mathcal{B}(\ell_2)/\mathcal{K}(\ell_2)$ denotes the quotient map). These projections satisfy the hypothesis of Proposition 5.3.2, so there are disjoint $\gamma_1, \gamma_2 \in \omega_2^{\mathbb{N}}$ such that $B_{\gamma_1}B_{\gamma_2}$ is non-compact, which contradicts the fact that $T(\chi_{\gamma_1})T(\chi_{\gamma_2}) = T(\chi_{\gamma_1}\chi_{\gamma_2}) = T(0) = 0$.

To see that there is no *-embedding of $\ell_\infty(\mathcal{Q}(\ell_2))$ into $\mathcal{Q}(\ell_2)$ we use the facts that $c_0(\mathfrak{c})$ embeds into ℓ_∞/c_0 (Lemma 5.2.4) and ℓ_∞/c_0 embeds into $\mathcal{Q}(\ell_2)$ (the natural embedding is given by $[(a_n)_{n\in\mathbb{N}}] \mapsto [A]$, where $A \in \mathbb{B}(\ell_2), A((c_n)_{n\in\mathbb{N}}) = (a_n c_n)_{n\in\mathbb{N}}$ for $(c_n)_{n\in\mathbb{N}} \in \ell_2$ i.e. A is the infinite diagonal matrix with entries $(a_n)_{n\in\mathbb{N}}$ on the diagonal), which implies that $\ell_\infty(c_0(\omega_2)) = \ell_\infty(c_0(\mathfrak{c}))$ embeds into $\ell_\infty(\mathcal{Q}(\ell_2))$. □

Now we will show an application of this result in the context of corona algebras of tensor products. Let us recall important definitions.

Definition 5.3.4. A C*-algebra $\mathcal{M}(\mathcal{A}) \supseteq \mathcal{A}$ is called the **multiplier algebra** of a C*-algebra $\mathcal{A}$, if $\mathcal{A}$ is an essential ideal in $\mathcal{M}(\mathcal{A})$ (i.e. $\mathcal{A}^\perp_{\mathcal{M}(\mathcal{A})} = \{0\}$, where $\mathcal{A}^\perp_{\mathcal{D}} = \{x \in \mathcal{D} : \mathcal{A}x = \{0\}\}$) and for every C*-algebra $\mathcal{D}$ containing $\mathcal{A}$ as an ideal the identity map $Id\colon \mathcal{A} \to \mathcal{B}(\mathcal{A})$ has a unique extension to $\mathcal{D}$ with kernel $\mathcal{A}^\perp_{\mathcal{D}}$.

The quotient algebra $\mathcal{Q}(\mathcal{A}) = \mathcal{M}(\mathcal{A})/\mathcal{A}$ is called the **corona** of $\mathcal{A}$.

It is well-known that $\mathcal{M}(\mathcal{K}(\ell_2)) \equiv \mathcal{B}(\ell_2)$ and $\mathcal{Q}(\mathcal{K}(\ell_2)) \equiv \mathcal{Q}(\ell_2)$. If X is a locally compact Hausdorff space, then $\mathcal{M}(C_0(X)) \equiv C(\beta X)$ and $\mathcal{Q}(C_0(X)) \equiv C(\beta X \backslash X)$, and so the multiplier algebra should be seen as the non-commutative analogue of the Stone-Čech compactification of a topological space, and corona as the non-commutative analogue of the Stone-Čech remainder. For more information on multiplier algebras and coronas see [15] or [43].

We will use the characterization of multipliers algebras in terms of double centralizers.

Definition 5.3.5. A pair (L, R) of linear maps on a C*-algebra $\mathcal{A}$ is a **double centralizer** of $\mathcal{A}$, if $xL(y) = R(x)y$ for all $x, y \in \mathcal{A}$.

Lemma 5.3.6. *[15, Theorem II.7.3.4] If (L, R) is a double centralizer of a C*-algebra $\mathcal{A}$, then L, R are bounded operators and $\|L\| = \|R\|$. The set of all double centralizers of $\mathcal{A}$ form a C*-algebra with operations*

- $(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2)$,
- $(L_1, R_1)(L_2, R_2) = (L_2 L_1, R_1 R_2)$,
- $(L, R)^* = (R^*, L^*)$,
- $\|(L, R)\| = \|L\| = \|R\|$.

This C-algebra is isomorphic to $\mathcal{M}(\mathcal{A})$.*

Recall that $\mathcal{K}(\ell_2)$ is a nuclear C*-algebra i.e. for every C*-algebra $\mathcal{A}$ there is a unique tensor product $\mathcal{A} \otimes \mathcal{K}(\ell_2)$.

Lemma 5.3.7. *There is a *-embedding of $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$ into $\mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2)$.*

Proof. Let $I \colon \ell_\infty/c_0 \to \mathcal{Q}(\ell_2)$ be the quotient of the diagonal embedding described in the proof of Theorem 5.3.3. Then I induces an embedding $I_0 \colon c_0(\ell_\infty/c_0) \to \mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2)$. Namely

$$I_0((a_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} I(a_n) \otimes e_{n,n}$$

where $a_n \in \ell_\infty/c_0$ and $e_{n,m} \in \mathcal{K}(\ell_2)$ is given by

$$e_{n,m}((c_k)_{k\in\mathbb{N}})(l) = \begin{cases} c_m, & \text{if } l = n \\ 0, & \text{otherwise} \end{cases}$$

for $n, m, l \in \mathbb{N}$ and $(c_n)_{n\in\mathbb{N}} \in \ell_2$ i.e. $e_{n,m}$ is the matrix with entry 1 at position (n, m) and 0 at other positions (so called matrix unit).

Now fix $A \in \ell_\infty(\ell_\infty/c_0)$, $A = (a_n)_{n\in\mathbb{N}}$. Consider operators $L_A, R_A \colon \mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2) \to \mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2)$ defined by

$$L_A(q \otimes e_{n,m}) = I(a_n)q \otimes e_{n,m}$$
$$R_A(q \otimes e_{n,m}) = qI(a_m) \otimes e_{n,m}$$

i.e. L_A and R_A are multiplications by diagonal matrices (from the left and right side respectively) with entries $(a_n)_{n\in\mathbb{N}}$ on the diagonal.

It is easy to check that (L_A, R_A) is a double centralizer of $\mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2)$ and $I_\infty \colon \ell_\infty(\ell_\infty/c_0) \to \mathcal{M}(\mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2))$ given by $I_\infty(A) = (L_A, R_A)$ is a *-embedding. Moreover, $I_\infty(A) \in \mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2)$ if and only if $A \in c_0(\ell_\infty/c_0)$. Thus,

$$J \colon \ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0) \to \mathcal{Q}(\mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2))$$

given by $J([A]) = [I_\infty(A)]$ is well-defined and is a *-embedding. ☐

Theorem 5.3.8. *In the Cohen model $V^{\mathbb{P}}$ there is no *-embedding of $\mathcal{Q}(\mathcal{Q}(\ell_2) \otimes \mathcal{K}(\ell_2))$ into $\mathcal{Q}(\ell_2)$.*

Proof. By Lemma 5.2.5 the algebra $\ell_\infty(c_0(\mathfrak{c}))$ embeds into $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$. In $V^{\mathbb{P}}$ we have $\mathfrak{c} = \omega_2$, so by Theorem 5.3.3 there is no embedding of $\ell_\infty(\ell_\infty/c_0)/c_0(\ell_\infty/c_0)$ into $\mathcal{Q}(\ell_2)$. Apply Lemma 5.3.7 to finish the proof. ☐

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