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$\mathscr{D}$-modules on Rigid Analytic Varieties

PhD dissertation

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Author's declaration:

I hereby declare that this thesis is my own work.

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Abstract

This PhD thesis is devoted to the study of $\mathscr{D}$-modules on rigid analytic varieties, with emphasis on the case when the ground field is discretely valued and of equal characteristic zero. Our main result establishes finiteness of the de Rham cohomology for holonomic $\mathscr{D}_X$-modules in the case when X is a smooth, quasi-compact, quasi-projective rigid analytic variety over the field $k((t))$ (char $k = 0$). On the way we prove some smaller results about rings of differential operators and nonarchimedean Banach algebras. We believe that those results may be of independent interest. In the last chapter we present an approach to the study of differential operators on smooth algebraic curves via the valuation theory.

Streszczenie

Poniższa rozprawa poświęcona jest $\mathscr{D}$-modułom na rozmaitościach sztywnych. Najciekawsze rezultaty dotyczą przypadku, gdy ciało bazowe jest równej charakterystyki zero a waluacja jest dyskretna. Główny wynik pracy orzeka, że kohomologie de Rhama holonomicznych $\mathscr{D}_X$-modułów mają skończony wymiar, jeżeli X jest quasi-zwartą, quasi-separowalną, gładką rozmaitością sztywną nad ciałem $k((t))$ (char $k = 0$). Po drodze dowodzimy mniejszych rezultatów dotyczących pierścieni operatorów oraz niearchimedesowych algebr Banacha, które naszym zdaniem są ciekawe same w sobie. Ostatni rozdział stanowi próbę badania operatorów różniczkowych na gładkich krzywych algebraicznych przy pomocy teorii waluacji.

Contents

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Niechaj tam inni księgi piszą. Nawet
niechaj im sława dźwięczy jak wieża studzwonna,
ja ksiąg pisać nie umiem, a nie dbam o sławę -
serwus, madonna.

Konstanty Ildefons Gałczyński, *Serwus, Madonna*

Chapter 1

Introduction

This thesis is devoted to the study of $\mathscr{D}$-modules on rigid analytic varieties, and in particular to the study of a certain homological invariant called the de Rham cohomology. I tried to make it readable to non-experts, but (probably just like thousands of PhD students before me) I might have lost the ability to recognize what does a non-expert know. This is completely understandable. We devote several years to the study of a certain problem, on the way we master our knowledge in the subject and in the end we forgot about the difficulties that we had to overcame. While this makes writing this chapter particularly hard, I will try my best to convince the reader that the mathematical content of the following chapters is nontrivial, and, what is maybe more important, that it may be interesting to the general audience (of algebraic geometers). The central object studied in this thesis is

> the (1) de Rham cohomology of (2) holonomic $\mathscr{D}$-modules on (3) rigid analytic varieties in (4) equal characteristic zero.

It is very likely that the reader is familiar with some, or even all, of the phrases (1), (2), (3), and (4), however it is even more likely that he or she has never seen them all together in this configuration. In this chapter we try to briefly explain the meaning of these phrases separately and justify why it is interesting to put them all together.

1.1 Classical theory

In this section we explain the meaning of phrases (1)-(4) above. In the course of doing so, we briefly present the classical theory on which our work is built.

1.1.1 Rigid analytic varieties

We now explain (3) and (4). Rigid analytic varieties are analogues of complex analytic manifolds and real $\mathscr{C}^\infty$-manifolds over complete normed fields other that $\mathbb{C}$ and $\mathbb{R}$. Such fields are called nonarchimedean. If K is a nonarchimedean field then the inequality

$$|x+y| \leq \max\{|x|,|y|\}$$

holds for all $x,y \in K$. This implies that the set $\mathfrak{o}_K = \{|x| \leq 1\} \subset K$ is in fact a valuation ring and we say that K is of equal (resp. mixed) characteristic if the residue characteristic of $\mathfrak{o}_K$ is equal (resp. not equal) to the characteristic of K. Among many examples of nonarchimedean fields we mention the field $\mathbb{Q}_p$ of p-adic numbers and the field $\mathbb{C}((t))$ of formal Laurent series (the norm on the latter is determined by the order of vanishing at zero of a Laurent series: $|f| = \exp(-\mathrm{ord}_0(f))$). Building the analog of a reasonable theory of manifolds over nonarchimedean fields is surprisingly nontrivial and the main difficulty comes from the fact that these fields are usually totally disconnected and therefore the naive approach via local charts does not work. The correct approach is similar to the construction of algebraic schemes. First, we define the category of affinoid varieties to be the opposite category to the category of certain Banach algebras (called affinoid algebras) and then we embed this category into the category of locally ringed spaces. General rigid analytic varieties are obtained by gluing affinoid varieties inside this larger category. For example let us fix a nonarchimedean field K and let us consider the ring

$$K\langle x\rangle = \left\{\sum_{n\geq 0} a_n x^n \in K[[x]] : \lim_{n\to\infty} |a_n| = 0\right\}.$$

This ring (called the Tate algebra) is a Banach K-algebra with respect to the norm

$$|\sum_{n\geq 0} a_n x^n| = \max |a_n|$$

and the corresponding affinoid variety is called the Tate disc. While the approach above seems standard from the point of view of today's mathematics, it took some time to formalize it. The first satisfactory definition is due to Tate, who introduced rigid analytic varieties as certain spaces with Grothendieck topology. His formalism was later refined by Berkovich, and finally reformulated by Huber into the now standard language of adic spaces.

The usefulness of rigid analytic varieties was quickly discovered by number theorists, who were mostly interested in varieties defined over the fields of mixed characteristic (like the p-adic numbers). Nonarchimedean geometry is now one of the standard tools in the p-adic Hodge theory. The flexibility of the formalism of Huber was also noticed by the algebraic geometers working in the positive characteristic as it turns out that one can sometimes use adic spaces to overcome the lack of resolution of singularities in characteristic $p > 0$. These are, however, *not* the situations we are interested in in this thesis. Instead we focus on fields of equal characteristic zero, for example the field $\mathbb{C}((t))$ of formal Laurent series. One explanation for this choice is that the main theorems that we prove are simply false for other base fields. The more satisfactory one is that, while less popular than their p-adic counterparts, the rigid analytic varieties over fields of equal characteristic zero are still very interesting objects studied by first class mathematicians, and they often find applications in other branches of mathematics. For example the conjecture of C. Sabbah about singularities of vector bundles with connections on complex analytic manifolds has been solved by K. Kedlaya using the nonarchimedean geometry in equal characteristic zero.

1.1.2 De Rham cohomology and $\mathscr{D}$-modules

We now explain (1) and (2). The reader is probably familiar with the de Rham cohomology of $\mathscr{C}^\infty$-manifolds and complex analytic manifolds. In these settings the significance of the de Rham cohomology is explained by the de Rham theorem which asserts that the de Rham cohomology computes the singular cohomology of the underlying topological space, i.e.,

$$H^*_{\mathrm{dR}}(X) = H^*_{\mathrm{Sing}}(X).$$

If X is a smooth algebraic $\mathbb{C}$-variety then we can consider the associated complex analytic manifold X^{an}. In this situation we can also consider the algebraic de Rham cohomology which is built from the sheaf of Kähler differentials instead of the holomorphic cotangent sheaf on X^{an}. Grothendieck discovered that the algebraic de Rham cohomology agrees with the analytic one and we have a natural isomorphism

$$H^*_{\mathrm{dR}}(X) = H^*_{\mathrm{Sing}}(X^{an}).$$

This is quite amazing, because it turns out that we can compute a purely topological invariant of X^{an} using only the algebraic structure of X and without referring to the euclidean topology. Note that, since X^{an} has the homotopy type of a finite CW-complex, these spaces are finitely dimensional.

It turns out that representations of the fundamental group of X^{an} can also be studied in a purely algebraic manner. First, let X be a complex analytic manifold. Then a vector bundle with an integrable connection is a holomorphic vector bundle $\mathscr{E}$ together with a $\mathbb{C}$-linear map

$$\nabla : \mathscr{E} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{E}$$

that satisfies Leibniz's rule $\nabla(fm) = df \otimes m + f\nabla(m)$ and an extra integrability condition, which asserts that we can construct for $(\mathscr{E}, \nabla)$ an analogue of the de Rham complex. Now the sheaf of horizontal sections

$$\mathscr{E}^{\nabla} = \{m \in \mathscr{E} : \nabla(m) = 0\}$$

is a local system (i.e., a locally constant sheaf of finitely dimensional $\mathbb{C}$-vector spaces) on X. It is a standard fact from algebraic topology that the category of local systems is equivalent to the category of representations of the fundamental group of X. The Riemann–Hilbert correspondence asserts that the former is also equivalent to the category $\mathrm{MIC}(X)$ of vector bundles with integrable connections and thus we have a canonical isomorphism

$$\mathrm{Rep}(\pi_1(X), \mathbb{C}) = \mathrm{MIC}(X).$$

Again the notion of a connection can be translated to the algebraic category and if X is a smooth algebraic $\mathbb{C}$-variety then we can consider the category of algebraic vector bundles with algebraic integrable connections. This category contains a subcategory $\mathrm{MIC}_{reg}(X)$ of connections with regular singularities at infinity and by the theorem of Deligne

$$\mathrm{MIC}_{reg}(X) = \mathrm{Rep}(\pi_1(X^{an}), \mathbb{C}),$$

so again we were able to study a purely topological invariant of X^{an} using only the algebraic structure on X.

The notion of the de Rham complex and the integrable connection is so formal that it translates easily to any reasonable geometric situation in which the cotangent sheaf is well-defined. In particular, it makes sense to study these objects on rigid analytic varieties. The discussion above motivates this study: we can hope that by studying the de Rham cohomology we study what would be the singular cohomology without defining it. In any case, this is some cohomology theory that has the advantage of being easily defined.

The passage from integrable connections to $\mathscr{D}$-modules is very similar to the passage from vector bundles to (quasi-)coherent sheaves in algebraic geometry. While we are mostly interested in the study of vector bundles with connections, this category does not have enough objects to have the desired functorial properties. Therefore we enlarge it to the category of $\mathscr{D}_X$-modules, i.e., modules over the sheaf of differential operators on X. The latter contains the subcategory of holonomic $\mathscr{D}_X$-modules. On the one hand every vector bundle with integrable connection is a holonomic $\mathscr{D}_X$-module in a natural way and on the other hand the category of holonomic $\mathscr{D}_X$-modules is closed under many functorial operations that we can define for $\mathscr{D}$-modules. Therefore it is a very useful tool in the study of vector bundles with integrable connections and often when we want to prove some theorem about connections it is better to go all in and prove it for holonomic modules. This is the case in the main theorem of this thesis. In fact, even for the trivial connection $(\mathscr{O}_X, d)$ the theorems that we prove require some thought.

1.2 Examples of the de Rham cohomology

Since this thesis is mainly devoted to proving finiteness theorems about the de Rham cohomology we devote this section to a brief overview of the analogous statements in other settings. This further motivates our work and it also builds the intuition which should at least partially convince the reader that the theorems we prove are plausible.

1.2.1 Classical finiteness theorems

If X is either a $\mathscr{C}^\infty$ real manifold or a complex analytic manifold, then finiteness of the de Rham cohomology (with constant coefficients) is a consequence of the de Rham theorem, and if X is a smooth algebraic $\mathbb{C}$-variety then the finiteness follows from the comparison theorem of Grothendieck. This result is easily generalized to the situation when $\mathbb{C}$ is replaced by any field of characteristic zero. In what follows we focus on non-constant coefficients, i.e., on finiteness of the de Rham cohomology for integrable connections and, more generally, $\mathscr{D}$-modules.

The oldest case of finiteness of the de Rham cohomology concerns algebraic $\mathscr{D}$-modules on smooth complex algebraic varieties. If $(\mathscr{E}, \nabla) \in \mathrm{MIC}_{\mathrm{reg}}(X)$ then Deligne showed that

$$H^*_{\mathrm{dR}}(X, (\mathscr{E}, \nabla)) = H^*_{\mathrm{dR}}(X^{an}, (\mathscr{E}^{an}, \nabla^{an})).$$

Since the latter equals to the cohomology of the corresponding local system it has finite dimension. If $(\mathscr{E}, \nabla)$ is not regular then $H^*_{\mathrm{dR}}(X, (\mathscr{E}, \nabla))$ is still finite but the proof is more involved. If X is the affine n-space then the theorem of J. Bernstein asserts that algebraic holonomic $\mathscr{D}_X$-modules have finite dimensional de Rham cohomology groups. The original proof of this theorem is hard to find although some of Bernstein's ideas are presented in [Ber72]. A nice proof can be found in the book of J.E. Björk [Bjö79, Chapter 1, Theorem 6.1]. Bernstein's result has been later generalized to the case when X is a smooth complex algebraic variety and finally to the derived setting. The strongest version says that if $f : X \to Y$ is a morphism of smooth varieties then the $\mathscr{D}$-module theoretic direct image $\int_f : D^b_{qc}(\mathscr{D}_X) \to D^b_{qc}(\mathscr{D}_Y)$ restricts to the functor $\int_f : D^b_h(\mathscr{D}_X) \to D^b_h(\mathscr{D}_Y)$. The notation here is taken from the book of R. Hotta, K. Takeuchi, and T. Tanisaki (see [HTT08, Theorem 3.2.3]).

Another classical (although much less known) case is when $\mathscr{D}$ is the ring of differential operators over the ring of formal power-series over a field of characteristic zero, i.e., when $\mathscr{D} = K[\![x_1,\dots,x_n]\!][\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}]$. This theorem has been proven by A. van den Essen in 1980s in a series of papers ([vdE79], [vdE82], [vdE83a], [vdE83b], [vdE83c]). A very nice and short exposition of these results has been recently written by N. Switala [Swi17].

1.2.2 Finiteness theorems in rigid analytic geometry

We now discuss the de Rham cohomology in the rigid analytic setting. We try to show the analogies with the classical calculus and explain which fundamental properties fail in the nonarchimedean setting. We believe that this elementary considerations are useful for building the intuition behind the de Rham cohomology.

To start the discussion let us recall the 'fundamental theorem of calculus' which states that if $f : U \to \mathbb{R}$ is a continuous function defined on an open subset of $\mathbb{R}$ then there exists a differentiable function $F : U \to \mathbb{R}$ with $\frac{dF}{dx} = f$. If we consider an open interval $I = (0,1)$ as a $\mathscr{C}^\infty$ manifold and write $\mathscr{C}(I)$ for the $\mathbb{R}$-vector space of smooth functions on I then the usual de Rham complex is given by

$$\mathscr{C}^\infty(I) \xrightarrow{f \mapsto \frac{df}{dx}} \mathscr{C}^\infty(I).$$

The fundamental theorem asserts that this map is surjective. Therefore one can think of the classical de Rham cohomology as of the quantitative measure to what extent the fundamental theorem of calculus fails on a $\mathscr{C}^\infty$ manifold.

Similar reasoning works in the complex analytic setting. If we let $\mathbb{D} = \{|z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$, and we write $\mathscr{O}(\mathbb{D})$ for the $\mathbb{C}$-vector space of the holomorphic functions on $\mathbb{D}$ then the de Rham complex is

$$\mathscr{O}(\mathbb{D}) \xrightarrow{f \mapsto \frac{df}{dx}} \mathscr{O}(\mathbb{D}) \tag{1.1}$$

and the map is again surjective. These computations agree with the fact that de Rham cohomology coincides with the singular cohomology as both I and $\mathbb{D}$ are contractible.

One of the strange features of the rigid analytic geometry is that the Tate disc $\mathbb{B}$ introduced in the previous section, which can be very well interpreted as the closed unit disc is a perfectly good smooth rigid analytic variety. This is of course completely different from the complex analytic case because the closed complex disc has a boundary. We will later see that the de Rham complex for the structure sheaf of $\mathbb{B}$ is again

$$K\langle x\rangle \xrightarrow{f\mapsto \frac{df}{dx}} K\langle x\rangle. \tag{1.2}$$

Now the surjectivity of the map in (1.1) follows from the fact that if $\sum a_n z^n$ is a complex power series with the radius of convergence $=1$ then its anti-derivative $\sum \frac{a_n}{n+1} z^{n+1}$ has the same radius of convergence. This is a simple consequence of Hadamard's formula for the radius of convergence

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}. \tag{1.3}$$

While the formula (1.3) holds also for nonarchimedean fields it is not enough to conclude that the map in (1.2) is surjective. This is yet another strange feature of the nonarchimedean world. In complex analysis integration can at most enlarge the region in which a power series converges, meaning that it may happen that a power series f converges on $\mathbb{D}$ but not on $\overline{\mathbb{D}}$ and its anti-derivative converges on $\overline{\mathbb{D}}$, but not the other way around. This situation is reversed in the nonarchimedean setting. For example, let $K = \mathbb{Q}_p$ and let

$$f = \sum_{n\geq 0} p^n x^{p^n - 1}.$$

Since $|p| < 1$ we see that $f \in \mathbb{Q}_p\langle x\rangle$. On the other hand its anti-derivative

$$F = \sum_{n\geq 0} x^{p^n}$$

clearly does not converge for $|x| = 1$ and therefore the class of f in $H^1_{\mathrm{dR}}(\mathbb{B})$ is nonzero. More generally we can consider power series

$$f = \sum_{n\geq 0} a_n p^n x^{p^n - 1}$$

with $|a_n| = 1$. They cannot be integrated and it is easy to conclude that

$$\dim_{\mathbb{Q}_p} H^1_{\mathrm{dR}}(\mathbb{B}) = \infty.$$

This phenomenon does not occur if we assume that K is of equal characteristic zero. Since the residue field is of characteristic zero, we see that for any nonzero integer n we have $|n| = 1$. Therefore $f = \sum_{n\geq 0} a_n x^n \in K[[x]]$ is an element of $K\langle x\rangle$ if and only if $\lim |a_n| = 0$, which is a case if and only if $\lim \left|\frac{a_n}{n+1}\right| = 0$, i.e., if and only if the anti-derivative of f is an element of $K\langle x\rangle$. We conclude that $H^1_{\mathrm{dR}}(\mathbb{B}) = 0$ in this case. In fact, the above reasoning is easily generalized to higher dimensions. Let

$$A = K\langle x_1,\dots,x_n\rangle = \left\{ \sum_{|\alpha|\geq 0} a_\alpha x^\alpha \in K[[x_1,\dots,x_n]] : \lim_{|\alpha|\to\infty} |a_\alpha| = 0 \right\}.$$

This is the Tate algebra, which corresponds to the ring of global functions on the n-dimensional closed polydics $\mathbb{B}^n$. We will later see that the de Rham complex for the structure sheaf on this space is

$$A \to \bigoplus_{1\leq i\leq n} Adx_i \to \cdots \to \bigoplus_{1\leq i_1<\cdots<i_{n-1}\leq n} Adx_{i_1}\wedge\cdots\wedge dx_{i_{n-1}} \to Adx_1\wedge\cdots\wedge dx_n,$$

with the differential given by $\delta(f dx_I) = \sum_i \partial_i(f) dx_i \wedge dx_I$, i.e, it is precisely what one would expect. We now prove

Theorem 1.2.1 (Poincaré lemma)**.** *Let K be a nonarchimedean field of equal characteristic zero. Then $H^i_{\mathrm{dR}}(\mathbb{B}^n) = 0$ for all $i > 0$.*

*Proof (*Hartshorne*).* We follow the exposition from [Har75, Proposition 7.1]. The proof is by induction on n and the case $n = 1$ has already been settled. We write

$$\Omega^k = \bigoplus_{|I|=k} Adx_I,$$

where $I = (1 \leq i_1 < \cdots < i_k \leq n)$ and $dx_I = dx_{i_1}\wedge\cdots\wedge dx_{i_k}$. Assume that $\omega \in \Omega^k$ and $d\omega = 0$. We have to show that $\omega = d\eta$ for some η. We can write

$$\omega = \omega_1 \wedge dx_1 + \omega_2,$$

where neither ω_1 nor ω_2 contain dx_1. For $f = \sum a_\alpha x^\alpha \in K\langle x\rangle$ we define

$$\int f dx_1 = x_1 \sum_\alpha \frac{a_\alpha}{(\alpha_1+1)} x^\alpha.$$

Note that this is a well defined element of A precisely because K is of equal characteristic zero. The definition of the integral extends to all forms in an obvious way. The form

$$\omega - d\int \omega_1 dx_1$$

does not contain dx_1 and it differs from ω by an exact form, so we may assume that ω does not contain dx_1, i.e., that $\omega = \omega_2$. Write $\omega = \sum f_I dx_I$. Since $d\omega = 0$ and none of the dx_I contains dx_1, we see that also none of the f_I contains x_1. Therefore we can apply the inductive assumption to ω and we are done. □

The moral of above discussion is the following. First of all, our definition of the de Rham cohomology is poorly behaved in the case of fields of mixed characteristic and second of all, it seems to behave reasonably in the case of fields of equal characteristic zero. This partially explains why for most of the time we restrict our considerations to the the latter case. At this point we remark that there exists a reasonable version of the de Rham cohomology in the case of mixed characteristic, called the overconvergent de Rham cohomology, which is a very active area of research. We refer the interested reader to the work of K. Kedlaya [Ked06] and E. Große-Klönne [GK04], [GK02] for some results regarding the connections between the de Rham and the rigid cohomology. We also refer to the work of V. Ertl and A. Shiho [ES20] for some 'non-examples' of the finiteness of the de Rham cohomology.

1.2.3 The main theorem

The main theorem of this thesis is the following.

Theorem 1.2.2. *Let X be a quasi-compact, quasi-separated, smooth rigid analytic space over a discretely valued nonarchimedean field of equal characteristic zero. Then for any holonomic $\mathscr{D}_X$-module $\mathscr{M}$ and for all i we have* $\dim_K H^i_{\mathrm{dR}}(X, \mathscr{M}) < \infty$.

Since in the following chapters things get a bit technical, we are afraid that the idea behind the proof of this theorem could easily be missed within numerous lemmas and propositions. We therefore present this idea below on a simple example.

Let $K = \mathbb{C}((t))$, and let $X = \operatorname{Spa} K\langle x\rangle$ be the Tate disc. Assume that we are given a vector bundle with integrable connection on X, i.e., a free $K\langle x\rangle$-module M of finite rank together with a K-linear map $\nabla : M \to M$ that is continuous for the canonical topology on M and which satisfies *Leibniz's rule*

$$\nabla(f.m) = \frac{df}{dx}.m + f.\nabla m.$$

Then we have

$$H^0_{\mathrm{dR}}(X, M) = \ker \nabla, \quad H^1_{\mathrm{dR}}(X, M) = \operatorname{coker} \nabla.$$

We present the idea behind the proof of Theorem 1.2.2 in this special case. A natural approach to the problem of finiteness of the de Rham cohomology of (M,∇) would be the following. First we look for a *model*, i.e., a finitely generated $\mathfrak{o}_K\langle x\rangle$ submodule $M^\circ \subset M$ such that $M^\circ \otimes_{\mathfrak{o}_K} K = M$ and $\nabla(M^\circ) \subset M^\circ$. Then we consider the *reduction* of our model, i.e. the $\mathbb{C}[x]$-module $\overline{M} = M \otimes_{\mathfrak{o}_K} \mathbb{C}$ together with the induced connection $\overline{\nabla}$. Now $(\overline{M},\overline{\nabla})$ is a vector bundle with connection on the affine line over $\mathbb{C}$ and therefore it has finite dimensional de Rham cohomology by the classical theory. One can show (see Lemma 2.2.6) that finiteness of the de Rham cohomology of (M,∇) follows from finiteness of the de Rham cohomology of $(\overline{M},\overline{\nabla})$. Moreover, if we write χ_{dR} for the Euler characteristic with respect to the de Rham cohomology then

$$\chi_{\mathrm{dR}}(X,(M,\nabla)) = \chi_{\mathrm{dR}}(\mathbb{A}^1_{\mathbb{C}},(\overline{M},\overline{\nabla})).$$

Following this idea we now ask if given (M,∇) one can always find a model. The module M carries a canonical family of equivalent norms and it is easy to see that if (M,∇) has a model then the spectral radius of ∇ satisfies $|\nabla|_{\mathrm{sp},M} \leq 1$ (see [Ked10, Definition 6.1.3] for the notation). If we take, e.g.,

$$M = K\langle x\rangle e, \quad \nabla(f.e) = \left(\frac{df}{dx} - t^{-1} f\right).e \tag{1.4}$$

then $|\nabla|_{\mathrm{sp},M} = |t^{-1}| > 1$ and (M,∇) does not admit a model. Thus we have to refine our approach. We consider two rings

$$\mathcal{D} = K\langle x\rangle[\partial], \quad \text{and} \quad \widehat{\mathcal{D}} = K\langle x,\partial\rangle.$$

The former is the ring of differential operators on X and the latter (the completed Weyl algebra) is its completion with respect to the operator norm. The elements of $\mathcal{D}$ are represented as polynomials $\sum f_i\partial^i$ with $f_i \in K\langle x\rangle$ and the elements of $\widehat{\mathcal{D}}$ are represented as formal power series $\sum f_i\partial^i$ such that $\lim|f_i| = 0$. Any (M,∇) can be seen as a left $\mathcal{D}$-module with $\partial.m = \nabla(m)$, and if $|\nabla|_{\mathrm{sp},M} \leq 1$ then it is in fact a $\widehat{\mathcal{D}}$-module. This suggests a way of forcing M to have a model by setting

$$\widehat{M} = \widehat{\mathcal{D}} \otimes_{\mathcal{D}} M,$$

and studying $\widehat{M}$ instead. It is a small miracle (to which Section 3.3 is devoted) that this base-change operation preserves the dimensions of the de Rham cohomology groups. For example, if we take (M,∇) defined in (1.4) then the corresponding $\mathcal{D}$-module is

$$M = \mathcal{D}/\mathcal{D}(\partial - t^{-1}) = \mathcal{D}/\mathcal{D}(1 - t\partial).$$

Note that $1-t\partial$ is a unit in $\widehat{\mathcal{D}}$ and therefore $\widehat{M}=0$. This is fine since $\ker\nabla=0$ as $|\nabla(f)|=|t^{-1}||f|$ and $\operatorname{coker}\nabla=0$ as

$$\nabla\left(\sum_{i\geq 0} t^i\partial^i(f)\right)=\sum_{i\geq 0} t^i\partial^{i+1}(f)-t^{-1}\sum_{i\geq 0} t^i\partial^i(f)=t^{-1}f.$$

The price that we pay for passing from M to $\widehat{M}$ is that $\widehat{M}$ will not be a vector bundle in general. In fact it is rarely finitely generated over $\mathcal{D}$. On the other hand, if M is a $\mathcal{D}$-module of minimal dimension (i.e., an algebraic object corresponding to a holonomic $\mathscr{D}_X$-module) then $\widehat{M}$ is a $\widehat{\mathcal{D}}$-module of minimal dimension (see Lemma 3.3.3). The discussion above in the case of Tate's disc carries to Tate's polydiscs of arbitrary dimension. The first step of our proof of Theorem 1.2.2 is based on the careful study of modules of minimal dimension over completed Weyl algebras from which we conclude Theorem 1.2.2 for holonomic $\mathscr{D}$-modules on Tate's polydiscs. Reduction of the general case to this situation is a bit technical and it takes some work but the key idea is the one above.

1.2.4 What is wrong with Bernstein's proof?

A perfectly natural question one may ask when seeing Theorem 1.2.2 for the first time is why the proof of the analogous result of Bernstein for algebraic varieties (or any other known proof) does not carry to the nonarchimedean setting. Maybe we are reinventing the wheel, while the proof is already there? One heuristic answer to this question is that if we could adapt any proof from the classical algebraic geometry to our setting, then such proof would take into the account only the characteristic of the base field (and not the residual characteristic) and therefore Theorem 1.2.2 would be also valid for $\mathbb{Q}_p$, which is not the case as we have already seen. It is also easy to explain why Bernstein's proof does not carry over to the nonarchimedean world using elementary topology. This proof is ultimately reduced to the case when $X=\mathbb{A}^n$ is the affine n-space. In this situation the ring of differential operators is the Weyl algebra, i.e., the ring of differential operators

$$K[x_1,\dots,x_n,\partial_1,\dots,\partial_n],$$

and every element has a unique presentation of form

$$P=\sum a_{\alpha\beta}x^\alpha\partial^\beta,$$

with $a_{\alpha\beta}\in K$. Weyl algebra admits *Bernstein's filtration*

$$B^n=\left\{P: a_{\alpha\beta}=0 \text{ for } |\alpha|+|\beta|>n\right\},$$

and the proof is based on the study of this filtration (cf. [Bjö79, Chapter 1]). If we replace the affine line by Tate's polydiscs then the algebra of differential operators is

$$\mathcal{D}_n = K\langle x_1, \ldots, x_n\rangle[\partial_1, \ldots, \partial_n].$$

Now assume that we have a filtration of $\mathcal{D}_n$ by finitely dimensional K-vector spaces. Then the intersection of this filtration with Tate's algebra induces a filtration on the latter with the same properties. As we will soon see, Tate's algebras are Banach, and in particular they are complete metric spaces. As every finitely dimensional subspace is closed and has empty interior, their countable union cannot be the whole Tate algebra by Baire's theorem from elementary topology.

1.3 Structure of the thesis and overview of original results

1.3.1 Structure of the thesis

This thesis is divided into four chapter and you are about to finish reading the first one. In the second chapter we present general theory of rigid analytic varieties and $\mathscr{D}$-modules, and we introduce necessary technical tools from homological algebra. That chapter contains mostly preliminary results, although it also contains several original ideas, simply because we felt that discussing them in that chapter improves the presentation. The main results of the thesis are contained in the last two chapters. In the third chapter we prove Theorem 1.2.2. To do so we first study modules over completed Weyl algebras and then we study some more sophisticated properties of $\mathscr{D}$-modules on rigid analytic spaces. The last chapter is more loosely connected to the rest of the text. It presents an approach to the theory of differential operators on algebraic curves based on the valuation theory. One application of this approach is a new proof of Deligne's index formula for the de Rham cohomology of a vector bundle with connection on a smooth affine curve. $\mathscr{D}$-modules on rigid analytic varieties reappear in the last section, where we show how to compute an index of a differential operator on a smooth affinoid curve with smooth affine reduction and we work out some explicit formulas for the Euler characteristic of holonomic $\mathscr{D}$-modules.

I have learned from my advisors that even while working on a very particular problem one should always have in mind the big picture. In other words, this thesis could be shorter. It is not because, whenever possible, we try to connect presented material to other fields of mathematics. This is especially visible in the second chapter. While such approach makes the manuscript longer it also makes the exposition less dry and motivates our work, so advantages and disadvantages cancel out.

1.3.2 Overview of the original results

As a general rule, whenever a discussed result is not original we try to either attribute it properly, or (if it is the case) emphasise that it is considered ‘folklore’. Another rule that we try to follow is not to give proofs of known results (and to provide references instead), but we are less strict with this rule. Sometimes we give proofs because we could not provide suitable references and sometimes we sketch them for the sake of better presentation.

The first chapter is introductory. The second chapter contains mainly preliminary results which are classical. Our main references for the nonarchimedean part are books of Bosch–Güntzer–Remmert [BGR84], Huber [Hub96], and Fresnel–van der Put [FvdP04]. The part devoted to homological algebra follows Weibel [Wei94], and in the $\mathscr{D}$-module part we mostly follow Hotta–Takeuchi–Tanisaki [HTT08], Mebkhout [Meb89], and Mebkhout–Narváez Macarro [MNM91]. There are still some new results in this chapter. In the section devoted to homological algebra we consider Lemmas 2.2.6, 2.2.7, 2.2.13, and 2.2.14 our own inventions. We mention however that these results are rather elementary (although perhaps non-obvious) and we do not exclude possibility that some of their variants are known to experts. This is surely the case with Lemma 2.2.14 (cf. Remark 2.2.15). All the lemmas listed above appear also in our preprint [Rą24b]. We also consider the content of subsection 2.3.3 to be at least partially original. There we prove the following.

Theorem 1.3.1 (cf. Theorem 2.3.12)**.** *Let K be a nonarchimedean field and let A be a noetherian Banach K-algebra. Assume that*

(1) *A is reduced, i.e., $\bigcap_{\mathfrak{p}\in\operatorname{Spec} A}\mathfrak{p} = \{0\}$.*

(2) *For every minimal prime ideal $\mathfrak{p}\subset A$, either $A/\mathfrak{p}$ is not a field or the field extension $K\subset A/\mathfrak{p}$ is finite.*

Then every K-linear differential operator on A is continuous.

Our proof, based on the work of Jewell–Sinclair [Sin75], [JS76] from classical functional analysis, seems to be completely new in the nonarchimedean setting.

The third chapter contains the main original results. These results are the content of our two preprints [Rą24b], [Rą24a] and we follow these papers quite faithfully. The majority of the results in this chapter is author's own invention, although several preliminary results are not original. We mention the most important ones. The main goal is

Theorem 1.3.2 (cf. Theorem 1.2.2)**.** *Let X be a quasi-compact, quasi-separated, smooth rigid analytic space over a discretely valued nonarchimedean field of equal characteristic zero. Then for any holonomic $\mathscr{D}_X$-module $\mathscr{M}$ and for all i we have $\dim_K H^i_{\mathrm{dR}}(X,\mathscr{M}) < \infty$.*

To prove it we study modules over the ring $\mathcal{D}_n$ of differential operators of the n-dimensional Tate algebra and over the completed Weyl algebra $\widehat{\mathcal{D}}_n$. We write $\widehat{M} = \widehat{\mathcal{D}}_n \otimes_{\mathcal{D}_n} M$. First, we prove

Theorem 1.3.3 (cf. Theorem 3.1.1)**.** *Let K be a discretely valued nonarchimedean field of equal characteristic zero and let M be a finitely generated left $\widehat{\mathcal{D}}_n$-module. Then the following conditions are equivalent:*

(1) *M is of minimal dimension.*

(2) *There exists a lattice $L \subset M$ such that $\overline{L}$ is a $\overline{\mathcal{D}}_n$-module of minimal dimension.*

(3) *For any lattice $L \subset M$ the reduction $\overline{L}$ is a $\overline{\mathcal{D}}_n$-module of minimal dimension.*

If these equivalent conditions are satisfied, then moreover

A) The semisimplification of $\overline{L}$ does not depend on L and only on M.

B) We have $\dim_K H^i_{dR}(M) < \infty$ for all i and the equality $\chi_{dR}(M) = \chi_{dR}(\overline{L})$ holds.

Then, we reduce the proof of Theorem 1.2.2 to Theorem 3.1.1. The main ingredient of this reduction is the following Lemma.

Lemma 1.3.4 (cf. Lemma 3.3.3)**.** *Let M be a finitely generated left $\mathcal{D}_n$-module.*

(1) *If M is of minimal dimension then so is $\widehat{M}$.*

(2) *The complexes $\mathbf{DR}^\bullet{}_{\mathcal{D}_n}(M)$ and $\mathbf{DR}^\bullet{}_{\widehat{\mathcal{D}}_n}(\widehat{M})$ are quasi-isomorphic.*

This lemma allows us to prove Theorem 1.2.2 for globally presented $\mathscr{D}$-modules on Tate's polydiscs. To deal with the general case we study $\mathscr{D}$-module theoretic direct image along a Zariski closed embedding.

Lemma 1.3.5 (cf, Lemma 3.4.2)**.** *Let $i : X \hookrightarrow Y$ be a Zariski closed embedding of smooth rigid analytic varieties. Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module. Then*

(1) *If both X and Y admit global coordinate systems and $\mathscr{M}$ is globally finitely presented then so is $i_+\mathscr{M}$.*

(2) *The left $\mathscr{D}_Y$-module $i_+\mathscr{M}$ is coherent.*

(3) *If $\mathscr{M}$ is holonomic then so is $i_+\mathscr{M}$.*

(4) *There exists a natural K-linear quasi-isomorphism of complexes*

$$i_*\mathbf{DR}^\bullet{}_X(\mathscr{M}) \to \mathbf{DR}^\bullet{}_Y(i_+\mathscr{M})[\dim X - \dim Y].$$

The main difficulty there is the construction in part (4). Although it seems classical, to our best knowledge the explicit construction of the desired quasi-isomorphism does not appear in the literature.

The fourth chapter is devoted to the study of differential operators on algebraic curves. It contains mostly new results which as for today (13.06.2024) are not yet the content of any preprint. The whole chapter is influenced by the approach presented in Kedlaya's book [Ked10]. The first subsection contains preliminary results which are all classical. There we follow Hartshorne's book [Har77]. Then we show that if K is a function field of a smooth algebraic curve over k then any k-valuation ν on K extends to the ring $\mathcal{D}_K$ of k-linear differential operators on K and we show the following.

Proposition 1.3.6 (cf Proposition 4.2.6)**.** *Let k be a field of characteristic zero and let $P \in \mathcal{D}_K$. Then*

(1) *For all $a \in K$ we have*

$$\nu(P(a)) \geq \nu(P) + \nu(a).$$

(2) *There exists a finite subset $S \subset \mathbb{Z}$ (depending on P) such that the equality*

$$\nu(P(a)) = \nu(P) + \nu(a)$$

holds whenever $\nu(a) \in \mathbb{Z} \setminus S$.

(3) *The number $\nu(P)$ is independent of the choice of a uniformizer of R_ν.*

We use the proposition above and the Riemann-Roch theorem to conclude main result of the chapter, i.e., the following theorem.

Theorem 1.3.7 (cf. Theorem 4.2.8)**.** *If $P \in \mathcal{D}_K$ is nonzero, then its index as a k-linear endomorphism of A satisfies*

$$\chi(P;A) = \sum_{i=1}^{r} \nu_i(P).$$

In particular this index exists, i.e., the kernel and the cokernel of P have finite dimensions over k.

We remark that an easy special case of this theorem appears in the book of Katz [Kat90] (cf. Propositions 4.3.1, 4.3.2). As an application of theorem above we give a new proof of Deligne's index formula (cf. Formula 4.2.11). The last section of the chapter is devoted to examples. Examples in the affine case are due to Katz, but we give new proofs based on Theorem 4.2.8. Examples in the affinoid case are new and follow from our previous results: Theorem 4.2.6 and Lemma 2.2.6.

1.4 Conventions and notation

For future reference, we record our basic setup.

1.4.1 General principles for the notation

Throughout the text we try to stick to the following notational rules.

(1) By letters $K, L, \ldots$ we usually denote a field. If K is nonarchimedean we usually write $\mathfrak{o}_K$ for its valuation ring, $\mathfrak{m} \subset \mathfrak{o}_K$ for the maximal ideal, and $k = \mathfrak{o}_K/\mathfrak{m}$ for the residue field.

(2) Rings are usually denoted by letters $R, S, \ldots$, or $A, B, \ldots$. Modules are usually denoted by letters $M, N, \ldots$. If A is a commutative K-algebra then we write $\mathcal{D}_A$ for the Grothendieck ring of K-linear differential operators. This is not misleading since K is always clear from the context.

(3) Topological spaces are denoted by letters $X, Y, \ldots$ and their open subsets are denoted by $U, V, \ldots$. For most of the time X stands for a smooth rigid analytic variety. Closed embeddings are usually denoted by $i : X \hookrightarrow Y$.

(4) We use 'mathscr' font to denote sheaves. For example $\mathscr{O}_X$ stands for the structure sheaf, $\mathscr{T}_X$ stands for the tangent sheaf, and $\mathscr{D}_X$ denotes the sheaf of differential

operators on X. One exception here is that we write Ω_X^k for the sheaf of differential k-forms and we write $\omega_X = \Omega_X^{\dim X}$ for the canonical sheaf.

(5) A norm on a normed space is always written as $|-|$. This may be potentially misleading, because sometimes we consider several normed spaces at the same time, but we always clarify in the text to which norm we refer, whenever there is a potential notational conflict.

(6) We do not use any special environment for introducing new definitions, because there are too many of those and it would make the text unreadable. Instead we use the *italic* font to mark that we introduce a new definition. For example the *free group of rank n* is by definition the fundamental group of the bouquet of n circles.

1.4.2 Multi-index notation

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When dealing with differential operators it is often convenient to use the multi-index notation which we here recall. A *multi-index* is an n-tuple

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n.$$

The *length* of α is

$$|\alpha| = \alpha_1 + \dots + \alpha_n.$$

We write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all i. If this is the case then the *binomial coefficient* is defined as

$$\binom{\alpha}{\beta} = \prod_i \binom{\alpha_i}{\beta_i}.$$

Let $K[x_1, \dots, x_n]$ be the polynomial ring over some field K and let $\partial_i = \frac{\partial}{\partial x_i}$ $(i = 1, \dots, n)$. We denote

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \qquad \partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}.$$

If $I = (1 \leq i_1 < \dots < i_k \leq n)$ is an ordered k-tuple we write $|I| = k$. It is convenient to write differential forms as

$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

We also use the above notation more generally, when X admits a local coordinate system $x_1, \dots, x_n$. In this situation we usually write $\partial_1, \dots, \partial_n$ for the dual basis of $dx_1, \dots, dx_n$.

Chapter 2

General theory

In this chapter we present theories of rigid analytic varieties and $\mathscr{D}$-modules in the generality needed for our purposes. Since our approach to the latter is mostly via homological algebra, we also recall some preliminary algebraic results. Most of the content of this chapter is not original. Our contribution consists of algebraic Lemmas 2.2.6, 2.2.7, 2.2.13, and 2.2.14 and of the discussion on the automatic continuity of differential operators (Theorem 2.3.12).

2.1 Rigid analytic varieties

In this section we recall basic results concerning rigid analytic varieties. For most of the time we follow [BGR84], [Hub96], and [FvdP04].

2.1.1 Functional analysis over nonarchimedean fields

A normed field $(K, |\quad|)$ is called *nonarchimedean* if it is complete with respect to the metric induced by the norm, and if for all $x, y \in K$ the *strong triangle inequality* (also called the *ultrametric inequality*)

$$|x+y| \leq \max\{|x|, |y|\}$$

holds. In what follows we simply write K to denote a nonarchimedean field. The subset

$$\mathfrak{o}_K = \{x \in K : |x| \leq 1\} \subset K$$

is a subring of K. It is a valuation ring and we say that K is *discretely valued* if $\mathfrak{o}_K$ is a discrete valuation ring. We denote by $\mathfrak{m} \subset \mathfrak{o}_K$ the maximal ideal and by $k = \mathfrak{o}_K/\mathfrak{m}$ the residue field. We say that K if *of equal characteristic* if $\operatorname{char} K = \operatorname{char} k$. Otherwise we say that K is *of mixed characteristic*. In this thesis we are mostly interested in discretely valued nonarchimedean fields of equal characteristic zero.

Example 2.1.1. The most basic examples of nonarchimedean fields are

(1) (finite extensions of) $\mathbb{Q}_p$ with (the unique extension of) the p-adic norm $|m| = p^{-v_p(m)}$.

(2) The field $k((t))$ of *formal Laurent series* over a field k with the norm given by $|f| = \exp(-\operatorname{ord}_0(f))$, where $\operatorname{ord}_0(f)$ stands for the order of the zero/pole of f at the origin.

(3) If K is any nonarchimedean field then the completion of its algebraic closure is again nonarchimedean (This is *Krasner's lemma* cf. [BGR84, page 146]).

The fields from examples (1) and (2) are discretely valued while the field in the last example is not. The p-adic fields are of mixed characteristic while the fields of formal Laurent series are of equal characteristic ($= \operatorname{char} k$). We write $\mathbb{C}_p$ for the completed algebraic closure of $\mathbb{Q}_p$.

Example 2.1.2. Let K be a discretely valued nonarchimedean field of equal characteristic and let $\varpi \in \mathfrak{o}_K$ be a uniformizer. Then $K \simeq k((\varpi))$ by Cohen's structure theorem. If $k \subset k'$ is a field extension then $k((\varpi)) \subset k'((\varpi))$ is an extension of nonarchimedean fields. Since most of the properties discussed in this thesis can be equally well checked after a base-change along a field extension, a reader who prefers more concrete mathematics may assume from now on that $K = \mathbb{C}((t))$. This will not decrease a generality of the discussed results in any major way.

A *Banach space* over K is a normed K-vector space V complete with respect to the metric induced by the norm. Here and elsewhere we require the norm to satisfy the strong triangle equality. We set

$$|V| = \{|v| : v \in V\} \subset \mathbb{R}_{\geq 0}.$$

If $S \subset V$ is any subset of V then we write $\operatorname{cl}(S)$ for its topological closure in V. If V_1, V_2 are Banach spaces over K and $L : V_1 \to V_2$ is a (linear) operator then it is continuous if and only if its *operator norm*

$$|L| = \sup\left\{\frac{|Lv_1|}{|v_1|} : v_1 \in V_1 \setminus \{0\}\right\}$$

is finite. In this situation one also says that L is *bounded* and we will use these terms interchangeably. The operator norm depends on the norms on V_1, V_2. If $V_1 = V_2 = V$ then the *spectral radius* of L is defined by the *Gelfand formula*

$$|L|_{\mathrm{sp}} = \inf_{n \geq 1} |L^n|^{\frac{1}{n}} = \lim_{n \to \infty} |L^n|^{\frac{1}{n}}.$$

The spectral radius depends only on the equivalence class of a norm on V, where two norms $|-|_1, |-|_2$ are said to be *equivalent* if there exist constants $C_1, C_2 > 0$ such that

$$C_1|-|_1 \leq |-|_2 \leq C_2|-|_2.$$

Continuity is the most basic property of a linear operator and it is useful to have some tools to test it. First of all, as in the classical case, we have

Proposition 2.1.3 (Closed Graph Theorem, [Sch02, Proposition 8.5])**.** *Let $\varphi : V \to W$ be a linear operator between two Banach spaces. The following are equivalent.*

(1) *φ is continuous.*

(2) *The graph $\Gamma(\varphi) = \{(v, \varphi(v)) : v \in V\} \subset V \times W$ is closed.*

(3) *If $\{v_n\} \subset V$ is a sequence such that $\lim v_n = 0$ and the limit $\lim \varphi(v_n)$ exists, then the latter limit is zero.*

Using the closed graph theorem C. E. Rickart [Ric50] introduced an invariant that measures how far φ is from being continuous. The *separating space* of $\varphi : V \to W$ is defined as

$$\mathfrak{S}(\varphi) = \{w \in W : \text{there exists a sequence } \{v_n\} \subset V \text{ with } \lim v_n = 0 \text{ and } \lim \varphi(v_n) = w\}$$

This is a linear subspace of W. The following properties are considered 'folklore' (cf. [Sin75, page 166]). Since we were not able to provide a suitable reference, we attach the proof for completeness.

Lemma 2.1.4. *Let $\varphi : V \to W$ be a (not necessarily continuous) operator between K-Banach spaces. Then*

(1) *φ is continuous if and only if $\mathfrak{S}(\varphi) = 0$.*

(2) *$\mathfrak{S}(\varphi) \subset W$ is a closed subspace.*

(3) *The composition of* φ *with the natural projection* $\pi : W \to W/\mathfrak{S}(\varphi)$ *is continuous.*

(4) *If* $T : W \to Z$ *is a continuous operator between Banach spaces then the composition* $T\varphi$ *is continuous if and only if* $T\mathfrak{S}(\varphi) = \{0\}$.

(5) *If* $T : W \to Z$ *is a continuous operator between Banach spaces then* $\mathrm{cl}(T\mathfrak{S}(\varphi)) = \mathfrak{S}(T\varphi)$.

(6) *If* $R : V \to V$ *and* $L : W \to W$ *are continuous operators such that* $\varphi R = L\varphi$ *then* $L\mathfrak{S}(\varphi) \subset \mathfrak{S}(\varphi)$.

Proof. (1) is just a reformulation of the equivalence of (1) and (3) in the closed graph theorem. For (2) assume that $\{w_n\} \subset \mathfrak{S}(\varphi)$ is a Cauchy sequence and let w be its limit in W. We have to show that $w \in \mathfrak{S}(\varphi)$. By the definition of $\mathfrak{S}(\varphi)$ for every positive integer n there exists $v_n \in V$ with $|v_n| \leq \frac{1}{n}$ and $|\varphi(v_n) - w_n| \leq \frac{1}{n}$. Then

$$|w - \varphi(v_n)| = |(w - w_n) + (w_n - \varphi(v_n))| \leq |w - w_n| + \frac{1}{n}.$$

We conclude that $\lim \varphi(v_n) = w$, i.e., that $w \in \mathfrak{S}(\varphi)$. For (3) first observe that by (2) the quotient $W/\mathfrak{S}(\varphi)$ is naturally a Banach space with the norm given by

$$|\pi(w)| = \inf\left\{|w + w'| : w' \in \mathfrak{S}(\varphi)\right\}.$$

To check that $\pi\varphi$ is continuous we have to check that $\mathfrak{S}(\pi\varphi) = 0$. Let $\{v_n\}$ be a sequence in V tending to zero, and assume that $\lim \pi\varphi(v_n) = \pi(w)$. Replacing $\{v_n\}$ by a subsequence we may assume that

$$|\pi\varphi(v_n) - \pi(w)| < \frac{1}{n}$$

for all n. By construction there exist $w_n \in \mathfrak{S}(\varphi)$ such that

$$|\varphi(v_n) + w_n - w| \leq \frac{1}{n}.$$

On the other hand, since $w_n \in \mathfrak{S}(\varphi)$ there exist $u_n \in V$ with $|u_n| \leq \frac{1}{n}$ and $|\varphi(u_n) - w_n| \leq \frac{1}{n}$. Therefore

$$|\varphi(v_n + u_n) - w| = |(\varphi(v_n) + w_n - w) + (\varphi(u_n) - w_n)| \leq \frac{2}{n}.$$

We conclude that $w \in \mathfrak{S}(\varphi)$, i.e., that $\pi(w) = 0$ and therefore $\mathfrak{S}(\pi\varphi) = 0$ as claimed. To prove (4) we only need to check that if $T\mathfrak{S}(\varphi) = \{0\}$ then the composition $T\varphi$ is continuous. This is a straightforward consequence of (3) because we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "\pi\varphi"'] & W \arrow[r, "T"] \arrow[d, "\pi"] & Z \\ & W/\mathfrak{S}(\varphi) \arrow[ur, dotted, "\overline{T}"'] & \end{tikzcd}$$

The dotted arrow exists (and is continuous) by assumption and the composition $\pi\varphi$ is continuous by (3). Therefore $T\varphi = \overline{T}\pi\varphi$ is continuous. We prove (5) as in [Sin75, page 166]. The inclusion $T\mathfrak{S}(\varphi) \subset \mathfrak{S}(T\varphi)$ is straightforward and the latter is closed by (2). Therefore we only need to verify that $\mathfrak{S}(T\varphi) \subset \mathrm{cl}(T\mathfrak{S}(\varphi))$. Consider the projection $\pi_Z : Z \to Z/\mathrm{cl}(T\mathfrak{S}(\varphi))$. We have $\pi_Z T\mathfrak{S}(\varphi) = \{0\}$ by construction and therefore the composition $\pi_Z T\varphi$ is continuous by (4). As π_Z is continuous we obtain from (4) that $\pi_Z\mathfrak{S}(T\varphi) = \{0\}$ and thus $\mathfrak{S}(T\varphi) \subset \ker \pi_Z = \mathrm{cl}(T\mathfrak{S}(\varphi))$. Assertion (6) is an easy computation. □

2.1.2 Nonarchimedean Banach algebras

If A is a K-algebra and a Banach space, then we say that it is a *Banach K-algebra* if $|a_1a_2| \leq |a_1||a_2|$ for all $a_1, a_2 \in A$ and $|1| = 1$. If this is the case and M is a normed A-module then we call it a *Banach module* if it is complete and $|am| \leq |a||m|$ for all $a \in A$ and $m \in M$.

Example 2.1.5. The K-algebra

$$K\langle x_1, \ldots, x_n\rangle = \left\{ \sum_{|\alpha|\geq 0} a_\alpha x^\alpha \in K[[x_1, \ldots, x_n]] : \lim_{|\alpha|\to\infty} |a_\alpha| = 0 \right\}$$

is called the (n-dimensional) *Tate algebra* over K. It carries the *Gauss norm*

$$\left| \sum_{|\alpha|\geq 0} a_\alpha x^\alpha \right| = \max_\alpha |a_\alpha|$$

which makes it into a Banach K-algebra. The ring

$$\mathfrak{o}_K\langle x_1, \ldots, x_n\rangle = \{ f \in K\langle x_1, \ldots, x_n\rangle : |f| \leq 1\}$$

is called the *integral Tate algebra*. These objects can be constructed in a purely algebraic manner. We have an isomorphism of topological rings

$$\mathfrak{o}_K\langle x_1, \ldots, x_n\rangle = \varprojlim_{k\geq 0} (\mathfrak{o}_K[x_1, \ldots, x_n]/\varpi^{k+1})$$

where $\varpi \in \mathfrak{o}_K$ is some pseudo-uniformizer and on the right hand side we consider the (ϖ)-adic topology. The Tate algebras are known to be noetherian (cf. [BGR84, page 164]). This is not completely obvious for general K but is quite clear if K is discretely valued. In this situation $\mathfrak{o}_K[x_1,\ldots,x_n]$ is noetherian (by Hilbert's basis theorem, because $\mathfrak{o}_K$ is) and thus

$$K\langle x_1,\ldots,x_n\rangle = \left(\varprojlim_{k\geq 0}(\mathfrak{o}_K[x_1,\ldots,x_n]/\varpi^{k+1})\right)[\varpi^{-1}]$$

is a localization of a ϖ-adic completion of a noetherian ring.

Let A be a (commutative) Banach K-algebra. We set

$$A^\circ = \{a \in A : \sup_{n\geq 0}|a^n| < \infty\}.$$

It is a subring of A because in the nonarchimedean norm we have $|\binom{n}{j}| \leq 1$ and therefore

$$|(a-b)^n| = \left|\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}\right| \leq \max_{0\leq i,j\leq n}|a^i||b^j|.$$

We say that $a \in A$ is *power-bounded* if $a \in A^\circ$ and we call A° the *ring of power-bounded elements* of A. We also set

$$A^{\circ\circ} = \{a \in A : \lim_{n\to\infty}|a^n| = 0\}.$$

It is an ideal in A°. Its elements are called *topologically nilpotent*. The ring

$$\widetilde{A} = A^\circ/A^{\circ\circ}$$

is called the *reduction* of A. If $\varphi : A \to B$ is a bounded homomorphism of Banach K-algebras then clearly $\varphi(A^\circ) \subset B^\circ$ and $\varphi(A^{\circ\circ}) \subset B^{\circ\circ}$. Therefore the assignment

$$A \mapsto \widetilde{A}$$

is a functor from the category of Banach K-algebras (and bounded K-algebra homomorphisms) to the category of k-algebras.

One nice property of noetherian Banach algebras is the following lemma which we use quite frequently.

Lemma 2.1.6 ([BGR84, Proposition 2, page 164])**.** *Let A be a (commutative) noetherian Banach algebra. Then every ideal in A is closed.*

In particular, any ideal in the Tate algebra is closed. If $A = K\langle x_1,\dots,x_n\rangle/I$, then we call A an *affinoid algebra*. Since it is a quotient of a Banach algebra by a closed ideal it is itself a Banach algebra, although it is not clear that the topological structure is independent from the choice of a presentation for A. This ambiguity will be soon clarified. First, we recall the basic algebraic properties of affinoid algebras.

Lemma 2.1.7. *Let A be an affinoid K-algebra. Then*

(1) *A is noetherian,*

(2) *if $m \subset A$ is a maximal ideal then the field extension $K \subset A/\mathfrak{m}$ is finite.*

Proof. The claims follow from [BGR84, Proposition 3, page 222], and [BGR84, Lemma 2, page 261]. □

We now deal with the topological structure of affinoid algebras. First let us recall that if $K \subset L$ is a finite extension of fields and K is nonarchimedean then there exists a unique extension of the norm on K to L which makes L into a nonarchimedean field. Now let $\mathfrak{m} \subset A$ be a maximal ideal in an affinoid algebra. By Lemma 2.1.7 the extension $K \subset A/\mathfrak{m}$ is finite and therefore for every $f \in A$ the value $|f \bmod \mathfrak{m}| \in |A/\mathfrak{m}|$ is well defined. We define the *supremum norm* of f to be

$$|f|_{\sup} = \sup_{\{\mathfrak{m}\subset A\}} |f \bmod \mathfrak{m}|$$

We also set

$$\rho(f) = \inf_{n\geq 1} |f^n|^{\frac{1}{n}} = \lim_{n\to\infty} |f^n|^{\frac{1}{n}}.$$

This is the spectral radius of the K-linear map $A \to A$ given by $a \mapsto fa$.

Lemma 2.1.8. *Let A be an affinoid K-algebra. Then every Banach K-algebra norm on A is equivalent to the one induced from the presentation $A = K\langle x_1,\dots,x_n\rangle/I$. Moreover, if A is reduced then*

(1) *$f \mapsto |f|_{\sup}$ is a K-Banach norm on A with the property that $|f| \in |\overline{K}|$ (here $\overline{K}$ is the algebraic closure of K).*

(2) *We have $|f|_{\sup} = \rho(f)$.*

(3) *We have the equalities*

$$A^{\circ} = \{a \in A : \rho(a) \leq 1\},$$

and

$$A^{\circ\circ} = \{a \in A : \rho(a) < 1\}.$$

Proof. See [BGR84, Proposition 2, Page 229], [BGR84, Proposition 2, Proposition 3, page 240], and [BGR84, Proposition 3, page 241]. □

Example 2.1.9. If A is the Tate algebra then A° is the integral Tate algebra and $\widetilde{A} = k[x_1, \dots, x_n]$. Moreover, the supremum norm on A agrees with the Gauss norm.

Remark 2.1.10. It is a classical result (cf. [ST74]) that every noetherian Banach $\mathbb{C}$-algebra has finite dimension as a $\mathbb{C}$-vector space, and therefore almost all interesting Banach $\mathbb{C}$-algebras are not noetherian. This is one of many differences between the classical and nonarchimedean functional analysis, where one rarely studies Banach K-algebras that are not noetherian. In fact, every Banach algebra discussed in this thesis will be noetherian.

2.1.3 Rigid analytic varieties

In this subsection we recall the basic definitions regarding Huber adic spaces. This theory has been in recent years developed in a very big generality but we use it in a very restricted case and therefore we only state definitions and necessary properties in generality needed later.

We start by considering the category of triplets $(X, \mathscr{O}_X, \{v_x\}_{x\in X})$, where X is a topological space, $\mathscr{O}_X$ is a sheaf of topological rings and v_x is an equivalence class of valuations on $\mathscr{O}_{X,x}$. A *morphism* $(X, \mathscr{O}_X, \{v_x\}_{x\in X}) \to (Y, \mathscr{O}_Y, \{v_y\}_{y\in Y})$ of such triplets is a map $f : X \to Y$ of topological spaces such that the induced map $\varphi : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism of sheaves of topological rings and valuations $v_{f(x)}$ and $v_x \circ \varphi$ are equivalent for all $x \in X$.

Remark 2.1.11. We write valuations multiplicatively. In particular, nonarchimedean norms introduced in the previous subsection (and, more generally, nonarchimedean seminorms) are examples of rank one valuations, provided they are multiplicative. If A is a ring and $x : A \to \Gamma \cup \{0\}$ is a valuation then we write $|a(x)|$ to denote $x(a)$. This notation is motivated by the following. If A is an affinoid K-algebra, $a \in A$, and $\mathfrak{p} \subset A$ is a maximal ideal then we would like to interpret (as in the classical theory of algebraic varieties) a as a function on the maximal spectrum of A. The problem is that (since K does not need to be algebraically closed) $L = A/\mathfrak{p}$ may be a finite extension of K and thus the value $a(\mathfrak{p}) = [a] \in A/\mathfrak{p}$ is well defined only up to the action of the Galois group $\mathrm{Gal}(L/K)$. On the other hand, the extension of the nonarchimedean norm from K to L is unique, so the value $|a(\mathfrak{p})|$ is well defined.

If B is a topological ring then we say that a valuation $x : B \to \Gamma \cup \{0\}$ is *continuous* if for any other valuation $\gamma \in \Gamma \cup \{0\}$ the sets

$$x^{-1}\left(\{\gamma' \in \Gamma : \gamma' \leq \gamma\}\right) \subset B$$

are open in B.

If A is an affinoid K-algebra then there is a natural way of assigning to A a triplet as above, denoted $\operatorname{Spa} A$. The points of this space are described as

$$\operatorname{Spa} A = \{\text{continuous valuations } x \text{ on } A : |a(x)| \leq 1 \text{ for all } a \in A^\circ\}$$

We say that a subset $U \subset \operatorname{Spa} A$ is *rational* if there exist $f_1, \ldots, f_n, g \in A$ such that

$$U = \{x \in \operatorname{Spa} A : |f_i(x)| \leq |g(x)| \neq 0 \text{ for all } i\}.$$

We declare these subsets to be generators for the topology on $\operatorname{Spa} A$. In fact, the rational subsets form a basis for the topology on $\operatorname{Spa} A$ (see [Hub96, page 39]). For a rational subset U as above we set

$$\mathscr{O}_{\operatorname{Spa} A}(U) = A\langle x_1, \ldots, x_n\rangle / (\{f_i - x_i g : i = 1, \ldots, n\})$$

These algebras are Banach K-algebras in a natural way and they depend only on U (and not on the choice of $f_1, \ldots, f_n, g$). In particular, we have

$$\mathscr{O}_{\operatorname{Spa} A}(\operatorname{Spa} A) = A.$$

Finally, if $x : A \to \Gamma \cup \{0\}$ is a continuous valuation then it extends in a unique way to the valuation $v_x : \mathscr{O}_{X,x} \to \Gamma \cup \{0\}$. The triplet $(\operatorname{Spa} A, \mathscr{O}_{\operatorname{Spa} A}, \{v_x\}_x)$ constructed in this way is called an *affinoid variety*. For the details of the construction above we refer the reader to [Hub96, pages 38-39]. We say that X is a *rigid analytic K-variety* if it is locally isomorphic to an affinoid variety.

Example 2.1.12. The n-dimensional *Tate polydisc* is by definition the rigid analytic variety

$$\mathbb{B}^n = \operatorname{Spa} K\langle x_1, \ldots, x_n\rangle.$$

If $n = 1$ we simply call $\mathbb{B}^1$ the *Tate disc*. Tate polydiscs are local (in the étale sense) models for the (smooth) rigid analytic varieties and thus they play a role similar to complex polydiscs in the theory of complex manifolds, or to open balls in the theory of $\mathscr{C}^\infty$ real manifolds. It would probably be more suggestive to denote the polydisc as $\mathbb{D}^n$ by analogy to the complex analytic theory but since this thesis is mostly devoted to $\mathscr{D}$-modules, the letter D is already used in too many different contexts. Therefore we settle for $\mathbb{B}^n$ by analogy to the $\mathscr{C}^\infty$ manifolds.

Remark 2.1.13. Tate polydiscs are by far the most important rigid analytic varieties in this thesis. All the main results remain meaningful and nontrivial in this case and actually most of the 'heavy lifting' is done precisely when $X = \mathbb{B}^n$.

For the future reference we recall that a topological space X is *quasi-compact* if every open cover of X has a finite subcover. Moreover X is *quasi-separated* if intersection of any two open quasi-compacts subsets is again quasi-compact. We note (see [Hub96, page 39]) that affinoid varieties are quasi-compact and quasi-separated. A morphism of rigid analytic varieties is a *Zariski closed embedding* if it locally corresponds to a surjective morphism of affinoid algebras.

We will now recall the notion of *reduction* of an affinoid variety. Let $X = \operatorname{Spa} A$ be such variety and let $x \in \operatorname{Spa} A$. Consider the subset

$$\mathfrak{p}_x = \{a \in A : |a(x)| = 0\} \subset A.$$

It is easy to verify that this is a closed prime ideal in A. In particular, X induces a nonarchimedean norm on the quotient field of $A/\mathfrak{p}_x$. We write $\kappa(x)$ to denote the completion of this field with respect to the induced norm. The map $A \to \kappa(x)$ induces the map $\widetilde{A} \to \widetilde{\kappa(x)}$. The *reduction map* is defined as

$$\pi : \operatorname{Spa} A \to \operatorname{Spec} \widetilde{A}; \quad x \mapsto \ker\left(\widetilde{A} \to \widetilde{\kappa(x)}\right).$$

We have the following

Lemma 2.1.14 ([Ber90, Proposition 2.4.4]). *With the above notation*

(1) *π is surjective.*

(2) *Let X be an irreducible component of* $\operatorname{Spec} \widetilde{A}$ *and let $\widetilde{\eta}$ be its generic point. Then there exists a unique $\eta \in \operatorname{Spa} A$ with $\pi(\eta) = \widetilde{\eta}$.*

(3) *If $|A|_{\sup} = |K|$ then $\widetilde{\kappa(\eta)} = \mathscr{O}_{\operatorname{Spec} \widetilde{A}, \widetilde{\eta}}$.*

Remark 2.1.15. Much of the foundational work in the theory of $\mathscr{D}$-modules on rigid analytic varieties has been done by Z. Mebkhout and L. Narváez Macarro in their paper [MNM91]. This work predates the work of Huber and has been written using the formalism of Tate, where open coverings are replaced by the so called admissible coverings. By [Hub96, 1.1.11] any affinoid variety in the sense of Tate can be seen as an affinoid variety in the sense of Huber and any admissible covering corresponds to an open covering. Therefore the work of Mebkhout and Narváez Macarro translates without changes to the formalism of Huber.

2.1.4 Quasi-coherent modules

In this subsection we recall the basic constructions regarding (quasi)coherent modules on rigid analytic varieties. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *coherent* if it is locally finitely presented, i.e., if there exists an open covering $X = \bigcup_i U_i$ such that for each index i there exists an exact sequence

$$\mathscr{O}_{U_i}^{\oplus a_2} \to \mathscr{O}_{U_i}^{\oplus a_1} \to \mathscr{F}_{|U_i} \to 0.$$

The category of coherent $\mathscr{O}_X$-modules enjoys many properties of the category of coherent modules on algebraic schemes. It is the case because if M is a finitely generated Banach module over an affinoid algebra then it turns out that the topological structure is already encoded in the algebraic structure. We recall below the basic results in this direction.

Lemma 2.1.16 ([BGR84, Chapter 3.7.3, Proposition 3 and Corollary 5])**.** *Let A be an affinoid K-algebra and let M be finitely generated A-module. Then*

(1) *There exists unique (up to equivalence) norm on M which makes M into a Banach A-module.*

(2) *Every homomorphism of finitely generated A-modules is continuous with respect to the topology induced by this equivalence class of norms.*

If we consider an affine scheme $Y = \operatorname{Spec} R$ then every R-module M gives a rise to a sheaf $\widetilde{M}$ on Y with the property that if $U_f = \operatorname{Spec} R[f^{-1}]$ is a distinguished open affine subset then

$$\widetilde{M}(U_f) = R[f^{-1}] \otimes_R M.$$

This is indeed a sheaf because the map $R \to R[f^{-1}]$ is flat. A similar result holds for affinoid varieties. We have

Lemma 2.1.17 ([BGR84, Chapter 7.3.2, Corollary 6])**.** *Let $X = \operatorname{Spa} A$ be an affinoid variety and let $U \subset X$ be an affinoid subdomain. Then the natural map $A \to \mathscr{O}_X(U)$ is flat.*

The fundamental theorem regarding coherent modules on rigid analytic varieties is due to Tate.

Theorem 2.1.18 (Tate's acyclicity theorem, [BGR84, Chapter 8.2.1, Theorem 1])**.** *Let X be an affinoid variety and let $\{U_i\}$ be a finite open covering of X by affinoid subdomains. Then the Čech complex*

$$0 \to \mathscr{O}_X(X) \to \bigoplus_i \mathscr{O}_X(U_i) \to \bigoplus_{i<j} \mathscr{O}_X(U_i \cap U_j) \to \dots \tag{2.1}$$

is exact.

As a corollary we obtain the following.

Theorem 2.1.19. *Let $X = \operatorname{Spa} A$ be an affinoid variety and let M be a (not necessarily finitely generated) A-module. Then*

(1) *The presheaf on affinoid subdomains*

$$\widetilde{M} : U \mapsto \mathscr{O}_X(U) \otimes_A M$$

is a sheaf.

(2) $H^i(X, \widetilde{M}) = 0$ *for all* $i > 0$.

Proof. Since the map $A \to \mathscr{O}_X(U)$ is flat by Lemma 2.1.17, both claims follow from tensoring the exact complex (2.1) with M. □

Remark 2.1.20. At this point we remark that every coherent sheaf on an affinoid variety is necessarily of form $\widetilde{M}$ for some finitely generated module M. This is no longer true if we are interested in what would be quasi-coherent modules, i.e., modules that on every affinoid open are locally of form $\widetilde{M}$ for some possibly not finitely generated M. This difference between the categories of affine schemes and affinoid varieties will cause some technical difficulties later.

2.1.5 Differential forms

We now discuss modules and sheaves of differential forms and the author feels that he should explain himself for this rather long discussion. We have decided to write it mainly because the book of Fresnel and Van der Put [FvdP04] is a very standard reference and since we want to use a slightly different definition (which in our opinion is much more natural) that the one given there, such definition needs to be justified. In particular the construction from [FvdP04] is used by Mebkhout and Narvaéz Macarro in their work [MNM91], which is one of our main sources.

First of all, it is clear that the algebraic definition of Kähler differentials is not well-suited for our purposes.

Example 2.1.21 ([FvdP04, Remarks 3.6.2]). Consider the Tate algebra $A = K\langle x\rangle$, where K is a field of characteristic zero. It is reasonable to expect that the module of K-differentials on A is Adx. Now let L be the fraction field of A. Then L has infinite transcendence dimension over K , so

$$\Omega^{\text{Kähler}}_{A/K} \otimes_K L = \Omega^{\text{Kähler}}_{L/K}.$$

It is an L-vector space of infinite dimension. It follows that $\Omega^{\text{Kähler}}_{A/K}$ is not even finitely generated as an A-module.

To fix this problem one usually considers instead the *universal finite differential module* $\Omega^f_{A/K}$, which is defined (for all K-algebras A, and K not necessarily nonarchimedean) by the universal property, that there exists a K-derivation $d : A \to \Omega^f_{A/K}$, and for any K-derivation $\delta : A \to M$ with M finitely generated there exists a unique A-module homomorphism that makes the diagram

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "d"] & M \\ \Omega^f_{A/K} \arrow[ur] & \end{tikzcd}$$

commutative. In other words, $\Omega^f_{A/K}$ satisfies the same universal property that the module of Kähler differentials, but only with respect to finitely generated A-modules. This definition is well-suited for affinoid algebras because of the following result.

Proposition 2.1.22 ([FvdP04, Theorem 3.6.1])**.** *If A is an affinoid K-algebra, then the module $\Omega^f_{A/K}$ exists and is finitely generated. Moreover,*

(1) *If $A = K\langle x_1,\dots,x_n\rangle$ then*
$$\Omega^f_{A/K} = \bigoplus_{i=1}^n Adx_i.$$

(2) *If $A = K\langle x_1,\dots,x_n\rangle/(f_1,\dots,f_m)$ then*
$$\Omega^f_{A/K} = \bigoplus_{i=1}^n Adx_i/(df_1,\dots,df_m).$$

While useful from the technical point of view, the definition of finite differentials does not seem to be the right one from the point of view of general theory. When dealing with Banach K-algebras it seems to be very reasonable to take into account the topological structure and the above definition is well-behaved only because Lemma 2.1.16 asserts that for affinoid K-algebras the topological structure on a finitely generated module is determined by the algebra.

Example 2.1.23. We will later see (cf. Example 2.3.10) that there exists a lot of $\mathbb{Q}_p$-linear derivations of $\mathbb{C}_p$ and that all of them are discontinuous. This shows that $\Omega^f_{\mathbb{C}_p/\mathbb{Q}_p}$ is nonzero. On the other hand, $\mathbb{C}_p$ plays the role of the algebraic closure of $\mathbb{Q}_p$ in the nonarchimedean geometry, so by analogy to the Kähler differentials we would expect any reasonable construction of differentials to be zero for the extension $\mathbb{Q}_p \subset \mathbb{C}_p$.

Let A be a Banach K-algebra. A *universal continuous differentials module* is a Banach A-module $\Omega^{\mathrm{cont}}_{A/K}$ together with a continuous derivation $d : A \to \Omega^{\mathrm{cont}}_{A/K}$ such that for any Banach A-module M and any continuous derivation $\delta : A \to M$ there exist a unique continuous homomorphism $\Omega^{\mathrm{cont}}_{A/K} \to M$ making the diagram

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "d"'] & M \\ \Omega^{\mathrm{cont}}_{A/K} \arrow[ur] & \end{tikzcd}$$

commutative.

Lemma 2.1.24 (cf. [FvdP04, Remarks 3.6.2])**.** *Let A be an affinoid K-algebra. Then* $\Omega^{\mathrm{cont}}_{A/K} = \Omega^f_{A/K}$.

Proof. We claim that: (1) if $\delta : A \to M$ is a continuous derivation into a Banach A-module, then its image is contained in a finitely generated submodule of M, and (2) the universal derivation $d : A \to \Omega^f_{A/K}$ is continuous (by Lemma 2.1.22 $\Omega^f_{A/K}$ is a finitely generated A-module so it has a natural Banach module structure by Lemma 2.1.16). Since every A-module homomorphism from a finitely generated A-module to a Banach A-module is continuous (cf. [Hub96, Page 76]), this will show that $\Omega^f_{A/K}$ satisfies the universal property of $\Omega^{\mathrm{cont}}_{A/K}$.

To show (1) let us consider a projection $\pi : K\langle x_1 \ldots, x_n\rangle \to A$ and let $X_i = \pi(x_i)$. Then $X_1, \ldots, X_n$ *topologically generates* A, i.e., every element $a \in A$ can be written as a convergent power series $a = \sum_\alpha a_\alpha X^\alpha$ with $a_\alpha \in K$. By the continuity of δ we have

$$\delta(a) = \sum_{i=1}^{n} \left(\sum_{\alpha} a_\alpha \alpha_i X^{\alpha - \alpha_i e_i} \right) \delta(X_i),$$

and therefore the image of δ is contained in the A-module spanned by $\delta(X_1), \ldots, \delta(X_n)$. To show (2) first note that the map

$$d : K\langle x_1, \ldots, x_n\rangle \to \Omega^f_{K\langle x_1,\ldots,x_n\rangle/K} = \bigoplus_{i=1}^{n} K\langle x_1, \ldots, x_n\rangle dx_i$$

is continuous. By Lemma 2.1.22 we have a commutative diagram

$$\begin{array}{ccc} K\langle x_1,\dots,x_n\rangle & \xrightarrow{d} & \Omega^f_{K\langle x_1,\dots,x_n\rangle/K} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ A & \xrightarrow{\quad d \quad} & \Omega^f_{A/f} \end{array}$$

and π' is continuous as it is a homomorphism of finitely generated $K\langle x_1,\dots,x_n\rangle$-modules. Therefore $d\pi$ is continuous and thus d is continuous because π is a quotient map by the open mapping theorem. □

Example 2.1.25. We will show that $\Omega^{\mathrm{cont}}_{\mathbb{C}_p/\mathbb{Q}_p} = 0$. Let V be a Banach $\mathbb{Q}_p$-vector space and let $\gamma : \mathbb{C}_p \to V$ be a continuous derivation. We have to show that γ is zero, and to do so we only need to show that for every $b \in \overline{\mathbb{Q}}_p$ we have $\gamma(b) = 0$. As b is algebraic over $\mathbb{Q}_p$ there exists minimal polynomial equation

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0, \tag{2.2}$$

where $a_i \in \mathbb{Q}_p$. Differentiating this equation we obtain

$$\left(nb^{n-1} + (n-1)a_{n-1}b^{n-2} + \cdots + a_1\right)\gamma(b) = 0.$$

The expression in the bracket is nonzero by the minimality of (2.2), because the characteristic of $\mathbb{Q}_p$ is zero. Therefore $\gamma(b) = 0$. This example illustrates that the universal continuous differential module is better suited for the study of nonarchimedean Banach algebras than the universal finite differential module.

As a straightforward consequence of Lemma 2.1.24 we obtain the following.

Proposition 2.1.26. *Let A be an affinoid K-algebra. Then every K-linear derivation of A is continuous.*

We finish this subsection by giving the definition of the *cotangent sheaf*.

Lemma 2.1.27 ([Hub96, Page 78])**.** *Let X be a rigid analytic variety. Then there exists a coherent sheaf Ω^1_X on X with the property that for every affinoid open subset $U \subset X$ we have $\Omega^1_X(U) = \Omega^{\mathrm{cont}}_{\mathscr{O}_X(U)/K}$.*

We further define *sheaves of differential k-forms* as $\Omega^k_X = \bigwedge^k \Omega^1_X$, and the *tangent sheaf* $\mathscr{T}_X = \mathscr{H}om_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)$.

2.1.6 Smoothness

If X is a rigid analytic variety then we say that it is *smooth* if the cotangent sheaf Ω^1_X is locally free of rank $n = \dim X$. The notion of an étale morphism of rigid analytic varieties is defined by analogy to algebraic geometry (cf. [Hub96, Chapter 1.6]). For our purposes we only need to know that to give an étale morphism from a smooth variety X to a Tate polydisc is equivalent to giving exact forms $dx_1, \dots, dx_n$ that form a basis of Ω^1_X. Following this idea, if $U \subset X$ is an open affinoid subset then we say that the elements $x_1, \dots, x_n \in \mathscr{O}_X(U)$ form a *local coordinate system* on U if $\Omega^1_{X|U}$ is a free $\mathscr{O}_U$-module and the elements $dx_1, \dots, dx_n \in \Omega^1_X(U)$ form a basis for $\Omega_{X|U}$. In this case we also say that U *admits a coordinate system*. If X itself admits a coordinate system we say that it *admits a global coordinate system*. Note that by the definition this implies that X is affinoid. Since every point $x \in X$ has a neighbourhood that admits such a morphism we see that open subsets of X admitting a coordinate system form a basis for the topology on X. These definitions may be generalized to the relative case when $i : X \hookrightarrow Y$ is a Zariski closed embedding. We say that such an embedding admits a coordinate system if there exists a coordinate system $y_1, \dots, y_n$ on Y such that X is cut out by the ideal $\mathcal{I} = (y_{r+1}, \dots, y_n)$ and the images of $y_1, \dots, y_r$ in $\mathscr{O}_Y/\mathcal{I} = \mathscr{O}_X$ form a coordinate system on X. The following lemma has been taken from the notes of B. Zavyalov [Zav, Lemma 5.8] (see also [Sta24, Tag 0FUE] for an analogous statement for algebraic schemes).

Lemma 2.1.28. *Let $i : X \hookrightarrow Y$ be a Zariski closed immersion of smooth rigid analytic varieties. Then for every $x \in X$ there exists an affinoid open $x \in U_x \subset Y$ and an étale morphism $h : U_x \to \mathbb{B}^{\dim Y}$ such that the following diagram is cartesian*

$$\begin{tikzcd} U_x \cap X \arrow[r] \arrow[d] & \mathbb{B}^{\dim X} \arrow[d] \\ U_x \arrow[r] & \mathbb{B}^{\dim Y}. \end{tikzcd}$$

The vertical arrow on the left is induced by i and the vertical arrow on the right is the inclusion of the vanishing locus of the first $(\dim Y - \dim X)$ *coordinates.*

Lemma 2.1.28 implies that for every Zariski closed embedding $X \hookrightarrow Y$ there exists an open covering $\{U_i\}$ of Y such that embeddings $X \cap U_i \hookrightarrow U_i$ admit coordinate systems for all i. We use this basic observation several times throughout the text without mentioning it explicitly.

2.2 Homological algebra

In this section we collect results from homological algebra that will be used in the following chapters.

2.2.1 Rings and modules

Let R be a ring (here and everywhere else a *ring* is an associative unital ring). The *opposite ring* of R is defined to be R as an additive group with the multiplication given by

$$r_1 \odot r_2 := r_2 r_1$$

for $r_1, r_2 \in R$. We write R^{op} to denote the opposite ring. If M is a left (resp. right) R-module then it is also a right (resp. left) R^{op} in a natural way. We simply define

$$m.r := rm \qquad (\text{resp. } r.m := mr)$$

for all $r \in R^{\mathrm{op}}$ and $m \in M$. We write $\mathrm{Mod}(R)$ for the category of left R-modules. By the above observation the category of right R-modules is naturally equivalent to the category of left R^{op}-modules. Therefore we slightly abuse the notation and we write $\mathrm{Mod}(R^{\mathrm{op}})$ to denote the category of right R-modules. We also denote by $\mathrm{Mod}_f(R)$ the category of finitely generated left R-modules and by $D^b_f(R)$ the bounded derived category of left R-modules with finitely generated cohomology.

An *involution* on a ring R is a ring homomorphism

$$\iota : R \to R^{\mathrm{op}} : r \mapsto r^t$$

such that $\iota^2 = \mathrm{Id}$. Such a homomorphism is necessarily an isomorphism and thus if ι exists then the categories $\mathrm{Mod}(R)$ and $\mathrm{Mod}(R^{\mathrm{op}})$ are equivalent.

Example 2.2.1. Let $R = \mathrm{M}_n(K)$ be the matrix ring. Then the map which assigns to each matrix its transpose is an involution of R. This justifies the notation $r \mapsto r^t$.

If M is a left R-module then its *projective dimension* is defined as the length of its minimal projective resolution

$$\mathrm{pd}(M) = \min\{n : \text{there exists a projective resolution } 0 \to P_n \to \cdots \to P_0 \to M \to 0\}.$$

We recall the following lemma.

Lemma 2.2.2 ([Wei94, Lemma 4.1.6]). *Let M be a left R-module. The following are equivalent.*

(1) $\mathrm{pd}(M) \leq d$.

(2) $\mathrm{Ext}_R^{d+1}(M,N) = 0$ *for all left R-modules N.*

(3) $\mathrm{Ext}_R^k(M,N) = 0$ *for all left R-modules N and all $k > d$.*

The *left global dimension* of R is then defined as

$$l.gl.\dim(R) = \sup\{\mathrm{pd}(M) : M \in \mathrm{Mod}(R)\}.$$

The *right global dimension* is defined as $l.gl.\dim(R^{\mathrm{op}})$. In general the left and the right global dimensions of R may be different but if R is left and right noetherian then they are equal (see [Wei94, Exercise 4.1.1]). In this case we simply talk about the *global dimension* of R

$$gl.\dim(R) = l.gl.\dim(R) = l.gl.\dim(R^{\mathrm{op}}).$$

From the definition $gl.\dim(R)$ is either a nonnegative integer or infinity. In the first case we say that R *has a finite global dimension*.

2.2.2 Duality and modules of minimal dimension

For the purpose of this subsection we assume that R is a left and right noetherian ring of finite global dimension. We set $n = gl.\dim(R)$. Following [MNM91, 1.2] we say that a finitely generated left (resp. right) R-module M is *of minimal dimension* if either $M = 0$ or

$$\mathrm{grade}_R(M) := \inf\{i : \mathrm{Ext}_R^i(M,R) \neq 0\} = n.$$

For such module we set

$$M^* = \mathrm{Ext}_R^n(M,R)$$

Note that by assumption $\mathrm{pd}(M) \leq n$ and therefore by Lemma 2.2.2 $\mathrm{Ext}_R^i(M,R) = 0$ for all $i \neq n$. We call M^* the *dual* of M. This name is justified by the following (well known) lemma. Since it is an important ingredient in the proof of Theorem 3.1.1 we sketch the proof for completeness.

Lemma 2.2.3. *Let M be a left (resp. right) R-module of minimal dimension. Then M^* is a right (resp. left) R-module of minimal dimension and $M^{**} = M$.*

Proof. It is well known that if P is a finitely generated projective left (resp. right) module, then its dual $P^\vee = \mathrm{Hom}_R(P,R)$ is a finitely generated projective right (resp. left) module and the natural map $P \to P^{\vee\vee}$ is an isomorphism. Let M be a left (resp. right) R-module of minimal dimension.

Since R is noetherian we know that M admits a finite projective resolution by finitely generated projective modules. Let $P_\bullet$ be such resolution and let $Q_\bullet = \mathrm{Hom}_R(P_{-\bullet},R)[n]$. We have $H_i(Q_\bullet) = \mathrm{Ext}_R^{n-i}(M,R)$ and therefore $Q_\bullet$ is a projective resolution of M^*. By reflexivity of finite projective modules we have $P_\bullet = \mathrm{Hom}_R(Q_{-\bullet},R)[n]$ and therefore

$$\mathrm{Ext}_R^i(M^*,R) = H_{n-i}(P_\bullet) = \begin{cases} 0 \text{ if } i \neq n \\ M \text{ if } i = n \end{cases} \tag{2.3}$$

This shows that M^* is of minimal dimension and that $M^{**} = M$. □

It is convenient to consider the derived version of the duality discussed above. Under our assumptions we have a well defined *duality functor*

$$\mathbb{D}_R = \mathbf{R}\mathrm{Hom}_R(-,R) : D_f^b(R) \to D_f^b(R^{\mathrm{op}}).$$

Its basic properties are contained in the following lemma.

Lemma 2.2.4. *With the above notation and assumptions the following hold.*

(1) *We have an equality $M = \mathbb{D}_{R^{\mathrm{op}}}\mathbb{D}_R(M)$.*

(2) *The duality functor is an equivalence of categories satisfying $\mathbb{D}_{R^{\mathrm{op}}}\mathbb{D}_R = \mathrm{Id}$.*

(3) *The natural map $\mathbf{R}\mathrm{Hom}_R(M^\bullet,N^\bullet) \to \mathbf{R}\mathrm{Hom}_{R^{\mathrm{op}}}(\mathbb{D}_R(N^\bullet),\mathbb{D}_R(M^\bullet))$ is an isomorphism.*

(4) *Let $R \to S$ be a ring homomorphism of left and right noetherian rings of finite global dimension. Assume that S is flat as a right R-module. Then for any $M^\bullet \in D_f^b(R)$ we have $\mathbb{D}_S(S \otimes_R M^\bullet) = \mathbb{D}_R(M^\bullet) \otimes_R S$.*

Proof. Parts (1)–(3) are well known, see for example [Meb89, p. 49], [HTT08, D.4]. Possibly (4) is known to the experts but we were unable to provide a suitable reference, so we give a proof.

If M is a finitely generated (and thus finitely presented by noetherianity) R-module then $\mathrm{Hom}_R(M,R) \otimes_R S = \mathrm{Hom}_S(S \otimes_R M, S)$ since S is R-flat. Under our assumptions every object in $D_f^b(R)$ is represented by a bounded complex of finitely generated projective modules. If $M^\bullet$ is such complex then

$$\mathbf{R}\mathrm{Hom}_R(M^\bullet,R) \otimes_R S = \mathbf{R}\mathrm{Hom}_S(S \otimes_R M^\bullet, S),$$

which finishes the proof. □

Example 2.2.5. If M is of minimal dimension then definitions of duality for modules of minimal dimension and duality on the derived category are connected by the formula $\mathbb{D}_R(M) = M^*[-n]$.

2.2.3 Euler characteristic of a complex

In this subsection K is a field. In Lemma 2.2.6, where we additionally assume that K is nonarchimedean and discretely valued, we follow our usual notational conventions.

Let $C^\bullet$ be a bounded complex of K-vector spaces and assume that $\dim_K H^i(C^\bullet) < \infty$ for all i. The *Euler characteristic* of $C^\bullet$ is then defined as an alternating sum

$$\chi(C^\bullet) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_K H^i(C^\bullet).$$

One of the basic properties of the Euler characteristic is that it is additive on exact sequences in the sense that if

$$0 \to C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to 0$$

is a short exact sequence of complexes as above, then

$$\chi(C_2^\bullet) = \chi(C_1^\bullet) + \chi(C_3^\bullet).$$

The following is less obvious.

Lemma 2.2.6. *Let K be a discretely valued nonarchimedean field. Let $C^\bullet$ be a complex of complete (for the $\mathfrak{m}$-adic topology), torsion-free $\mathfrak{o}_K$-modules and assume that all k-vector spaces $H^i(C^\bullet \otimes_{\mathfrak{o}_K} k)$ have finite dimensions. Then*

(1) *All $\mathfrak{o}_K$-modules $H^i(C^\bullet)$ are finitely generated and therefore also all K-vector spaces $H^i(C^\bullet \otimes_{\mathfrak{o}_K} K)$ have finite dimensions.*

(2) *If $C^\bullet$ is bounded then*

$$\chi(C^\bullet \otimes_{\mathfrak{o}_K} k) = \chi(C^\bullet \otimes_{\mathfrak{o}_K} K) \tag{2.4}$$

i.e., the Euler characteristic of $C^\bullet$ on the special and the generic fibers are equal.

To prove Lemma 2.2.6 we need the following variant of Nakayama's Lemma.

Lemma 2.2.7. *Let*

$$V \to W \to Q \to 0$$

be an exact sequence of $\mathfrak{o}_K$-modules. Assume that V complete, W is separated and $Q \otimes_{\mathfrak{o}_K} k$ is finitely generated. Then Q is finitely generated.

Proof. Since the tensor product is right exact we have the following commutative diagram with exact rows.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d] & W \arrow[r, "\psi"] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\ \overline{V} \arrow[r, "\overline{\varphi}"] & \overline{W} \arrow[r, "\overline{\psi}"] & \overline{Q} \arrow[r] & 0 \end{tikzcd}$$

Pick generators $\overline{q}_1, \dots, \overline{q}_n \in \overline{Q} = Q \otimes_{\mathfrak{o}_K} k$ and let $q_1, \dots, q_n \in Q$ denote lifts of these elements to Q. Let $w_1, \dots, w_n \in W$ satisfy $\psi(w_i) = q_i$. To prove the lemma it suffices to show that for any $x \in W$ there exist $r_1, \dots, r_n \in \mathfrak{o}_K$ and $v \in V$ such that $w = \sum_{i=1}^n r_i w_i + \psi(v)$. Since $\overline{W}$ is generated modulo $\operatorname{im}\varphi$ by $\overline{w}_1, \dots, \overline{w}_n$, there exist $r_1^0, \dots, r_n^0 \in \mathfrak{o}_K$, $v_0 \in V$ and $x_1 \in W$ such that

$$x = \sum_{i=1}^n r_i^0 w_i + \varphi(v_0) + \varpi x_1$$

We can repeat this process for x_1 to find inductively elements $r_i^0, r_i^1, r_i^2, \dots \in \mathfrak{o}_K$, $v_0, v_1, \dots \in V$ and $x_1, x_2, \dots \in W$ such that for every $m \geq 1$

$$x = \sum_{i=1}^n \left(\sum_{j=0}^m \varpi^j r_i^j\right) w_i + \sum_{j=0}^m \varpi^j \varphi(v_j) + \varpi^{m+1} x_{m+1}.$$

Since $\mathfrak{o}_K$ is complete there also exist $r_i = \lim_{m\to\infty} \sum_{j=0}^m \varpi^j r_i^j$. Since V is complete there exists $v = \operatorname{im}_{m\to\infty} \sum_{j=0}^m \varpi^j v_j$ and therefore $\varphi(v) = \lim_{m\to\infty} \sum_{j=0}^m \varpi^j \varphi(v_j)$. Since W is separated we have

$$x - \sum_{i=1}^n r_i w_i - \varphi(x) \in \bigcap_{m\geq 1} \varpi^m W = \{0\}$$

and hence $x = \sum_{i=1}^n r_i w_i + \varphi(v)$. $\square$

Proof of Lemma 2.2.6. Recall that a module over a discrete valuation ring is flat if and only if it is torsion-free. In particular, images of d^i are also flat and we may invoke the Künneth formula [Wei94, Theorem 3.6.1]. We have exact sequences

$$0 \to H^i(C^\bullet) \otimes_{\mathfrak{o}_K} k \to H^i(C^\bullet \otimes_{\mathfrak{o}_K} k) \to \operatorname{Tor}_1^{\mathfrak{o}_K}(H^{i+1}(C^\bullet), k) \to 0. \tag{2.5}$$

To prove the first assertion we consider the exact sequences

$$C^{n-1} \xrightarrow{d^{n-1}} \ker d^n \to H^n(C^\bullet) \to 0 \tag{2.6}$$

By (2.5) and our assumptions the dimensions $\dim_k H^n(C) \otimes_{\mathfrak{o}_K} k$ are finite. Moreover by assumption C^n are all complete and thus $\ker d^n$ are separated modules as they are submodules of complete (and thus separated) modules. Therefore we may apply Lemma 2.2.7 to sequences (2.6) to conclude the first part of the lemma.

For the second part, recall that it follows from the classification of finitely generated modules over discrete valuation rings that if M is such module then

$$\dim_k M \otimes_{\mathfrak{o}_K} k - \dim_K M \otimes_{\mathfrak{o}_K} K = \dim_k \mathrm{Tor}_1^{\mathfrak{o}_K}(M,k). \tag{2.7}$$

Since $-\otimes_{\mathfrak{o}_K} K$ is the same as localization at ϖ, it is an exact functor. The first part of the lemma together with (2.5) and (2.7) imply formula (2.4). Indeed, we have

$$\begin{aligned}
\chi(C^\bullet \otimes_{\mathfrak{o}_K} k) &= \sum (-1)^i \dim_k H^i(C^\bullet \otimes_{\mathfrak{o}_K} k) \\
&= \sum (-1)^i \dim_k H^i(C^\bullet) \otimes_{\mathfrak{o}_K} k + \sum (-1)^i \dim_k \mathrm{Tor}_1^{\mathfrak{o}_K}(H^{i+1}(C^\bullet), k) \\
&= \sum (-1)^i (\dim_k H^i(C^\bullet) \otimes_{\mathfrak{o}_K} k - \dim_k \mathrm{Tor}_1^{\mathfrak{o}_K}(H^i(C^\bullet), k)) \\
&= \sum (-1)^i \dim_K H^i(C^\bullet) \otimes_{\mathfrak{o}_K} K \\
&= \sum (-1)^i \dim_K H^i(C^\bullet \otimes_{\mathfrak{o}_K} K) = \chi(C^\bullet \otimes_{\mathfrak{o}_K} K).
\end{aligned}$$

This finishes the proof. □

Consider a complex of K-vector spaces with only two nonzero entries

$$V^1 \xrightarrow{f} V^2. \tag{2.8}$$

We say that f *has an index* if both $\ker f$ and $\operatorname{coker} f$ are of finite dimension. The *index* of f is the Euler characteristic of the complex (2.8). It is usually denoted as $\chi(f; V^1, V^2)$ or, if $V^1 = V^2 = V$, as $\chi(f; V)$.

Example 2.2.8. If V^1, V^2 are finitely dimensional then f has an index and

$$\chi(f; V^1, V^2) = \dim_K V^1 - \dim_K V^2,$$

so in this situation the index does not depend on f.

The index enjoys all the arithmetic properties of the Euler characteristic. It also distributes additively with respect to the composition of linear maps, i.e., for any two morphisms $V^1 \xrightarrow{f^1} V^2$ and $V^2 \xrightarrow{f^2} V^3$ the equality

$$\chi(f^2 f^1; V^1, V^3) = \chi(f^1; V^1, V^2) + \chi(f^2; V^2, V^3)$$

holds, provided both f^1 and f^2 have indices.

Example 2.2.8 shows that the problem of computing an index of a linear map is the most interesting if V^1 and V^2 are of infinite dimension. This is in general a difficult problem but it can be sometimes reduced to the finitely dimensional case if both V^1 and V^2 are filtered and f preserves the filtration. By a *filtration* on a vector space V we understand an increasing family $F_\bullet V$ of subspaces of V such that $\bigcup_n F_n V = V$ and $F_m V = 0$ for $m \ll 0$. The following lemma is sufficient for our purposes.

Lemma 2.2.9. *Let $(V, F_\bullet V)$ be a filtered vector space and let $f : V \to V$ be an endomorphism. Assume that*

(1) *There exists an integer $\alpha \geq 0$ such that $f(F_n V) \subset F_{n+\alpha} V$ for all n.*

(2) *There exists an integer $\beta \geq 0$ such that for all $n \geq \beta$ the induced maps $\mathrm{gr}^F_n V \to \mathrm{gr}^F_{n+\beta} V$ are isomorphisms.*

Then f has an index if and only if its restriction $f : F_\beta V \to F_{\alpha+\beta} V$ has an index. If this is the case then these indices are equal.

Corollary 2.2.10. In the situation above and with additional assumption that $F_n V$ are all finite dimensional we have $\chi(\varphi; V) = \dim V_\beta - \dim V_{\alpha+\beta}$ (cf. Example 2.2.8).

To prove Lemma 2.2.9 we first recall the following (well known) property of morphisms of filtered vector spaces.

Lemma 2.2.11. *Let $(V, F_\bullet V)$, $(W, F_\bullet W)$ be two filtered vector spaces and let $f : V \to W$ be a linear map preserving these filtrations (i.e., $f(F_n V) \subset F_n W$ for all integers n). Assume that the induced morphisms $\mathrm{gr} f : \mathrm{gr}^F V \to \mathrm{gr}^F W$ is an isomorphism. Then f is an isomorphism.*

Proof. Without loss of generality $F_n V = 0$ for $n < 0$. Then also $F_n W = 0$ for $n < 0$ since otherwise $\mathrm{gr} f$ is not surjective. To prove that f is an isomorphism it suffices to show that it is an isomorphism for all $n \geq 0$. We do it by induction on n. The case $n = 0$ follows from our assumptions. In general our claim follows from the application of the snake lemma to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{n-1}V & \longrightarrow & F_nV & \longrightarrow & \mathrm{gr}^F V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{n-1}W & \longrightarrow & F_nW & \longrightarrow & \mathrm{gr}^F W & \longrightarrow & 0.
\end{array}$$

□

Proof of Lemma 2.2.9. Consider a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_\beta V & \longrightarrow & V & \longrightarrow & V/F_\beta V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_{\alpha+\beta} V & \longrightarrow & V & \longrightarrow & V/F_{\alpha+\beta} V & \longrightarrow & 0 \end{array}$$

Now the vector spaces $V/F_\beta V$ and $V/F_{\alpha+\beta}V$ are naturally filtered. By assumption (1) the map $V/F_\beta V \to V/F_{\alpha+\beta}V$ induced by f preserves this filtration and by assumption (2) the induced map on the graded pieces is an isomorphism. From Lemma 2.2.11 the map $V/F_\beta V \to V/F_{\alpha+\beta}V$ is an isomorphism. It follows from the snake lemma that we have natural isomorphisms

$$\ker(f) = \ker(F_\beta V \to F_{\alpha+\beta}V), \qquad \operatorname{coker}(f) = \operatorname{coker}(F_\beta V \to F_{\alpha+\beta}V),$$

and the proof follows. □

Remark 2.2.12. Our terminology of 'having an index' and the notation for it is taken from the work [Mal72] of B. Malgrange. We use it because it seems to be the most common among algebraic geometers. Since our work has some intersection with (nonarchimedean) functional analysis we remark that among people working in this area it is more common to call an operator f that has an index a *Fredholm operator* and to denote its index by $\operatorname{ind}(f)$ (cf. [Sch02, Chapter 22]).

2.2.4 Two technical lemmas

It is possible that Lemmas 2.2.13 and 2.2.14 below are known to the experts but we are not aware of any published proof in the form that we need. We will use these results in a very special case while proving Theorem 3.1.1, but since the proofs would be neither easier nor shorter after restricting to this special case, we present them in a more general setting.

For the purpose of this subsection we assume that B_0 is a (not necessarily commutative) ring and $\pi \in B_0$ is a central element that is not a zero divisor. We set $B = B_0[\pi^{-1}]$ and $\overline{B} = B_0/\pi B_0$. Because π is not a zero divisor the natural map $B_0 \to B$ is injective and we may write $B = \bigcup_{n\in\mathbb{Z}} \pi^n B_0$. A model example of this situation is when π is a uniformizer of some discrete valuation ring $\mathcal{O}$ and B_0 is a flat $\mathcal{O}$-algebra.

Lemma 2.2.13. *Let M be a right B_0-module that is π-torsion free and has a finite projective resolution by finitely generated modules. Then for each $i \geq 0$ there exist short exact sequences of left $\overline{B}$-modules*

$$0 \to \overline{B} \otimes_{B_0} \operatorname{Ext}^i_{B_0}(M, B_0) \to \operatorname{Ext}^i_{\overline{B}}(M \otimes_{B_0} \overline{B}, \overline{B}) \to \operatorname{Tor}_1^{B_0}(\overline{B}, \operatorname{Ext}^{i+1}_{B_0}(M, B_0)) \to 0.$$

The same holds for left B_0-modules with obvious modifications.

Proof. Note that $\overline{B}$ has a projective resolution

$$0 \to B_0 \xrightarrow{\times\pi} B_0 \to \overline{B} \to 0. \tag{2.9}$$

Thus for any right B_0-module M we have $\operatorname{Tor}_i^{B_0}(M, \overline{B}) = 0$ for $i \geq 2$ and

$$\operatorname{Tor}_1^{B_0}(M, \overline{B}) = \{m \in M : m\pi = 0\}.$$

In particular, if M is π-torsion free and if

$$P^\bullet = [0 \to P^{-n} \to \cdots \to P^{-1} \to P^0 \to 0]$$

is a projective resolution of M by finitely generated modules then

$$\overline{P}^\bullet = P^\bullet \otimes_{B_0} \overline{B}$$

is a projective resolution of $M \otimes_{B_0} \overline{B}$. Set

$$Q_\bullet = \operatorname{Hom}_{B_0}(P^\bullet, B_0). \tag{2.10}$$

This is a complex of finitely generated projective left B_0-modules and we have

$$H_i(Q_\bullet) = \operatorname{Ext}^{-i}_{B_0}(M, B_0). \tag{2.11}$$

On the other hand, we have

$$\operatorname{Ext}^{-i}_{\overline{B}}(M \otimes_{B_0} \overline{B}, \overline{B}) = H_i(\operatorname{Hom}(\overline{P}^\bullet, \overline{B})) = H_i(\overline{B} \otimes_{B_0} Q_\bullet) \tag{2.12}$$

Here the first equality holds because $\overline{P}^\bullet$ is a projective resolution of $M \otimes_{B_0} \overline{B}$ and the second equality holds because for any finitely generated projective right B_0-module P we have natural isomorphisms of left $\overline{B}$-modules

$$\overline{B} \otimes_{B_0} \operatorname{Hom}_{B_0}(P, B_0) = \operatorname{Hom}_{B_0}(P, \overline{B}) = \operatorname{Hom}_{\overline{B}}(P \otimes_{B_0} \overline{B}, \overline{B}).$$

Consider the following claim: *If $Q_\bullet$ is a bounded chain complex of finitely generated projective left B_0-modules then there exist exact sequences of left $\overline{B}$-modules*

$$0 \to \overline{B} \otimes_{B_0} H_j(Q_\bullet) \to H_j(\overline{B} \otimes_{B_0} Q_\bullet) \to \mathrm{Tor}_1^{B_0}(\overline{B}, H_{j-1}(Q_\bullet)) \to 0. \tag{2.13}$$

Once we have proven the claim we are done with the proof because of equalities (2.11) and (2.12). The fastest way to show existence of exact sequences (2.13) is to use Künneth's spectral sequence [Wei94, Theorem 5.6.4] (see also [Rot09, Theorem 10.90] for the formulation over noncommutative rings)

$$E^2_{i,j} = \mathrm{Tor}_i^{B_0}(\overline{B}, H_j(Q_\bullet)) \Rightarrow H_{i+j}(\overline{B} \otimes_{B_0} Q_\bullet) \tag{2.14}$$

and to note that because of the resolution (2.9) we have $\mathrm{Tor}^i(\overline{B}, -) = 0$ for $i \neq 0, 1$ and hence the spectral sequence degenerates to short exact sequences

$$0 \to E^2_{0,j} \to H_j(\overline{B} \otimes_{B_0} Q_\bullet) \to E^2_{1,j-1} \to 0. \tag{2.15}$$

The problem with this approach is that existence of the spectral sequence (2.14) is usually formulated with $\overline{B}$ replaced by an arbitrary right B_0-module. Therefore formally one needs to check that maps in sequences (2.15) are in fact $\overline{B}$-linear and not only additive (which is usually the case for tensor product of a left and a right module over a noncommutative ring). Alternatively, we can notice that if $d_\bullet$ is a differential in $Q_\bullet$ then as in the proof of [Wei94, Thm 3.6.1] we have the short exact sequence of complexes

$$0 \to \ker d_\bullet \otimes_{B_0} \overline{B} \to Q_\bullet \otimes_{B_0} \overline{B} \to \mathrm{im}\, d_\bullet \otimes_{B_0} \overline{B} \to 0. \tag{2.16}$$

This is again a consequence of description of $\mathrm{Tor}^i(-, \overline{B})$. Based on this observation we can copy the proof of [Wei94, Theorem 3.6.1] to prove our claim. Then $\overline{B}$-linearity is clear because the arrows in short exact sequences come from the long exact sequence associated to (2.16). □

We now briefly introduce a notion of a lattice in a generality needed for our applications. A *lattice* in a finitely generated B-module M is a finitely generated B_0-submodule $L \subset M$ such that $B \otimes_{B_0} L = L[\pi^{-1}] = M$. We set $\overline{L} = L/\pi L$ and call it a *reduction* of L. Recall that a B_0-module N is *of finite length* if it has a finite composition series

$$0 = N_0 \subset N_1 \subset \cdots \subset N_r = N$$

in which the factors N_i/N_{i-1} are simple modules. If N is of finite length then the module

$$N^{\mathrm{ss}} = \bigoplus_{i=1}^{r} N_i/N_{i-1}$$

does not depend on the choice of a composition series. We call this module the *semisimplification* of N.

Lemma 2.2.14. *Let M be a finitely generated left B-module and let $L_1, L_2 \subset M$ be two lattices. If $\overline{L}_1$ has finite length then so does $\overline{L}_2$ and they have isomorphic semisimplifications.*

Proof. Since $B = \bigcup_{n\in\mathbb{Z}} \pi^n B_0$, there exist integers $n, m \in \mathbb{Z}$ with $\pi^n L_2 \subset L_1 \subset \pi^m L_2$. Because $\overline{\pi^k L_i}$ is isomorphic to $\overline{L}_i$ we may assume that

$$L_2 \subset L_1 \subset \pi^{-n} L_2, \tag{2.17}$$

where $n \geq 1$. We prove the lemma by induction on n. We do the inductive step first. Assume that $n \geq 2$ and that the statement is true for $n-1$. Then the result holds for n because we have containments

$$L_2 \subset L_1 \cap \pi^{-n+1} L_2 \subset \pi^{-n+1} L_2$$

and

$$L_1 \cap \pi^{-n+1} L_2 \subset L_1 \subset \pi^{-1}(L_1 \cap \pi^{-n+1} L_2).$$

Therefore we only need to deal with the base for induction, i.e., with the case $n = 1$. We have

$$L_1 \subset \pi^{-1} L_2 \subset \pi^{-1} L_1 \tag{2.18}$$

Taking reductions of (2.17) (for $n = 1$) and of (2.18) gives exact sequences

$$\overline{L}_2 \xrightarrow{\varphi} \overline{L}_1 \xrightarrow{\psi} \overline{L}_2$$

and

$$\overline{L}_1 \xrightarrow{\psi} \overline{L}_2 \xrightarrow{\varphi} \overline{L}_1$$

where φ (resp. ψ) is the map induced by the inclusion $L_2 \subset L_1$ (resp. $L_1 \subset \pi^{-1} L_2$). Therefore we have exact sequences

$$0 \to \mathrm{im}\varphi \to \overline{L}_1 \to \mathrm{im}\psi \to 0 \tag{2.19}$$

and

$$0 \to \mathrm{im}\psi \to \overline{L}_2 \to \mathrm{im}\varphi \to 0. \tag{2.20}$$

If $0 \to N_1 \to N \to N_2 \to 0$ is a short exact sequence of modules then N has finite length if and only if N_1 and N_2 have finite length. If this is a case then $N^{\mathrm{ss}} = N_1^{\mathrm{ss}} \oplus M_2^{\mathrm{ss}}$. Therefore the result follows from existence of short exact sequences (2.19) and (2.20). $\square$

Remark 2.2.15. Lemma 2.2.14 above is a simple generalization of a classical observation that appears in many branches of mathematics. For example in algebraic geometry a variant of Lemma 2.2.14 for vector bundles with integrable connections is due to O. Gabber and may be found in a book of N. Katz [Kat90, Variant 2.5.2]. More recently similar a argument was used by A. Langer in [Lan22]. There is also a variant of Lemma 2.2.14 in representation theory of modular representations of finite groups (see [Sch13, Theorem 2.2.3]).

2.3 $\mathscr{D}$-modules

In this section we recall basic notions an properties of rings of differential operators and $\mathscr{D}$-modules.

2.3.1 Connections

The discussion in this subsection is very general and it applies to every locally ringed space $(X,\mathscr{O}_X)$ on which we can define a cotangent bundle Ω^1_X equipped with an exterior derivative $d:\mathscr{O}_X\to\Omega^1_X$. Its main purpose is to introduce the de Rham complex and the de Rham cohomology. In what follows we are only interested in the case when X is a smooth rigid analytic variety and Ω^1_X is the cotangent sheaf of continuous differentials but for the sake of motivation we list several situations to which the discussion below also applies

Example 2.3.1. Below one may take $(X,\mathscr{O}_X,\Omega^1_X)$ to be one of the following:

(1) $(X,\mathscr{O}_X)$ is a $\mathscr{C}^\infty$ real manifold and Ω^1_X is the sheaf of $\mathscr{C}^\infty$-differential forms.

(2) $(X,\mathscr{O}_X)$ is a complex manifold and Ω^1_X is the sheaf of holomorphic differential forms.

(3) $(X,\mathscr{O}_X)$ is a smooth algebraic variety and Ω^1_X is the sheaf of Kähler differentials.

(4) $(X,\mathscr{O}_X)$ is a smooth rigid analytic K-variety and Ω^1_X is the sheaf of continuous differential forms.

In what follows we write Ω^i_X for $\bigwedge^i\Omega^1_X$. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. A *connection* on $\mathscr{E}$ is an additive map

$$\nabla:\mathscr{E}\to\Omega^1\otimes\mathscr{E}$$

that satisfies the *Leibniz rule*

$$\nabla(fm) = df \otimes m + f\nabla(m)$$

for all local sections $f \in \mathscr{O}_X$, $m \in \mathscr{E}$. Note that in Examples 2.3.1 ∇ is automatically K-linear, where K is the base field. Given $(\mathscr{E}, \nabla)$ we can more generally consider maps

$$\nabla^i : \Omega^i_X \otimes \mathscr{E} \to \Omega^{i+1}_X \otimes \mathscr{E}$$

given by the formula

$$\nabla(\alpha \otimes m) = d\alpha \otimes m + (-1)^i \alpha \wedge \nabla(m).$$

We have $\nabla^0 = \nabla$ and it is straightforward to verify the *generalized Leibniz rule* which says that for all local sections $\alpha \in \Omega^i_X$, $\beta \in \Omega^j_X$, and $m \in \mathscr{E}$ we have

$$\nabla^{i+j}(\alpha \wedge \beta \otimes m) = d\alpha \wedge \beta \otimes m + (-1)^i \alpha \wedge \nabla^j(\beta \otimes m).$$

Let us write $R = \nabla^1\nabla^0$. Then

$$\begin{aligned} \nabla^{i+1}\nabla^i(\alpha \otimes m) &= \nabla^{i+1}\left(d\alpha \otimes m + (-1)^i \alpha \wedge \nabla(m)\right) \\ &= d^2\alpha \otimes m + (-1)^{i+1} d\alpha \wedge \nabla(m) \\ &+ (-1)^i d\alpha \wedge \nabla(m) + (-1)^{2i} \alpha \wedge R(m) \\ &= \alpha \wedge R(m). \end{aligned} \tag{2.21}$$

In particular, taking $i = 0$, we see that $R \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \Omega^2_X \otimes \mathscr{E})$. We call R the *curvature* of ∇ and we say that ∇ is *integrable* if $R = 0$. If this is the case then by (2.21)

$$\mathbf{DR}^\bullet_X(\mathscr{E}, \nabla) = (\Omega^\bullet_X \otimes \mathscr{E}, \nabla^\bullet) \tag{2.22}$$

is a complex. We call it the *de Rham complex* of $(\mathscr{E}, \nabla)$. Its (hyper)cohomology is the *de Rham cohomology* of $(\mathscr{E}, \nabla)$. We set

$$H^i_{\mathrm{dR}}(X, (\mathscr{E}, \nabla)) = \mathbb{H}^i(\mathbf{DR}^\bullet_X(\mathscr{E}, \nabla)).$$

We also write $\mathrm{MIC}(X)$ for the category of coherent $\mathscr{O}_X$-modules with integrable connections.

Remark 2.3.2. If ∇ is integrable then we can also consider a complex $\mathscr{E} \otimes \Omega^\bullet_X$ with the differential given by

$$\nabla^i_{\text{shuffle}}(m \otimes \alpha) = m \otimes d\alpha + \nabla(m) \wedge \alpha.$$

It is straightforward that the natural maps $\Omega^i_X \otimes \mathscr{E} \to \mathscr{E} \otimes \Omega^i_X$ yield an isomorphism of complexes. In a local coordinate system $x_1, \dots, x_n$ we have

$$\nabla^i_{\text{shuffle}}(m.dx_I) = \sum_i \partial_i.mdx_i \wedge dx_I, \tag{2.23}$$

where $\partial_i : \mathscr{E} \to \mathscr{E}$ is defined as the contraction of ∇ by the vector field $\partial_i \in \mathscr{T}_X$. Thus we recover the de Rham complex discussed in (2.31). In what follows, whenever we discuss computations concerning de Rham complexes in local coordinates, we will without mentioning identify the de Rham complex with (2.23).

We finish this subsection by giving examples of connections that appear in different branches of geometry. We hope that this discussion partially justifies our study of integrable connections (and $\mathscr{D}$-modules) on rigid analytic varieties.

Example 2.3.3. (1) The structure sheaf $\mathscr{O}_X$ carries a natural integrable connection defined by the exterior derivatve $d : \mathscr{O}_X \to \Omega^1_X$. If X is either a $\mathscr{C}^\infty$ real manifold or a complex manifold then by the de Rham Theorem $H^*_{\text{dR}}(X, (\mathscr{O}_X, d)) = H^*_{\text{Sing}}(X)$. If X is a smooth algebraic $\mathbb{C}$-variety then $H^*_{\text{dR}}(X, (\mathscr{O}_X, d)) = H^*_{\text{Sing}}(X^{an})$ by the theorem of Grothendieck (cf. [Gro66]).

(2) If (M, g) is a $\mathscr{C}^\infty$ real riemannian manifold then there is a distinguished connection ∇_{L-C} on the tangent bundle $\mathscr{T}_M$ called the *Levi-Civita connection*. The curvature $R(\nabla_{L-C})$ is the curvature of (M, g) so this connection is rarely integrable. In what follows we are interested only in integrable connections.

(3) If X is a complex analytic manifold then the *Riemann–Hilbert correspondence* establishes an equivalence of categories $\text{MIC}(X) = \text{Rep}(\pi_1(X), \mathbb{C})$. This equivalence has been later generalized by Deligne to the *Deligne–Riemann–Hilbert correspondence* $\text{MIC}_{reg}(X) = \text{Rep}(\pi_1(X^{an}), \mathbb{C})$. Here X is a smooth algebraic $\mathbb{C}$-variety and $\text{MIC}_{reg}(X)$ stands for the category of vector bundles with integrable connections that have regular singularites at infinity (cf. [Del70]). Finally, Kashiwara and Mebkhout generalized these results to the correspondence between the categories of regular holonomic $\mathscr{D}_X$-modules and perverse sheaves on X^{an}.

(4) The classical study of Picard–Fuchs equations is described in the modern language using the *Gauss-Manin connection* (cf. [ABC20]).

2.3.2 Rings of differential operators

In this subsection K is a field of characteristic zero. Let A be a commutative K-algebra. Then the ring of differential operators is defined as the subring

$$\mathcal{D}_A = \bigcup_{n\geq 0} \mathcal{D}_A^{\leq n} \subset \mathrm{Hom}_K(A,A),$$

where $\mathcal{D}_A^{\leq 0} = A$ and

$$\mathcal{D}_A^{\leq n} = \{P \in \mathrm{End}_K(A) : [P,f] \in \mathcal{D}_A^{\leq n-1} \text{ for all } f \in A\}. \tag{2.24}$$

We call elements of $\mathcal{D}_A$ *differential operators*. If $P \in \mathcal{D}_A^{\leq n} \setminus \mathcal{D}_A^{\leq n-1}$, then we say that P is a differential operator *of order n*.

Example 2.3.4. We recall the definition of the Weyl algebra. Let B be any commutative ring and let R be a free noncommutative B-algebra on symbols $x_1,\dots,x_n,\partial_1,\dots,\partial_n$. Consider the two-sided ideal $I \subset R$ generated by elements $[x_i,x_j]$, $[\partial_i,\partial_j]$, and $[\partial_i,x_j]-\delta_{ij}$, where $i,j = 1,\dots,n$. The n-th *Weyl algebra* over B is defined as

$$\mathbb{W}_n(B) = R/I.$$

If $A = K[x_1,\dots,x_n]$ is a polynomial ring over a field of characteristic zero, then $\mathbb{W}_n(K) = \mathcal{D}_A$.

We recall the following theorem of Matsumura in the formulation of Mebkhout and Narváez Macarro.

Lemma 2.3.5 (Matsumura, [MNM91, 1.2.2]). *Let A be a commutative noetherian K-algebra such that*

(1) *A is of equal dimension n.*

(2) *Residue fields for maximal ideals are algebraic extensions of K.*

(3) *There exist $x_1,\dots,x_n \in A$ and $\partial_1,\dots,\partial_n \in \mathrm{Der}_K(A,A)$ with $\partial_i(x_j) = \delta_{ij}$.*

Then $\mathrm{Der}_K(A,A)$ *is a free A module with a basis $\partial_1,\dots,\partial_n$.*

Note that in the situation of the lemma above we necessarily have $[\partial_i,\partial_j] = 0$. Indeed, since $\partial_1,\dots,\partial_n$ form a basis for $\mathrm{Der}_K(A,A)$ we may write

$$[\partial_i,\partial_j] = \sum_k f_{ij}^k \partial_k.$$

After evaluating both sides of this equation on $x_1,\dots,x_n$ we see that $f_{ij}^k = 0$ for all k.

Example 2.3.6. The assumptions of Lemma 2.3.5 are satisfied in various geometric situations. For example.

(1) If $X = \operatorname{Spa} A$ is a smooth affinoid K-variety (K is nonarchimedean of characteristic zero) then the properties (1) and (2) are satisfied. Assume that X admits an étale $X \to \mathbb{B}^n$ to some Tate's polydisc and let $\varphi : K\langle y_1, \dots, y_n\rangle \to A$ be the corresponding morphism of algebras. If we write $x_i = \varphi(y_i)$ then $\Omega^1_X = \bigoplus_{i=1}^n \mathscr{O}_X dx_i$. If we let $\partial_1, \dots, \partial_n$ be the dual basis of $\mathscr{T}_X$ then by definition

$$\partial_i(x_j) = \langle \partial_i, dx_j\rangle = \delta_{ij},$$

so the property (3) is also satisfied in this situation.

(2) Analogous considerations apply to the case when $X = \operatorname{Spec} A$ is a smooth affine $\mathbb{C}$-variety that admits an étale morphism to $\mathbb{A}^n_{\mathbb{C}}$.

(3) If A is either the ring of formal power series $\mathbb{C}[\![x_1, \dots, x_n]\!]$ or the ring $\mathbb{C}\{\{x_1, \dots, x_n\}\}$ of power series with nonzero radius of convergence then one may take $\partial_i = \frac{\partial}{\partial x_i}$.

Mebkhout and Narváez Macarro [MNM91] conclude from Lemma 2.3.5 that we have a direct sum decomposition

$$\mathcal{D}_A^{\leq n} = \bigoplus_{|\alpha| \leq n} A.\partial^\alpha \tag{2.25}$$

and therefore

$$\operatorname{gr}\mathcal{D}_A = A[\xi_1, \dots, \xi_n]$$

is a polynomial ring over A. We call ξ_i the *symbol* of ∂_i. While the fact that equality (2.25) holds under the assumptions of Lemma 2.3.5 seems to be 'folklore', it is not completely obvious. Because in what follows it is very important and we were not able to find a suitable reference, we sketch its proof for completeness.

Lemma 2.3.7. *Let A be a commutative K-algebra. Then*

(1) $\mathcal{D}_A^{\leq 1} = A \oplus \operatorname{Der}_K(A)$.

(2) *If* $\operatorname{char} K = 0$ *and there exist* $x_1, \dots, x_n \in A$ *and a free basis* $\partial_1, \dots, \partial_n$ *of* $\operatorname{Der}_K(A)$ *such that* $\partial_i(x_j) = \delta_{ij}$*, then equality* (2.25) *holds for all n.*

Proof. To verify (1) it suffices to show that if $P \in \mathcal{D}_A^{\leq 1}$ then $P - P(1)$ is a derivation. Set $a = P(1)$ and let $x, y \in A$. By the definition we have

$$[P,x](y) = y[P,x](1) = y(P-a)(x)$$

and thus

$$\begin{aligned}(P-a)(xy) &= P(xy) - xP(y) + xP(y) - axy \\ &= [P,x](y) + x(P-a)(y) \\ &= y(P-a)(x) + x(P-a)(y).\end{aligned}$$

So $P-a$ is a derivation. The proof of (2) is more involved. We work by induction on the order of a differential operator. Let P be a differential operator of order n. We define inductively $P_0 = P - P(1)$ and

$$P_i = P_{i-1} - \sum_{|\alpha|=i} \frac{1}{\alpha!} P_{i-1}(x^\alpha)\partial^\alpha.$$

Then $P_n(x^\alpha) = 0$ for $|\alpha| \le n$ and $P - P_n \in \bigoplus_{|\alpha|\le n} A.\partial^\alpha$. This shows that we can assume that $P(x^\alpha) = 0$ for $|\alpha| \le n$ from the beginning. We now show that $[P, x_i] = 0$. By the inductive assumption

$$[P, x_i] = Px_i - x_iP = \sum_{|\alpha|\le n-1} a_\alpha \partial^\alpha.$$

From the first equality we conclude that $[P,x_i](x^\alpha) = 0$ for $|\alpha| \le n-1$. On the other hand,

$$\partial^\beta(x^\alpha) = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta} & \text{if } \beta \le \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{2.26}$$

These two observations imply that $a_\alpha = 0$ for all α. We conclude that $P(x^\alpha f) = x^\alpha P(f)$ for all multi-indices α and all $f \in A$. Now let $f \in A$ be any element. Our goal is to show that $[P, f]$ is an operator of order zero. This would imply that $P \in \mathcal{D}_A^{\le 1}$ and then the proof follows from (1) and the initial assumption. Again by the inductive assumption we can write

$$[P, f] = \sum_{|\alpha|\le n-1} \frac{1}{\alpha!} f_\alpha \partial^\alpha.$$

We prove that $f_\alpha = 0$ for $|\alpha| \ge 1$ by induction. Assume that $f_\alpha = 0$ for $|\alpha| \le i-1$ and let $|\beta| = i$. On the one hand, since $P(1) = 0$ we have

$$x^\beta P(f) = x^\beta [P, f](1) = x^\beta f_0.$$

On the other hand, using formula (2.26), the inductive assumption, and the established properties of P we have

$$x^\beta P(f) = P(x^\beta f) = [P, f](x^\beta) = f_\beta + x^\beta f_0.$$

This shows that $[P, f] = f_0$ is indeed an operator of order zero. □

Remark 2.3.8. We often abuse the notation and write equality (2.25) as $\mathcal{D}_A = A[\partial_1, \dots, \partial_n]$. This notation should be understood geometrically as a choice of the coordinate system on the corresponding rigid analytic variety. Then $\partial_1, \dots, \partial_n$ should be understood as the partial derivatives with respect to this coordinate system.

For the future reference we recall the following.

Proposition 2.3.9. *Let A be a K-algebra that satisfies assumptions of Lemma 2.3.5. Then*

(1) *$\mathcal{D}_A$ is left and right noetherian and $gl.\dim \mathcal{D}_A = gl.\dim A$.*

(2) *If we identify $\mathcal{D}_A = A[\partial_1, \dots, \partial_n]$ then $\mathcal{D}_A$ has an involution given by*

$$\sum_\alpha f_\alpha \partial^\alpha \mapsto \left(\sum_\alpha f_\alpha \partial^\alpha\right)^t = \sum_\alpha (-1)^{|\alpha|} \partial^\alpha f_\alpha. \tag{2.27}$$

Proof. Both parts of the proposition are well known. The first one may be found in [MNM91, Théorème 1.4.4]. The second one is a direct computation based on the classical elementary formulas

$$\partial^\alpha f = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}(f) \partial^\beta$$

and

$$f \partial^\alpha = \sum_{\beta \le \alpha} (-1)^{|\alpha-\beta|} \binom{\alpha}{\beta} \partial^\beta \partial^{\alpha-\beta}(f)$$

(cf. [Sab, Exercise E.1.16]). □

2.3.3 Continuity of differential operators for Banach algebras

Let A be a Banach K-algebra. Since the topological structure is a part of the definition of A it is reasonable to restrict the study of K-linear differential operators on A to these which are continuous. In this subsection we clarify this ambiguity. It follows from Lemmas 2.1.26 and 2.3.7 that every K-linear operator on A is continuous if A is an affinoid algebra satisfying assumptions of Lemma 2.3.5. On the other hand, continuity of differential operators on noetherian Banach algebras is not automatic in general, as shown in the following example.

Example 2.3.10. Let $A = \mathbb{C}_p$. This is a noetherian Banach algebra over $\mathbb{Q}_p$. We know that $\mathbb{C}_p$ is algebraically closed and that it is the completion of $\overline{\mathbb{Q}}_p$, the algebraic closure of $\mathbb{Q}_p$. Therefore the field extension $\overline{\mathbb{Q}}_p \subset \mathbb{C}_p$ has transcendence degree at least one, and since these fields are of characteristic zero there exists a nonzero $\overline{\mathbb{Q}}_p$-linear derivation $\delta : \mathbb{C}_p \to \mathbb{C}_p$. Such derivation is also $\mathbb{Q}_p$-linear and it cannot be continuous because every continuous derivation that is zero on $\overline{\mathbb{Q}}_p$ is also zero on $\mathbb{C}_p$. In fact, every continuous $\mathbb{Q}_p$-derivation of $\mathbb{C}_p$ is zero (cf. Example 2.1.25).

Example 2.3.11. We now present a variation of the previous example. Let $A = \mathbb{C}_p\langle x\rangle/(x^2)$. Again, A is a noetherian Banach $\mathbb{Q}_p$-algebra. Let δ be the discontinuous nonzero derivation from the previous example and let $\delta' : f \mapsto \delta(f(0))x$. This is clearly $\mathbb{Q}_p$-linear and we have

$$\begin{aligned}\delta'(fg) &= \delta(fg(0))x \\ &= \delta(f(0))g(0)x + \delta(g(0))f(0)x \\ &= \delta'(f)g + \delta'(g)f.\end{aligned}$$

The last equality follows from the fact that $fx = f(0)x$ in A. We conclude that $\delta' : A \to A$ is a $\mathbb{Q}_p$-linear derivation that is not continuous.

Int turns out that if we assume that A is noetherian, then the above examples essentially show the only obstructions for continuity of differential operators.

Theorem 2.3.12. *Let K be a nonarchimedean field and let A be a noetherian Banach K-algebra. Assume that*

(1) *A is reduced, i.e., $\bigcap_{\mathfrak{p}\in \operatorname{Spec} A} \mathfrak{p} = \{0\}$.*

(2) *For every minimal prime ideal $\mathfrak{p} \subset A$, either $A/\mathfrak{p}$ is not a field or the field extension $K \subset A/\mathfrak{p}$ is finite.*

Then every K-linear differential operator on A is continuous.

Since by Lemma 2.1.7 affinoid algebras are noetherian and for every maximal ideal $m \subset A$ the extension $K \subset A/\mathfrak{m}$ is finite, we easily conclude from Theorem 2.3.12 the following.

Theorem 2.3.13. *If A is a reduced affinoid K-algebra then every K-linear differential operator on A is continuous.*

Since we are interested in smooth affinoid varieties, and those are adic spectra on reduced affinoid algebras, Theorem 2.3.13 is sufficient for our applications. Theorem 2.3.12 is a consequence of the general result of N. P. Jewell and A. M. Sinclair who studied automatic continuity of derivations of complex Banach algebra. While the result we want to use translates almost without changes to the nonarchimedean setting, it does not seem to be known to the general nonarchimedean audience. Therefore we include the proof of the following lemma for the convenience of the reader.

Lemma 2.3.14 (N. P. Jewell, A. M. Sinclair)**.** *Let $\varphi : V \to W$ be a (not necessarily continuous) operator. Assume that there exist continuous linear operators $\{R_n\}$ and $\{L_n\}$ on V and W respectively such that the operators*

$$\varphi R_n - L_n \varphi : V \to W$$

are continuous for all n. Then there exists a positive integer N such that for all $n \geq N$ the equality

$$\mathrm{cl}\,(L_1 \dots L_n \mathfrak{S}(\varphi)) = \mathrm{cl}\,(L_1 \dots L_N \mathfrak{S}(\varphi)) \tag{2.28}$$

holds.

We first prove a sublemma.

Lemma 2.3.15 ([JS76, Lemma 1])**.** *In Lemma 2.3.14 we may assume that the equality*

$$\varphi R_n - L_n \varphi = 0 \tag{2.29}$$

holds for all n

Proof. Let $V' = W' = V \oplus W$, and we let

$$\varphi' = \begin{pmatrix} \mathrm{Id} & 0 \\ \varphi & 0 \end{pmatrix}, \qquad R'_n = L'_n = \begin{pmatrix} R_n & 0 \\ \varphi R_n - L_n \varphi & L_n \end{pmatrix}.$$

Clearly, L'_n are continuous. It is a matter of a straightforward computation that $L'_n \varphi' - \varphi' R'_n = 0$ for all n and that

$$\mathfrak{S}(\varphi') = \{0\} \oplus \mathfrak{S}(\varphi),$$

and more generally

$$L'_1 \dots L'_n \mathfrak{S}(\varphi') = \{0\} \oplus L_1 \dots L_n \mathfrak{S}(\varphi).$$

Clearly, we have

$$\mathrm{cl}\,(\{0\} \oplus L_1 \dots L_n \mathfrak{S}(\varphi)) = \{0\} \oplus \mathrm{cl}\,(L_1 \dots L_n \mathfrak{S}(\varphi))$$

and therefore we can indeed assume that equalities (2.29) hold. □

Proof of Lemma 2.3.14. We may assume that equalities (2.29) hold for all n. The proof in this case is the content of [Sin75, Lemma 2.1] and we closely follow the exposition given there. We assume that the conclusion of Lemma 2.3.14 does not hold and our goal is to reach a contradiction. By Lemma 2.1.4 (6) we have inclusions $L_n\mathfrak{S}(\varphi) \subset \mathfrak{S}(\varphi)$ and therefore inclusions

$$\mathrm{cl}\,(L_1 \ldots L_{n-1}L_n\mathfrak{S}(\varphi)) \subset \mathrm{cl}\,(L_1 \ldots L_{n-1}\mathfrak{S}(\varphi)) \tag{2.30}$$

hold for all n. The first observation is that we can assume that these inclusions are always strict. Let $\{m_n\}$ be the sequence of integers for which inclusions in (2.30) are strict. By assumption, this is the case for infinitely many integers. We now let

$$L'_n = L_{m_n+1} \ldots L_{m_{n+1}}, \qquad R'_n = R_{m_n+1} \ldots R_{m_{n+1}}.$$

Then

$$L'_n\varphi = L_{m_n+1} \ldots L_{m_{n+1}}\varphi = \varphi R'_n = R_{m_n+1} \ldots R_{m_{n+1}} = \varphi R'_n$$

and the inclusion

$$\begin{aligned}
\mathrm{cl}\left(L'_1 \ldots L'_{n-1}L'_n\mathfrak{S}(\varphi)\right) &= \mathrm{cl}\,(L_1 \ldots L_{m_n}\mathfrak{S}(\varphi)) \\
&\subset \mathrm{cl}\,(L_1 \ldots L_{m_n-1}\mathfrak{S}(\varphi)) \\
&= \mathrm{cl}\left(L_1 \ldots L_{n-1}L_{m_{n-1}}\mathfrak{S}(\varphi)\right) \\
&= \mathrm{cl}\left(L'_1 \ldots L'_{n-1}L'_{n-1}\mathfrak{S}(\varphi)\right)
\end{aligned}$$

is strict by construction. We now proceed with the proof under the extra assumption that that all inclusions in (2.30) are strict. We may also assume that in the operator norm $|R_n| \leq 1$. Consider natural projections

$$\pi_n : W \to W/\mathrm{cl}\,(L_1 \ldots L_n\mathfrak{S}(\varphi)) = W_n.$$

The composition

$$\pi_n L_1 \ldots L_n \varphi : V \to W_n$$

is continuous by Lemma 2.1.4 (4) because we have

$$\pi_n L_1 \ldots L_n\mathfrak{S}(\varphi) = 0.$$

On the other hand, by the same lemma the composition

$$\pi_n L_1 \ldots L_{n-1}\varphi : V \to W_n$$

is not continuous, because if $\pi_n L_1 \dots L_{n-1}\mathfrak{S}(\varphi) = 0$ then we would have a containment $L_1 \dots L_{n-1}\mathfrak{S}(\varphi) \subset \mathrm{cl}(L_1 \dots L_n \mathfrak{S}(\varphi))$, contradicting our assumptions. Using discontinuity of the latter operator we can for every positive integer n find an element $v_n \in V$ such that $|v_n| \le 2^{-n}$ and

$$|\pi_n L_1 \dots L_{n-1}\varphi v_n| \ge n + |\pi_n L_1 \dots L_n \varphi| + \left|\pi_n \varphi \sum_{j=1}^{n-1} R_1 \dots R_{j-1} v_j\right|.$$

Now let

$$z = \sum_{n\ge 1} R_1 \dots R_{n-1} v_n.$$

We have

$$|z| \le \sum_{n\ge 1} |R_1 \dots\, R_{n-1}||v_n| \le \sum_{n\ge 1} 2^{-n} \le 1,$$

so this is a well defined element of V. Now for any positive integer n we compute that

$$\begin{aligned}
|\varphi z| \ge |\pi_n \varphi z| &= \left|\sum_{n\ge 1} \pi_n \varphi R_1 \dots R_{n-1} v_n\right| \\
&= \left|\pi_n \varphi R_1 \dots R_{n-1} v_n + \pi_n \varphi \sum_{j=1}^{n-1} R_1 \dots R_{j-1} v_j + \pi_n L_1 \dots L_n \varphi \sum_{j\ge n+1} R_{n+1} \dots R_{j-1} v_j\right| \\
&\ge |\pi L_1 \dots L_{n-1}\varphi v_n| - \left|\pi_n \varphi \sum_{j=1}^{n-1} R_1 \dots R_{j-1} v_j\right| - \left|\pi_n L_1 \dots L_n \varphi \sum_{j\ge n+1} R_{n+1} \dots R_{j-1} v_j\right| \\
&\ge n + |\pi_n L_1 \dots L_n \varphi| - |\pi_n L_1 \dots L_n \varphi| \left|\sum_{j\ge n+1} R_{n+1} \dots R_{j-1} v_j\right| \ge n.
\end{aligned}$$

Since the inequality $|\varphi z| \ge n$ cannot hold for all integers we reach the contradiction. □

We now prove Theorem 2.3.12 under the extra assumption that A is a domain.

Proof of Theorem 2.3.12 when A is a domain. Since every differential operator (and in fact every K-linear operator) acting on a finitely dimensional K-vector space is continuous, we may assume that A is not a field. The proof goes by induction on the order of the differential operator P. If this order is zero, then the theorem is clear, because in this situation P is a left multiplication by some element $f \in A$ and continuity of such an operator is a part of definition of a Banach algebra. Now assume that the theorem holds for operators of order $\le n-1$ and let P be an operator of order n. The key observation is that the separating space $\mathfrak{S}(P)$ is an ideal in A. Indeed, let $x \in \mathfrak{S}(P)$ and $f \in A$. By the definition $[P, f]$ is an operator of order $\le n-1$ and therefore it is continuous by the inductive assumption. Let

$\{x_n\}$ be a sequence in A tending to zero, such that $\lim P(x_n) = x$. Then fx_n also tends to zero and we have equality

$$[P, f](x_n) = P(fx_n) - fP(x_n).$$

From the continuity of $[P, f]$ we conclude that $\lim[P, f](x_n) = 0$, and therefore

$$\lim P(fx_n) = \lim fP(x_n) = fx,$$

i.e., $fx \in \mathfrak{S}(P)$. Now let $a \in A$ be a nonzero element that is not a unit. We apply Lemma 2.3.14 to

$$\varphi = P, \quad L_n = R_n = \text{left multiplication by } a^n.$$

By noetherianity of A every ideal in A is closed (Lemma 2.1.6). In particular, we can omit the closure in Lemma 2.3.14, and we conclude that there exists N such that

$$a^{N+1}\mathfrak{S}(P) = a^N\mathfrak{S}(P).$$

By the noetherianity, $a^N\mathfrak{S}(P)$ is a finitely generated A-module. Therefore from Nakayama's lemma and the above equality we conclude that there exist $b \in A$ with $(1-ab)a^N\mathfrak{S}(P) = 0$. Since a is not a unit and A is a domain, we conclude that $\mathfrak{S}(P) = 0$, i.e., P is continuous. This finishes the proof in the case when A is a domain. □

To prove Theorem 2.3.12 in the full generality we further exploit the noetherianity hypothesis. It is known that a noetherian ring has only finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. By the definition

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \sqrt{0}.$$

If A is reduced then this intersection is zero. On the other hand, the intersection $\cap_{j \neq i}\mathfrak{p}_j$ is nonzero if the minimal primes are nonzero. Otherwise we would have

$$\cap_{j \neq i}\mathfrak{p}_j \subset \mathfrak{p}_i,$$

which implies that $\mathfrak{p}_j \subset \mathfrak{p}_i$ for some $j \neq i$, contradicting the minimality.

Lemma 2.3.16. *Let A be a reduced noetherian K-algebra (over an arbitrary field) and let $P : A \to A$ be a differential operator. Then $P(\mathfrak{p}) \subset \mathfrak{p}$ for every minimal prime $\mathfrak{p} \subset A$, and the induced K-linear map $A/\mathfrak{p} \to A/\mathfrak{p}$ is a differential operator.*

Proof. Both claims are verified by induction on the order of P. The lemma is clear if this order is zero. We let $\mathfrak{q}$ denote the intersection of all minimal primes of A different from $\mathfrak{p}$. Then $\mathfrak{q}$ is nonzero by the above discussion and $\mathfrak{p} \cap \mathfrak{q} = \{0\}$ since A is reduced. Let $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$ be nonzero elements. Then $xy = 0$ and by the inductive assumption

$$[P,y](x) = P(yx) - yP(x) = -yP(x) \in \mathfrak{p}.$$

Since $y \notin \mathfrak{p}$ we conclude that $P(x) \in \mathfrak{p}$ and the first claim is verified. To verify the second claim let us write $\overline{P} : A_i \to A_i$ for the induced k-linear map. We show by induction on the order of P that it is a differential operator of order less or equal to the order of P. This follows from the straightforward equality $\overline{[P,f]} = [\overline{P}, \overline{f}]$. □

Proof of Theorem 2.3.12. Let $\mathfrak{p}_1 \dots \mathfrak{p}_n$ be all minimal primes of A and let $P : A \to A$ be a differential operator. By Lemma 2.3.16 the induced maps $P_i : A/\mathfrak{p}_i \to A/\mathfrak{p}_i$ are differential operators. All $\mathfrak{p}_i$ are closed by the noetherianity of A (Lemma 2.1.6) and thus $A/\mathfrak{p}_i$ are Banach K-algebras, which are domains. If $A/\mathfrak{p}_i$ is not a field then P_i is continuous by the special case of Theorem 2.3.12, which we have already shown. Otherwise, $K \subset A/\mathfrak{p}_i$ is a finite field extension and P_i is continuous because every K-linear map of finitely dimensional K-vector spaces is. Since

$$\{0\} = \bigcap_i \mathfrak{p}_i = \ker\left(A \to \prod_i A/\mathfrak{p}_i\right),$$

by the reducedness assumption, we have a commutative diagram with injective rows

$$\begin{tikzcd} A \arrow[r] \arrow[d, "P"] & \prod_i A/\mathfrak{p}_i \arrow[d, "(P_1,\dots,P_n)"] \\ A \arrow[r] & \prod_i A/\mathfrak{p}_i \end{tikzcd}$$

The right vertical arrow is continuous and therefore P is also continuous. □

Remark 2.3.17. Note that in this subsection we did not put any restrictions on the nonarchimedean field K. It may be of arbitrary characteristic and not necessarily discretely valued.

Remark 2.3.18. The continuity of derivations on complex (commutative) Banach algebras has been studied by many authors. The classical result of I. M. Singer and J. Wermer [SW55] states that every continuous derivation of such an algebra A has its image contained in the Jacobson radical of A In particular, if this radical is zero then there is no nonzero continuous derivations. This result has been later strengthened by M. P. Thomas [Tho88], who showed that the continuity assumption in the work of Singer–Wermer is superfluous. Therefore commutative Banach $\mathbb{C}$-algebras with zero Jacobson radical have no nonzero derivations. This is yet another example of a striking difference between the nonarchimedean and classical theory, because reduced affinoid algebras usually have plenty of (continuous by Theorem 2.3.13) derivations (for example $\frac{d}{dx} : K\langle x\rangle \to K\langle x\rangle$ is a nonzero derivation) and they are known to be Jacobson rings.

2.3.4 Modules over rings of differential operators

In this subsection we again assume that A satisfies assumptions of Lemma 2.3.5. If M is a left $\mathcal{D}_A$-module then its *de Rham complex* is defined as

$$\mathbf{DR}^{\bullet}_{\mathcal{D}_A}(M) = \left[M \to \bigoplus_{i=1}^{n} M dx_i \to \bigoplus_{i<j} M dx_i \wedge dx_j \to \cdots \to M dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \right] \tag{2.31}$$

with the differential given by

$$\delta^k(m.dx_I) = \sum_{j=1}^{n} \partial_j m.dx_j \wedge dx_I.$$

The *de Rham cohomology* of M is defined as the cohomology of this complex and denoted $H^i_{\mathrm{dR}}(M)$. This should be compared with formula (2.23).

We are interested in computing the cohomology of (2.31) when M is of minimal dimension. If M is a finitely generated left $\mathcal{D}_A$-module then there exists a *good filtration* $F_{\bullet}M$ on M, i.e., a filtration such that $\mathrm{gr}^F M$ is a finitely generated $A[\xi_1,\ldots,\xi_n]$-module (here ξ_i denotes the *symbol* of ∂_i). By [BGK+87, Thm V.2.2.2]

$$\mathrm{grade}_{\mathcal{D}_A}(M) + \dim \mathrm{gr}^F M = 2n, \tag{2.32}$$

where $\dim \mathrm{gr}^F M$ is by definition the Krull dimension of $B = \frac{A[\xi_1,\ldots,\xi_n]}{\sqrt{\mathrm{Ann}(\mathrm{gr}^F M)}}$. One often refers to the affine scheme Spec B as the *singular support* or the *characteristic variety* of M and denotes it by $\mathbf{CC}(M)$. It follows from the definition that if M is a nonzero finitely generated $\mathcal{D}_A$-module then

$$\dim \mathbf{CC}(M) = n$$

if and only if M is of minimal dimension.

Remark 2.3.19. If $X = \operatorname{Spec} A$ is a smooth affine $\mathbb{C}$-variety then the characteristic cycle has a nice geometric interpretation. In this situation we can identify $\operatorname{gr}\mathcal{D}_A$ with the algebra of functions on the (geometric) cotangent bundle T^*X. This is again a smooth affine variety and $\mathbf{CC}(M) \subset T^*X$ is an algebraic cycle. If $X = \operatorname{Spa} A$ is affinoid then $\operatorname{gr}\mathcal{D}_A$ is the algebra of functions on T^*X that are polynomial on the fibers, and thus it does not correspond to any nice geometric object. This is partially resolved by the $\widehat{D}$-module theory of Ardakov–Bode–Wadsley (cf. [AW19]). For our applications we are only interested in the equality (2.32) and the lack of its geometric interpretation is not a problem.

The category of $\mathcal{D}_A$-modules of minimal dimension has many nice properties.

Lemma 2.3.20. *Let M, M', M'' be finitely generated left $\mathcal{D}_A$-modules. Then*

(1) *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence then M is of minimal dimension if and only if M' and M'' are of minimal dimension.*

(2) *If $\operatorname{Ext}^i_{\mathcal{D}_A}(M, \mathcal{D}_A) \neq 0$, then $\operatorname{grade}_{\mathcal{D}_A}(\operatorname{Ext}^i_{\mathcal{D}_A}(M, \mathcal{D}_A)) \geq i$.*

(3) *$\operatorname{Ext}^n_{\mathcal{D}_A}(M, \mathcal{D}_A)$ is a right $\mathcal{D}_A$-module of minimal dimension (here $n = gl.\dim(A)$).*

(4) *If M is of minimal dimension then it has finite length as a $\mathcal{D}_A$-module.*

The same holds for right $\mathcal{D}_A$-modules.

Proof. The proofs of (1), (2) and (4) are in [MNM91, page 231]. For the proof of (3) it suffices to use (2). If $\operatorname{Ext}^n_{\mathcal{D}_A}(M, \mathcal{D}_A) = 0$ there is nothing to prove. Otherwise we have

$$n \geq \operatorname{grade}_{\mathcal{D}_A}(\operatorname{Ext}^n_{\mathcal{D}_A},(M, \mathcal{D}_A)) \geq n,$$

where the first inequality follows from $gl.\dim \mathcal{D}_A = n$ (Proposition 2.3.9). See also [Meb89, page 49] for the reference. ☐

Remark 2.3.21. The part (3) of Lemma 2.3.20 will be crucial for us. Together with Lemma 2.2.3 it gives a nice interpretation of the operation

$$M \mapsto \delta(M) = \operatorname{Ext}^n_{\mathcal{D}_A}(\operatorname{Ext}^n_{\mathcal{D}_A}(M, \mathcal{D}_A), \mathcal{D}_A)$$

in terms of the elementary linear algebra. By Lemma 2.2.3 we have $\delta(M) = M$ provided that M is of minimal dimension. In any case $\delta(M)$ is of minimal dimension by Lemma 2.2.3 and part (3) of Lemma 2.3.20. Thus if we write V for a K-vector space freely spanned by the isomorphism classes of finitely generated left $\mathcal{D}_A$-modules and $W \subset V$ for a subspace spanned by classes of modules of minimal dimension then the K-linear map

$$V \to V : [M] \mapsto [\delta(M)]$$

is a linear projection onto W.

If we take $A = K[x_1,\dots,x_n]$ then $\mathcal{D}_A = \mathbb{W}_n(K)$ is the n-th Weyl algebra (char $K = 0$). In this context the de Rham cohomology of modules of minimal dimension has been classically studied. We refer to [Bjö79, Ch. 1 Thm 6.1] for the proof of the following fundamental result.

Theorem 2.3.22 (Bernstein)**.** *Let M be a left $\mathbb{W}_n(K)$-module of minimal dimension. Then* $\dim_K H^i_{\mathrm{dR}}(M) < \infty$ *for all i.*

2.3.5 Coherent, finitely presented, and holonomic $\mathscr{D}$-modules

In this subsection we assume that X is a smooth rigid analytic variety over a nonarchimedean field K of characteristic zero. We define sheaves of differential operators of order $\leq n$ on X as a sheafified version of (2.24), i.e., as

$$\mathscr{D}_X^{\leq n} = \{P \in \mathscr{E}\mathrm{nd}_K(\mathscr{O}_X) : [P,f] \in \mathscr{D}_X^{\leq n-1} \text{ for all } f \in \mathscr{O}_X\}.$$

These sheaves are coherent because if $x_1,\dots,x_n$ is a coordinate system on an open subset $U \subset X$ and $\partial_1,\dots,\partial_n$ is the dual basis to the basis $dx_1,\dots,dx_n$ then $\mathscr{D}_U^{\leq n}$ is the coherent sheaf associated to the finitely generated module $\mathcal{D}^{\leq n}_{\mathscr{O}_X(U)}$. Finally, we define the sheaf of differential operators on X as

$$\mathscr{D}_X = \bigcup_{n\geq 0} \mathscr{D}_X^{\leq n}.$$

Alternatively, $\mathscr{D}_X$ can be defined as the subsheaf of $\mathscr{E}nd_K(\mathscr{O}_X)$ generated as a sheaf of K-algebras by $\mathscr{O}_X$ and $\mathscr{T}_X$. This follows from Lemmas 2.3.5 and 2.3.7, and from Theorem 2.3.13. The latter description shows that giving a left $\mathscr{D}_X$-module structure on an $\mathscr{O}_X$-module $\mathscr{E}$ is equivalent to giving a K-linear map

$$\nabla : \mathscr{T}_X \to \mathscr{E}nd_K(\mathscr{E},\mathscr{E}); \qquad \theta \mapsto \nabla_\theta$$

that satisfies for all local sections $f \in \mathscr{O}_X$, $\theta \in \mathscr{T}_X$ the following conditions.

(1) $\nabla_{f\theta} = f\nabla_\theta$

(2) $\nabla_\theta(f-) = \theta(f) + f\nabla_\theta$

(3) $\nabla_{[\theta_1,\theta_2]} = [\nabla_{\theta_1}, \nabla_{\theta_2}]$.

Further, it is easy to see that the first two conditions are equivalent to giving a connection $\nabla : \mathscr{E} \to \Omega^1_X \otimes \mathscr{E}$ and the last condition is equivalent to vanishing of the curvature of ∇. The passage from connections to $\mathscr{D}$-modules is given by contracting ∇ with elements of $\mathscr{T}_X$, i.e., by setting

$$\nabla_\theta : \mathscr{E} \xrightarrow{\nabla} \Omega^1_X \otimes \mathscr{E} \xrightarrow{\theta \otimes \mathrm{Id}} \mathscr{E}.$$

The passage in the other direction is easily done in local coordinates. One sets

$$\nabla(m) = \sum dx_i \otimes \nabla_{\partial_1}(m)$$

and checks that this description is coordinate-independent. Since equivalence of these two constructions is well known we omit the details and we refer the interested reader to [ABC20, Chapter 2].

The notions of a coherent and globally finitely presented left (or right) $\mathscr{D}_X$-module is clear. Namely, a $\mathscr{D}_X$-module is *globally finitely presented* if it admits a finite presentation

$$\mathscr{D}_X^{\oplus a_2} \to \mathscr{D}_X^{\oplus a_1} \to \mathscr{M} \to 0,$$

and it is *coherent* if there exists an open covering $X = \bigcup_i U_i$ such that $\mathscr{M}_{|U_i}$ is finitely presented for all i. Finally, we say that a left coherent $\mathscr{D}_X$-module is *holonomic* if there exists an open covering of X by affinoids $\{U_i\}$ such that each U_i admits a coordinate system and $\mathscr{M}(U_i)$ are $\mathcal{D}_{\mathscr{O}_X(U_i)}$-modules of minimal dimension.

Remark 2.3.23. If we replace X by a smooth algebraic $\mathbb{C}$-variety then all the above definitions translate *mutatis mutandis* to that setting. By equality (2.32) our notion of holonomicity agrees with the classical definition in terms of the dimension of the characteristic cycle.

Example 2.3.24 (cf. [HTT08, Example 2.2.4]). Let $(\mathscr{E}, \nabla)$ be a vector bundle with an integrable connection on X and let $U \subset X$ be an open affinoid subset that admits a coordinate system and trivializes $\mathscr{E}$. We set $M = \mathscr{E}(U)$. Consider a filtration

$$F_iM = \begin{cases} 0 & i < 0, \\ M & i \geq 0. \end{cases}$$

It is clearly a good filtration and if $\mathrm{gr}\mathcal{D}_A = A[\xi_1,\dots,\xi_n]$ ($A = \mathscr{O}_X(U)$), then $\mathrm{gr}^F M = M$ as an A-module and the action of $\xi_1,\dots,\xi_n$ is trivial on $\mathrm{gr}^F M$. We conclude that

$$\mathrm{gr}^F M = (A[\xi_1,\dots,\xi_n]/(\xi_1,\dots\xi_n))^{\oplus \mathrm{rk}\,\mathscr{E}}$$

as an $A[\xi_1,\dots,\xi_n]$-module. Therefore M is of minimal dimension by equality (2.32). Hence, $(\mathscr{E},\nabla)$ corresponds to a holonomic $\mathscr{D}_X$-module. In particular, the structure sheaf $\mathscr{O}_X$ is a holonomic $\mathscr{D}_X$-module in a natural way.

If $\mathscr{M}$ is a coherent left $\mathscr{D}_X$-module then we define its *de Rham complex* as

$$\mathbf{DR}_X^\bullet(\mathscr{M}) = \left[\mathscr{M}\otimes_{\mathscr{O}_X}\Omega_X^0 \to \cdots \to \mathscr{M}\otimes_{\mathscr{O}_X}\Omega_X^{\dim X}\right] \tag{2.33}$$

with the differential defined (locally, in a coordinate system) as

$$d(m\otimes\omega) = m\otimes d\omega + \sum_{i=1}^n \partial_i m\otimes(dx_i\wedge\omega) \tag{2.34}$$

If we interpret $\mathscr{M}$ as a sheaf with an integrable connection then the above de Rham complex coincides with (2.22). The *de Rham cohomology* of $\mathscr{M}$ is defined as the (hyper)cohomology of (2.33) and it is denoted by $H^i_{\mathrm{dR}}(X,\mathscr{M})$.

We now recall the *Spencer complex*.

$$\mathbf{Sp}_X^\bullet(\mathscr{D}_X) = \left[\mathscr{D}_X\otimes_{\mathscr{O}_X}\bigwedge^n \mathcal{T}_X \to \cdots \to \mathscr{D}_X\otimes_{\mathscr{O}_X}\bigwedge^1 \mathcal{T}_X \to \mathscr{D}_X\otimes_{\mathscr{O}_X}\bigwedge^0 \mathcal{T}_X\right] \tag{2.35}$$

with the differential given by

$$\begin{aligned}P\otimes\theta_1\wedge\cdots\wedge\theta_k \mapsto &\sum_i (-1)^{i+1}P\theta_i\otimes\theta_1\wedge\cdots\wedge\widehat{\theta_i}\wedge\cdots\wedge\theta_k\\ &+\sum_{i<j}(-1)^{i+j}P\otimes[\theta_i,\theta_j]\wedge\theta_1\wedge\cdots\wedge\widehat{\theta_i}\wedge\cdots\wedge\widehat{\theta_j}\wedge\cdots\wedge\theta_k.\end{aligned}$$

Our convention is that the complex (2.35) is concentrated in degrees $[-n,0]$. The following is well known.

Lemma 2.3.25. *With the notation aboveL*

(1) *The Spencer complex (2.35) is resolution of $\mathscr{O}_X$ by locally free left $\mathscr{D}_X$-modules.*

(2) *For any coherent left $\mathscr{D}_X$ module $\mathscr{M}$ we have a natural isomorphism of complexes*

$$\mathbf{DR}^{\bullet}{}_X(\mathscr{M}) = \mathscr{H}om_{\mathscr{D}_X}(\boldsymbol{Sp}_X^{-\bullet}(\mathscr{D}_X), \mathscr{M}). \tag{2.36}$$

Proof. The first claim is verified by noticing that (2.35) is a complex of filtered $\mathscr{D}_X$-modules and that the associated graded is the Koszul complex of $\mathscr{O}_X$ (cf. [HTT08, Lemma 1.5.27]).

For the second claim note that for any coherent left $\mathscr{D}_X$-module $\mathscr{M}$ we have natural isomorphisms

$$\begin{aligned} \Omega_X^k \otimes_{\mathscr{O}_X} \mathscr{M} &= \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \Omega_X^k \otimes \mathscr{M}) \\ &= \mathscr{H}om_{\mathscr{O}_X}(\bigwedge^k \mathscr{T}_X, \mathscr{M}) \\ &= \mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^k \mathscr{T}_X, \mathscr{M}) \\ &= \mathscr{H}om_{\mathscr{D}_X}(\mathbf{Sp}^{-k}(\mathscr{D}_X), \mathscr{M}). \end{aligned}$$

given by the formula

$$\alpha \otimes m \mapsto [P \otimes \theta \mapsto P\langle \alpha, \theta \rangle m].$$

A standard computation in local coordinates shows that these isomorphisms yield an isomorphism of complexes. $\square$

2.3.6 $\mathcal{D}_A$-modules versus $\mathscr{D}_X$-modules

In this subsection $X = \operatorname{Spa} A$ is a smooth affinoid variety that admits a local coordinate system. In this setting we investigate the relations between $\mathcal{D}_A$-modules and $\mathscr{D}_X$-modules.

Let M be a finitely generated left $\mathscr{D}_X(X)$-module. We define a presheaf on affinoid subdomains

$$\widetilde{M} : U \mapsto \mathscr{D}_X(U) \otimes_{\mathscr{D}_X(X)} M$$

Lemma 2.3.26. *Assume that $X = \operatorname{Spa} \mathrm{A}$ admits a global coordinate system. Then:*

(1) *For every finitely generated left $\mathcal{D}_A$-module M the presheaf $\widetilde{M}$ is a sheaf.*

(2) $H^i(X, \widetilde{M}) = 0$ *for* $i > 0$.

(3) $M \mapsto \widetilde{M}$ *establishes an equivalence of categories between finitely generated $\mathcal{D}_A$-modules and globally finitely presented $\mathscr{D}_X$-modules. The quasi-inverse is given by* $\mathscr{M} \mapsto \Gamma(X, \mathscr{M})$.

(4) $H^i_{\mathrm{dR}}(M) = H^i_{\mathrm{dR}}(X, \widetilde{M})$.

(5) *The coherent $\mathscr{D}_X$-module $\mathscr{M} = \widetilde{M}$ is holonomic if and only if M is a $\mathcal{D}_A$-module of minimal dimension.*

Proof. Clearly A satisfies assumptions of Lemma 2.3.5 and for every affinoid subdomain $U \subset X$ we have $\mathscr{D}_X(U) = \mathcal{D}_A = \mathscr{O}_X(U)[\partial_1, \dots, \partial_n]$. We conclude parts (1)–(3) of the lemma from Tate's acyclicity theorem 2.1.18. Let $X = \bigcup_{i=1}^N U_i$ be a finite covering of X by affinoid subdomains. Then Tate's acyclicity theorem states that the Čech complex

$$\mathscr{C}^\bullet(\{U_i\}, \mathscr{O}_X) = \left[0 \to \mathscr{O}_X(X) \to \bigoplus_{i=1}^{n} \mathscr{O}_X(U_i) \to \cdots \right]$$

is exact. On the other hand, under our assumptions we have

$$\mathscr{D}_X(U) \otimes_{\mathscr{D}_X(X)} M = \mathscr{O}_X(U) \otimes_A M.$$

Since the maps $A \to \mathscr{O}_X(U)$ are known to be flat (Lemma 2.1.17) we deduce that the complex

$$\mathscr{C}^\bullet(\{U_i\}, \widetilde{M}) = \mathscr{C}^\bullet(\{U_i\}, \mathscr{O}_X) \otimes_A M$$

is exact. This shows that the presheaf $\widetilde{M}$ is in fact a sheaf and it has no higher sheaf cohomology, i.e., that (1) and (2) hold.

To prove (3) we show that our functor is fully faithful and essentially surjective. Let M be a finitely generated $\mathcal{D}_A$-module. By noetherianity it is finitely presented and we have a presentation

$$\mathcal{D}_A^{\oplus a_2} \to \mathcal{D}_A^{\oplus a_1} \to M \to 0,$$

which induces a presentation

$$\mathscr{D}_X^{\oplus a_2} \to \mathscr{D}_X^{\oplus a_1} \to \widetilde{M} \to 0.$$

Note that the latter is really a presentation because we have

$$\mathscr{D}_X(U) \otimes_{\mathcal{D}_A} M = (\mathscr{O}_X(U) \otimes_A \mathcal{D}_A) \otimes_{\mathcal{D}_A} M = \mathscr{O}_X(U) \otimes_A M$$

and therefore $M \mapsto \widetilde{M}$ is an exact functor as the functor $\mathscr{O}_X(U) \otimes_A (-)$ is exact. Now we have

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{D}_X, \widetilde{N}) = \widetilde{N}(X) = N$$

and therefore both $\mathrm{Hom}_{\mathscr{D}_X}(\widetilde{M},\widetilde{N})$ and $\mathrm{Hom}_{\mathcal{D}_A}(M,N)$ can be realized as the kernel of the same homomorphism

$$N^{\oplus a_1} \to N^{\oplus a_2}.$$

This shows that our functor is fully faithful. To check that it is essentially surjective we use exactness once again. Let

$$\mathscr{D}_X^{\oplus a_2} \to \mathscr{D}_X^{\oplus a_1} \to \mathscr{M} \to 0$$

and let M be the cokernel of the corresponding map $\mathscr{D}_X(X)^{\oplus a_2} \to \mathscr{D}_X(X)^{\oplus a_1}$. From the exactness of $M \mapsto \widetilde{M}$ we obtain an exact sequence

$$\mathscr{D}_X^{\oplus a_2} \to \mathscr{D}_X^{\oplus a_1} \to \widetilde{M} \to 0.$$

It follows from the fully-faithfulness that $\widetilde{M} = \mathscr{M}$ as they both must be the cokernel of the same homomorphism. This finishes the proof of (3).

To prove (4) note that

$$\mathbf{DR}^{\bullet}{}_{\mathcal{D}_A}(M) = \Gamma(X, \mathbf{DR}^{\bullet}{}_X(\widetilde{M}))$$

and therefore we only need to check that for $i > 0$ we have $H^i(X, \widetilde{M} \otimes_{\mathscr{O}_X} \Omega_X^j) = 0$. Since Ω_X^j is globally free of finite rank this follows from (2).

To prove (5) it is sufficient to check that if $\mathscr{M}$ is holonomic then M is of minimal dimension. If M is a finitely generated $\mathcal{D}_A$-module and $U \subset X$ is an affinoid subdomain then from the flatness of $\mathscr{D}_X(U)$ over $\mathcal{D}_A$ we get (see Lemma 2.2.4 (4))

$$\mathrm{Ext}^i_{\mathscr{D}_X(U)}(\widetilde{M}(U), \mathscr{D}_X(U)) = \mathrm{Ext}^i_{\mathcal{D}_A}(M, \mathcal{D}_A) \otimes_{\mathcal{D}_A} \mathscr{D}_X.$$

Then an argument with Tate's acyclicity theorem, analogous to the one given above shows that the presheaf

$$\mathscr{N}_i : U \mapsto \mathrm{Ext}^i_{\mathcal{D}_A}(M, \mathcal{D}_A) \otimes_{\mathcal{D}_A} \mathscr{D}_X(U)$$

is a sheaf of *right* $\mathscr{D}_X$-modules. In particular, we have

$$\mathscr{N}_i(X) = \mathrm{Ext}^i_{\mathcal{D}_A}(M, \mathcal{D}_A).$$

Now assume that $\mathscr{M}$ is holonomic and $i \neq \dim X$. Let $\{U_j\}$ be the covering of X from the definition of holonomicity. Then

$$\mathscr{N}_i(U_j) = \mathrm{Ext}^i_{\mathscr{D}_X(U_j)}(\widetilde{M}(U_j), \mathscr{D}_X(U_j)) = 0$$

and therefore $\mathscr{N}_i(X) = 0$. $\square$

Remark 2.3.27. Let $Y = \operatorname{Spec} A$ be a smooth affine variety over $\mathbb{C}$. Then there is an equivalence of categories between the category of coherent left $\mathscr{D}_Y$-modules and the category of finitely generated left $\mathcal{D}_A$-modules. Under this equivalence holonomic $\mathscr{D}_Y$-modules correspond to modules of minimal dimension. More generally, we say that a smooth algebraic $\mathbb{C}$-variety Y is D-affine if the functor $\Gamma(Y,-)$ is exact and it induces an equivalence of categories between the category of $\mathscr{D}_Y$-modules that are $\mathscr{O}_Y$-quasi-coherent and the category of $\mathscr{D}_Y(Y)$-modules. It is natural to expect that smooth affinoid varieties are also D-affine in the sense that the functor $M \mapsto \widetilde{M}$ of Lemma 2.3.26 gives a desired equivalence of categories. Unfortunately, we do not know if it is true even for the Tate polydiscs. The problem is that, contrary to the situation in classical algebraic geometry, quasi-coherent $\mathscr{O}_X$-modules on affinoid varieties need not be globally generated. If we knew that smooth affinoids are in fact D-affine some arguments in the next chapter could be simplified.

2.3.7 Side-changing operations and direct image along closed embedding

Let X be a smooth rigid analytic variety. We write $\omega_X = \det \Omega_X$ for the canonical line bundle on X. It is known (cf. [HTT08, p. 19]) that ω_X is in fact a right $\mathscr{D}_X$-module with the differential structure induced by the Lie derivative. More precisely, $\mathscr{T}_X$ acts on ω_X by

$$\begin{aligned}((\mathrm{Lie} X).\omega)(X_1,\dots,X_n) = {} & X(\omega(X_1,\dots,X_n)) \\ & - \sum_{i=1}^{n} \omega(X_1,\dots,[X,X_i],\dots,X_n),\end{aligned}$$

where $X, X_1, \dots, X_n \in \mathscr{T}_X$ and $\omega \in \omega_X$. One verifies that the natural action of $\mathscr{O}_X$ on ω_X together with the right action of $\mathscr{T}_X$ given by

$$\omega.X := -(\mathrm{Lie} X)\omega$$

extends to a right $\mathscr{D}_X$-module structure on ω_X. More generally, if $\mathscr{M}$ (resp. $\mathscr{M}'$) is a left (resp. right) $\mathscr{D}_X$-module then $\omega_X \otimes_{\mathscr{O}_X} \mathscr{M}$ (resp. $\mathcal{H}om_{\mathscr{O}_X}(\omega_X, \mathscr{M}')$) carries a natural structure of a right (resp. left) $\mathscr{D}_X$-module and these constructions are inverse to each other (cf. [HTT08, p. 20]). The right (resp. left) structure is given by

$$(\omega \otimes m)X = \omega X \otimes m - \omega \otimes Xt \quad (\text{resp. } (X\varphi)(\omega) = -\varphi(\omega)X + \varphi(\omega X)).$$

If $x_1, \dots, x_n$ is a coordinate system on X, $\partial_1, \dots, \partial_n$ are the corresponding derivations and $dx_1 \wedge \cdots \wedge dx_n$ is the corresponding section of ω_X then the passage from left to right $\mathscr{D}_X$-

modules is given by the involution (2.27), i.e., by the formula

$$dx_1 \wedge \cdots \wedge dx_n \otimes m.P = dx_1 \wedge \cdots \wedge dx_n \otimes P^t m \tag{2.37}$$

Operations described above are called the *side-changing operations*. We need the following easy lemma.

Lemma 2.3.28. *Assume that X admits a global coordinate system. Then the right $\mathscr{D}_X$-module $\omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is free of rank one. Similarly, the left $\mathscr{D}_X$-module $\mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee$ is also free.*

Proof. Let us set $\mathscr{M} = \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X$. We fix a coordinate system $x_1, \ldots, x_n$ and let $e = dx_1 \wedge \cdots \wedge dx_n \otimes 1$. The action of $\mathscr{D}_X$ on $\mathscr{M}$ is defined by action of the tangent sheaf given by

$$(f dx_1 \wedge \cdots \wedge dx_n \otimes Q).\theta = -\theta(f) dx_1 \wedge \cdots \wedge dx_n \otimes Q - f dx_1 \wedge \cdots \wedge dx_n \otimes \theta Q.$$

We conclude that for $P \in \mathscr{D}_X$ we have

$$(dx_1 \wedge \cdots \wedge dx_n \otimes 1).P = dx_1 \wedge \cdots \wedge dx_n \otimes P^t$$

and it is easy to see that $\mathscr{M} = e.\mathscr{D}_X$ is free of rank one. $\square$

Let $i : X \hookrightarrow Y$ be a Zariski closed embedding of smooth rigid analytic varieties. Let us recall the definition of the *transfer modules*. We have

$$\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}\mathscr{D}_Y = i^*\mathscr{D}_Y.$$

This is an $(\mathscr{D}_X, i^{-1}\mathscr{D}_Y)$-bimodule with the $\mathscr{D}_X$-module structure given by the chain rule (cf. [HTT08, p. 23]). We also have

$$\mathscr{D}_{Y\leftarrow X} = \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y} \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}\omega_Y^\vee.$$

This is a $(i^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodule with the structure induced by the side-changing operations. The *direct image* of a left (resp. right) $\mathscr{D}_X$-module is defined as

$$i_+\mathscr{M} = i_*(\mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{D}_X} \mathscr{M})$$

(resp. $i_*(\mathscr{M} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y})$).

Remark 2.3.29. While our notation is widely used, another common notation for the direct image, used for example in [Meb89], [HTT08], is $\int_i \mathscr{M}$.

It is a standard computation (cf. [HTT08, p. 24]) that (for a closed embedding) the transfer modules are locally free over $\mathscr{D}_X$. Moreover, since i is affine i_* is an exact functor. It follows that i_+ is an exact functor. The same computation shows that if $y_1,\dots,y_n$ is a coordinate system for the closed embedding $i : X \to Y$ such that X is cut out by the ideal $\mathcal{I} = (y_{r+1},\dots,y_n)$ then

$$i_+\mathscr{M} = \bigoplus_{i_{r+1},\dots,i_n} i_*\mathscr{M}.\partial_{r+1}^{i_{r+1}}\dots\partial_n^{i_n} = i_*\mathscr{M}[\partial_{r+1},\dots,\partial_r]. \tag{2.38}$$

If moreover $Y = \operatorname{Spa} A$ and $X = \operatorname{Spa} B$ with $B = A/\mathcal{I}$ then the above choice of the coordinate system induces the homomorphism

$$\mathcal{D}_A = A[\partial_1,\dots,\partial_n] \to B[\partial_1,\dots,\partial_n] = \mathcal{D}_B[\partial_{r+1},\dots,\partial_n]$$

and the $\mathscr{D}_Y$-module structure on (2.38) is the one induced by this homomorphism. Finally, we mention that the formation of direct images commutes with the side-changing operations in the sense that we have a commutative diagram (cf. [HTT08, p. 23])

$$\begin{tikzcd} \operatorname{Mod}(\mathscr{D}_X) \arrow[r, "\omega_X\otimes -"] \arrow[d, "i_+"] & \operatorname{Mod}(\mathscr{D}_X^{\mathrm{op}}) \arrow[d, "i_+"] \\ \operatorname{Mod}(\mathscr{D}_Y) \arrow[r, "\omega_Y\otimes -"] & \operatorname{Mod}(\mathscr{D}_Y^{\mathrm{op}}) \end{tikzcd} \tag{2.39}$$

Remark 2.3.30. In this subsection we referred mostly to the definitions and basic properties of $\mathscr{D}$-modules contained in the first chapter of the book [HTT08] of Hotta–Takeuchi–Tanisaki. This book deals with $\mathscr{D}$-modules on complex algebraic varieties but it is clear that all the stated facts translate *mutatis mutandis* to our setting as they are formal consequences of the fact that $\mathscr{D}_X$ is generated by the tangent bundle. In the following section we need to be more careful.

2.3.8 Cohomological descent

To reduce the proof of Theorem 1.2.2 to the affinoid case we need to use some spectral sequences. Let X be a smooth rigid analytic variety and let $X = \bigcup_{i=1}^N U_i$ be an open cover. We define

$$\mathscr{U}^0 = \bigsqcup_i U_i$$

and

$$\mathscr{U}^n = \underbrace{\mathscr{U}^0 \times_X \mathscr{U}^0 \times_X \cdots \times_X \mathscr{U}^0}_{(n+1)\text{-times}}.$$

In other words $\mathscr{U}^n$ are disjoint sums of the intersections of $(n+1)$ elements from the open cover of X. We refer to [Del74, 5.3.3.3] for the proof of the following lemma.

Lemma 2.3.31. *Let $\mathscr{F}^\bullet$ be a complex of sheaves of abelian groups on X. Then there exists a spectral sequence*

$$E_1^{p,q} = H^q(\mathscr{U}^p, \mathscr{F}^\bullet_{|\mathscr{U}^p}) \implies H^{p+q}(X, \mathscr{F}^\bullet).$$

Obviously, if $\mathscr{F}^\bullet$ in a complex sheaves of K-vector spaces then the maps in the spectral sequence are K-linear. By taking $\mathscr{F}^\bullet$ the de Rham complex of some $\mathscr{D}_X$-module we obtain the following.

Lemma 2.3.32. *Let $X = \bigcup_{i=1}^N U_i$ and let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module. Then there exists a spectral sequence of K-vector spaces*

$$E_1^{p,q} = \bigoplus_{1 \le i_1, \dots, i_p \le N} H^q_{\mathrm{dR}}(U_{i_1} \cap \dots \cap U_{i_p}, \mathscr{M}_{|U_{i_1} \cap \dots \cap U_{i_p}}) \implies H^{p+q}_{\mathrm{dR}}(X, \mathscr{M})$$

Proof. This is a direct consequence of Lemma 2.3.31 and the definition of the de Rham cohomology. □

Chapter 3

Finiteness of the de Rham cohomology

3.1 Introduction

In this chapter we prove Theorem 1.2.2. To do so we first recall the notion of completed Weyl algebra and we study modules of minimal dimension over such algebras. Then we reduce the proof of Theorem 1.2.2 to the analogous result about such modules, i.e., to Theorem 3.1.1 below. The exposition very closely follows the one given in our two recent preprints [Rą24b], and [Rą24a]. For the rest of this chapter K stands for a discretely valued, nonarchimedean field of equal characteristic zero.

Let us briefly explain the structure behind the proof, which is summarized in Figure (3.1) below. If $X = \operatorname{Spa} K\langle x_1, \ldots, x_n\rangle$ is the Tate polydisc then we set $\mathcal{D}_n = \Gamma(X, \mathscr{D}_X)$ and we write $\widehat{\mathcal{D}}_n$ for the completion of this ring with respect to the operator norm. We also write $\widehat{\mathcal{D}}_n^\circ = \{P \in \widehat{\mathcal{D}}_n : |P| \leq 1\}$ and $\overline{\mathcal{D}}_n = \widehat{\mathcal{D}}_n^\circ \otimes_{\mathfrak{o}_K} k$. First, we study modules of minimal dimension over $\widehat{\mathcal{D}}_n$, and we prove the following.

Theorem 3.1.1. *Let K be a discretely valued nonarchimedean field of equal characteristic zero and let M be a finitely generated left $\widehat{\mathcal{D}}_n$-module. Then the following conditions are equivalent:*

(1) *M is of minimal dimension.*

(2) *There exists a lattice* $L \subset M$ *such that* $\overline{L}$ *is a* $\overline{\mathcal{D}}_n$*-module of minimal dimension.*

(3) *For any lattice* $L \subset M$ *the reduction* $\overline{L}$ *is a* $\overline{\mathcal{D}}_n$*-module of minimal dimension.*

If these equivalent conditions are satisfied, then:

A) The semisimplification of $\overline{L}$ *does not depend on* L *and only on* M*.*

B) We have $\dim_K H^i_{dR}(M) < \infty$ *for all* i *and the equality* $\chi_{dR}(M) = \chi_{dR}(\overline{L})$ *holds.*

To prove Theorem 1.2.2 we first observe that the natural map $\mathcal{D}_n \to \widehat{\mathcal{D}}_n$ is flat. This observation allows us to deduce the theorem for globally presented $\mathscr{D}_X$-modules on Tate's polydiscs from Theorem 3.1.1. Next we prove Theorem 1.2.2 for globally generated holonomic $\mathscr{D}_X$-modules on a smooth affinoid variety X with a global coordinate system (i.e., an étale morphism to some polydisc). Since X is affinoid there exists a Zariski closed embedding $i : X \hookrightarrow Y$ of X into a polydisc Y. We study the $\mathscr{D}$-module theoretic direct image $i_+\mathscr{M}$ to conclude the assertion by reducing to the previous case (see Proposition 3.4.1). Finally, to prove Theorem 1.2.2 in full generality we cover X by open affinoid subsets such that on each of these subsets assumptions of the previous case hold and we use cohomological descent to conclude finiteness of the de Rham cohomology from its finiteness on each of the open subsets.

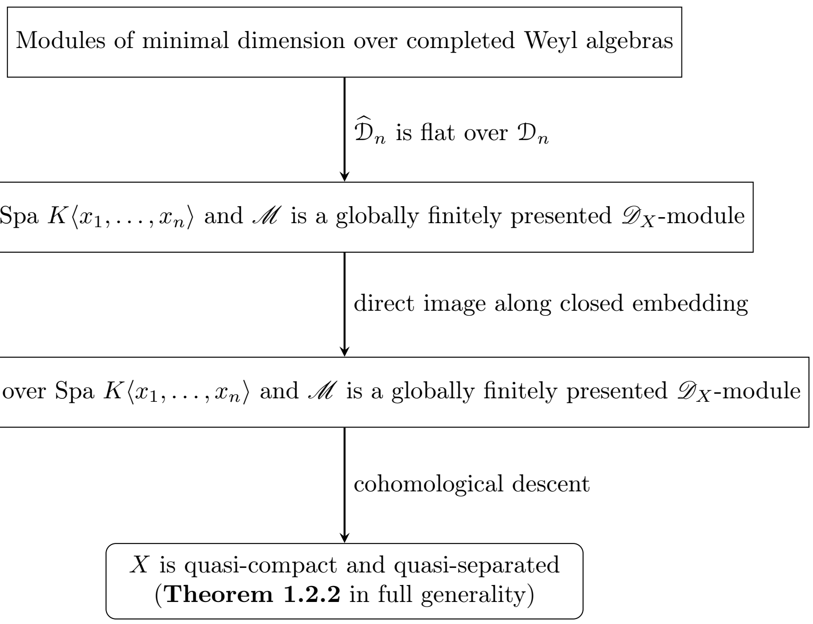

Figure 3.1: The logical structure of the proof of Theorem 1.2.2.

The reduction of the problem described above justifies the generality in which Theorem 1.2.2 is stated, i.e., the use of holonomic $\mathscr{D}$-modules. Although in practice one is usually interested in the de Rham cohomology of vector bundles with integrable connections, or even just the de Rham cohomology of the trivial vector bundle $(\mathscr{O}_X, d)$, the approach sketched above quickly leads to the more general category of holonomic $\mathscr{D}$-modules. Indeed, for a closed embedding $i : X \hookrightarrow Y$ the direct image $i_+(\mathscr{O}_X, d)$ is a holonomic $\mathscr{D}_Y$-module that is not a vector bundle as it is supported on a proper closed subset. On the other hand, it is hard to think of a larger category of $\mathscr{D}$-modules that is natural to work with and enjoys the property of finiteness of the de Rham cohomology, although it is worth mentioning that as a byproduct of our proof one can fairly easily construct examples of non-holonomic $\mathscr{D}$-modules with finite de Rham cohomology. One such example will be discussed in Example 3.3.4.

Remark 3.1.2. It would be interesting to compare our result with other known variants of the de Rham cohomology used in the nonarchimedean setting. We expect an analogue of Theorem 1.2.2 to be also valid (with K of equal characteristic zero) for the overconvergent de Rham cohomology used by the p-adic geometers and for the de Rham cohomology of the $\widehat{D}$-modules from the recent work of K. Ardakov, A. Bode, and S. Wadsley ([AW19], [AW18], [ABW21]). Perhaps such results could be derived directly from our Theorem 1.2.2. It is also natural to ask if the assumption that K is discretely valued can be dropped. We believe that it is true but it requires revisiting the proofs in [Rą24b] and it seems to require further work.

3.2 Completed Weyl algebras

In this section we recall the notion of completed Weyl algebra and we study the category of modules of minimal dimension over this algebra. The main result of this section is the proof of Theorem 3.1.1.

3.2.1 Preliminary definitions

Let K be a discretely valued nonarchimedean field of equal characteristic zero and let $\mathfrak{o}_K$ be its valuation ring. We also fix a uniformizer $\varpi \in \mathfrak{o}_K$. The notion of the n-th Weyl algebra over $\mathfrak{o}_K$ was recalled in Example 2.3.4. Then n-th *completed Weyl algebra* over $\mathfrak{o}_K$

is defined as

$$\widehat{\mathcal{D}}_n^\circ = \varprojlim \frac{\mathbb{W}_n(\mathfrak{o}_K)}{\varpi^{s+1}\mathbb{W}_n(\mathfrak{o}_K)}, \tag{3.1}$$

and the n-th *completed Weyl algebra* over K is defined as

$$\widehat{\mathcal{D}}_n = \widehat{\mathcal{D}}_n^\circ \otimes_{\mathfrak{o}_K} K. \tag{3.2}$$

Similarly to the Tate algebra, the completed Weyl algebra can be defined in two equivalent ways. The algebraic definition was given above. To give the functional-analytic definition we first fix some notation that is used throughout this chapter. Let us set

$$\mathcal{D}_n = \mathcal{D}_{K\langle x_1,\ldots,x_n\rangle} = K\langle x_1,\ldots,x_n\rangle[\partial_1,\ldots,\partial_n].$$

The natural action of $\mathcal{D}_n$ on the Tate algebra is continuous and therefore the Gauss norm induces an operator norm on $\mathcal{D}_n$. One checks that

$$|\sum f_\alpha \partial^\alpha| = \max|f_\alpha|.$$

We define $\mathcal{D}_n^\circ = \{P \in \mathcal{D}_n : |P| \le 1\}$. Then $\mathcal{D}_n^\circ$ is an $\mathfrak{o}_K$-algebra such that $\mathcal{D}_n^\circ \otimes_{\mathfrak{o}_K} K = \mathcal{D}_n$. Moreover, we have

$$\frac{\mathcal{D}_n^\circ}{\varpi^{k+1}\mathcal{D}_n^\circ} = \frac{\mathbb{W}_n(\mathfrak{o}_K)}{\varpi^{k+1}\mathbb{W}_n(\mathfrak{o}_K)}$$

and thus

$$\varprojlim \frac{\mathcal{D}_n^\circ}{\varpi^{k+1}\mathcal{D}_n^\circ} = \varprojlim \frac{\mathbb{W}_n(\mathfrak{o}_K)}{\varpi^{k+1}\mathbb{W}_n(\mathfrak{o}_K)} = \widehat{\mathcal{D}}_n^\circ,$$

and $\widehat{\mathcal{D}}_n = \widehat{\mathcal{D}}_n^\circ \otimes_{\mathfrak{o}_K} K$. We have natural containments

$$\mathbb{W}_n(\mathfrak{o}_K) \subset \mathcal{D}_n^\circ \subset \widehat{\mathcal{D}}_n^\circ$$

and

$$\mathbb{W}_n(K) \subset \mathcal{D}_n \subset \widehat{\mathcal{D}}_n.$$

From this construction it is clear that the operator norm on $\mathcal{D}_n$ extends to the norm on $\widehat{\mathcal{D}}_n$. The elements of $\widehat{\mathcal{D}}_n$ are written as power series

$$\sum_\alpha f_\alpha \partial^\alpha, \quad f_\alpha \in K\langle x_1,\ldots,x_n\rangle, \quad \lim_{|\alpha|\to\infty} |f_\alpha| = 0.$$

The norm $|\sum_\alpha f_\alpha \partial^\alpha| = \max|f_\alpha|$ makes $\widehat{\mathcal{D}}_n$ into a (noncommutative) Banach K-algebra and $\widehat{\mathcal{D}}_n^\circ$ is the unit ball with respect to this norm. We also denote

$$\overline{\mathcal{D}}_n = \widehat{\mathcal{D}}_n^\circ / \varpi \widehat{\mathcal{D}}_n^\circ.$$

It is clear from the construction that $\overline{\mathcal{D}}_n = \mathbb{W}_n(k)$ is the n-the Weyl algebra over the residue field of K but we believe that the 'overline' notation is more intuitive in the following considerations.

If M is a left $\widehat{\mathcal{D}}_n$-module then it is also a $\mathcal{D}_n$-module and as such it has a de Rham complex and the de Rham cohomology groups defined by formulas (2.31).

Remark 3.2.1. Algebraic properties of completed Weyl algebras have been studied by many authors, for example by L. Narváez Macarro in [NM98], and more recently by A. Pangalos in [Pan08]. Since the construction of the completed Weyl algebra is parallel to the construction of the Tate algebra with the polynomial ring replaced by the Weyl algebra (cf. Example 2.1.5 and the discussion above), the name *Weyl-Tate algebra* is also sometimes used to denote these objects. Our notation $\widehat{\mathcal{D}}_n$ (pronounced 'D-hat') should not be confused with the ring $\widehat{D}$ (pronounced 'D-cap') of Ardakov–Bode–Wadsley (cf. [AW19], [AW18], [ABW21]). Since the use of 'hat' to denote a completion of a ring is widely used we believe that our notation is justified.

3.2.2 Algebraic properties of completed Weyl algebras

In this subsection we discuss some basic algebraic properties of completed Weyl algebras. We show that they are left and right noetherian and that $gl.\dim \widehat{\mathcal{D}}_n = n$. We also establish some basic properties of $\widehat{D}_n^\circ$-modules that are needed in the proof of Theorem 3.1.1.

Lemma 3.2.2. *Both $\widehat{\mathcal{D}}_n^\circ$ and $\widehat{\mathcal{D}}_n$ are left and right noetherian.*

Proof. Note that the Weyl algebra $\mathbb{W}_n(\mathfrak{o}_K)$ is left and right noetherian. Indeed, the associated graded ring of the *Bernstein filtration*

$$F_n\mathbb{W}_n(\mathfrak{o}_K) = \bigoplus_{|\alpha|+|\beta|\leq n} a_{\alpha\beta}x^\alpha\partial^\beta$$

is the polynomial ring in $2n$ variables over $\mathfrak{o}_K$. Since the valuation on K is discrete, $\mathfrak{o}_K$ is noetherian and therefore so is any polynomial ring over $\mathfrak{o}_K$. We can apply [HTT08, Prop. D.1.4] which states that if the associated graded ring is noetherian then so is the original ring.

It is well known in the commutative case that for a noetherian ring R its I-adic completion is again noetherian. While this does not need to be the case for noncommutative rings, it follows from [McC79, Proposition 2.1.] that the theorem remains true if I is a two sided ideal generated by a single central element. Because $\mathbb{W}(\mathfrak{o}_K)$ is left and right noetherian and ϖ is central we conclude that $\widehat{\mathcal{D}}_n^{\circ}$ is left and right noetherian. Then $\widehat{\mathcal{D}}_n$ is left and right noetherian because it is a localization of $\widehat{\mathcal{D}}_0$ at ϖ. This proves the lemma.

$\square$

We also need the following properties of $\widehat{\mathcal{D}}_n^{\circ}$-lattices (lattices were defined in Subsection 2.2.4). We write $\overline{L} = L \otimes_{\mathfrak{o}_K} k$.

Lemma 3.2.3. *Let L be a finitely generated left $\widehat{\mathcal{D}}_n^{\circ}$-module. Then:*

(1) *L is complete in the ϖ-adic topology.*

(2) *If $\overline{L} = 0$ then $L = 0$.*

Proof. Since ϖ is central we can use the same reasoning as in the case of commutative noetherian rings. Since by [Row88, p. 413] the Artin–Rees Lemma holds for finitely generated left $\widehat{\mathcal{D}}_n^{\circ}$-modules, we can proceed as in [AM16, Ch. 10] to check that if $I = (\varpi)$ and L is a finitely generated left $\widehat{\mathcal{D}}_n^{\circ}$-module, then (as $\widehat{\mathcal{D}}_n^{\circ}$ is complete)

$$\widehat{L}^I = \widehat{\mathcal{D}}_n^{\circ} \otimes_{\widehat{\mathcal{D}}_n^{\circ}} L = L.$$

This proves the first assertion of the lemma. Since $L = \widehat{L}^I$ is separated, the second assertion follows from Nakayama's lemma for separated modules. $\square$

Lemma 3.2.4 (A. Pangalos)**.** *$gl.\dim \widehat{\mathcal{D}}_n = n$.*

Remark 3.2.5. We remark that Lemma 3.2.4 follows from the PhD thesis of A. Pangalos [Pan08]. More precisely, Proposition 3.1.3 of op. cit. gives a bound gl.dim$(\widehat{\mathcal{D}}_n) \geq n$ and Proposition 4.3.6 gives a bound gl.dim$(\widehat{\mathcal{D}}_n) \leq n$. Since this thesis is not published we give a slightly different proof below.

Proof of Lemma 3.2.4. We only have to show that $gl.\dim \widehat{\mathcal{D}}_n \leq n$ since the inequality $gl.\dim \widehat{\mathcal{D}}_n \geq n$ follows directly from the existence of the Spencer resolution which is obtained by tensoring the Spencer resolution for $\mathcal{D}_n$ with $\widehat{\mathcal{D}}_n$ (see the proof of Lemma 2.2.4). To do so, it suffices to show that $\mathrm{Ext}^{n+1}_{\widehat{\mathcal{D}}_n}(M,N) = 0$ for all finitely generated $\widehat{\mathcal{D}}_n$-modules M,N. Indeed, N can be assumed to be finitely generated, as it is a direct limit of finitely generated modules, and the direct limit commutes with Ext on the second variable (cf. [BGK+87, Theorem 2.4.3, Pages 189-190]). M can be assumed to be finitely generated (or even of form $\widehat{\mathcal{D}}_n/I$) by [Wei94, Theorem 4.1.2 and Lemma 4.1.6].

We first show that the global dimension is bounded. In this step we follow the main source used by A. Pangalos, i.e., [LvO96], and our argument is based on his proof. Set $R = \widehat{\mathcal{D}}_n$ and $F_nR = \varpi^{-n}\widehat{\mathcal{D}}^\circ_n$. We claim that this filtration is *faithful* (cf. [LvO96, Definition 12, page 45]), i.e., that $F_{-1}R$ is contained in the Jacobson radical of F_0R. To see this fix a maximal left ideal $\mathfrak{m}$. If $\mathfrak{m}$ does not contain $F_{-1}R$ then $\mathfrak{m}+F_{-1}R = F_0R$ and thus $1 = x+\varpi y$ for some $x \in \mathfrak{m}$ and $y \in F_0R$. This cannot be the case since $x = 1-\varpi y$ is clearly a unit in F_0R with the inverse $\sum_{j\geq 0}\varpi^j y^j$. Now set $\widetilde{R} = \widehat{\mathcal{D}}^\circ_n[X,X^{-1}]$ (the Laurent polynomial ring in one commuting variable X). This ring is clearly noetherian by Hilbert's basis theorem since $\widehat{\mathcal{D}}^\circ_n$ is (by Lemma 3.2.2), and agrees with the definition of $\widetilde{R}$ given in [LvO96, Page 36]. This shows that the assumptions of [LvO96, Theorem 12, Page 68] are satisfied and we obtain

$$gl.\dim R \leq gl.\dim \mathrm{gr}^F R = \mathbb{W}_n(k)[X,X^{-1}].$$

The latter has finite global dimension by Hilbert's syzygy theorem [Rot09, Theorem 8.37], since $\mathbb{W}_n(k)$ has finite global dimension by Lemma 2.3.9.

The second step in our proof is to show that $\mathrm{Ext}^d_{\widehat{\mathcal{D}}_n}(M,\widehat{\mathcal{D}}_n) = 0$ for $d \geq n+1$. Let $L \subset M$ be a lattice. By Lemma 2.2.13 we have exact sequences

$$0 \to \mathrm{Ext}^d_{\widehat{\mathcal{D}}^\circ_n}(L,\widehat{\mathcal{D}}^\circ_n)\otimes\overline{\mathcal{D}}_n \to \mathrm{Ext}^d_{\overline{\mathcal{D}}_n}(\overline{L},\overline{\mathcal{D}}_n),$$

and the term on the right hand side vanishes for $d \geq n+1$ ($\overline{\mathcal{D}}_n = \mathbb{W}_n(k)$). Since the module on the left hand side is a finitely generated right $\widehat{\mathcal{D}}^\circ_n$-module (by noetherianity of this ring), we conclude from Lemma 3.2.3 that

$$\mathrm{Ext}^d_{\widehat{\mathcal{D}}^\circ_n}(L,\widehat{\mathcal{D}}^\circ_n) = 0.$$

Since localization commutes with Ext for finitely generated modules we conclude that

$$\mathrm{Ext}^d_{\widehat{\mathcal{D}}_n}(M,\widehat{\mathcal{D}}_n) = \mathrm{Ext}^d_{\widehat{\mathcal{D}}^\circ_n}(L,\widehat{\mathcal{D}}^\circ_n)\otimes_{\mathfrak{o}_K} K = 0.$$

Once we know that the global dimension is finite and that equality $\mathrm{Ext}^i_{\widehat{\mathcal{D}}_n}(M,\widehat{\mathcal{D}}_n) = 0$ holds for all $i > n$ and all finitely generated $\widehat{\mathcal{D}}_n$-modules, we can proceed as in the proof of [BGK+87, Theorem 2.4.3, Pages 189-190]. Assume that $gl.\dim(\widehat{\mathcal{D}}_n) = d > n$. To reach a contradiction we need to show that $\mathrm{Ext}^d_{\widehat{\mathcal{D}}_n}(M,N) = 0$ for all finitely generated $\widehat{\mathcal{D}}_n$-modules M,N. We have a short exact sequence

$$0 \to K \to \widehat{\mathcal{D}}_n^{\oplus r} \to N \to 0,$$

where K is again finitely generated by noetherianity. We conclude that

$$\mathrm{Ext}^d_{\widehat{\mathcal{D}}_n}(M,N) = \mathrm{Ext}^{d+1}_{\widehat{\mathcal{D}}_n}(K,M) = 0.$$

□

3.2.3 Proof of Theorem 3.1.1

Now we prove Theorem 3.1.1.

Proof of Theorem 3.1.1. First we prove that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). We show condition A) as a part of the second implication. Then we show B).

(1) $\Longrightarrow$ (2). This is the most tricky part of the proof. It suffices to prove that for any *right* $\widehat{\mathcal{D}}_n$-module N of minimal dimension its dual $N^* = \mathrm{Ext}^n_{\widehat{\mathcal{D}}_n}(N, \widehat{\mathcal{D}}_n)$ has a lattice with reduction of minimal dimension. Indeed, by Lemma 2.2.3 we then may take $N = M^*$ which is of minimal dimension and satisfies $N^* = M^{**} = M$.

Let $V \subset N$ be some lattice (*a priori* with reduction that is possibly not of minimal dimension). By Lemma 2.2.13 applied to $B_0 = \widehat{\mathcal{D}}^\circ_n$ and $\pi = \varpi$ we have an inclusion

$$0 \to \overline{\mathcal{D}}_n \otimes_{\widehat{\mathcal{D}}^\circ_n} \mathrm{Ext}^n_{\widehat{\mathcal{D}}^\circ_n}(V, \widehat{\mathcal{D}}^\circ_n) \to \mathrm{Ext}^n_{\overline{\mathcal{D}}_n}(\overline{V}, \overline{\mathcal{D}}_n).$$

The key observation is that the module on the left is of minimal dimension. Indeed, since $\overline{\mathcal{D}}_n = \mathbb{W}_n(k)$ and $\overline{V}$ is finitely generated, by part (3) of Lemma 2.3.20 we know that the module on the right hand side is of minimal dimension. Therefore so is the module on the left hand side by part (1) of the same lemma. Now set

$$T = \{m \in \mathrm{Ext}^n_{\widehat{\mathcal{D}}^\circ_n}(V, \widehat{\mathcal{D}}^\circ_n) : \varpi^k m = 0 \text{ for some k}\}.$$

This is a left $\widehat{\mathcal{D}}^\circ_n$-module because ϖ is central in $\widehat{\mathcal{D}}^\circ_n$. We define L as the quotient of $\mathrm{Ext}^n_{\widehat{\mathcal{D}}_0}(V, \widehat{\mathcal{D}}_0)$ by T, so that it fits into the short exact sequence

$$0 \to T \to \mathrm{Ext}^n_{\widehat{\mathcal{D}}_0}(V, \widehat{\mathcal{D}}_0) \to L \to 0. \tag{3.3}$$

We will show that L is the desired lattice, i.e., that

(a) $K \otimes_{\mathfrak{o}_K} L = N^*$ and the natural map $L \to N^*$ is injective,

(b) L is a finitely generated $\widehat{\mathcal{D}}^\circ_n$-module,

(c) $\overline{\mathcal{D}}_n \otimes_{\widehat{\mathcal{D}}^\circ_n} L$ has minimal dimension.

To show (a) we note that $K \otimes_{\mathfrak{o}_K} -$ coincides with the localization at ϖ. In particular, it is an exact functor, it commutes with Ext and (by construction) $K \otimes_{\mathfrak{o}_K} T = 0$. Tensoring (3.3) with K we get that $K \otimes_{\mathfrak{o}_K} L = N^*$. The natural map $L \to M$ is injective by construction because its kernel consists precisely of ϖ-torsion of L. Recall that by Lemma 3.2.2 we know that $\widehat{\mathcal{D}}^\circ_n$ is left and right noetherian. From noetherianity we conclude that because V was finitely generated so is $\mathrm{Ext}^n_{\widehat{\mathcal{D}}^\circ_n}(V, \widehat{\mathcal{D}}^\circ_n)$. Then L is also finitely generated because by (3.3) it is a quotient of a finitely generated module. This shows (b). Since tensoring is right exact we have an exact sequence of left $\overline{\mathcal{D}}_n$-modules

$$\overline{\mathcal{D}}_n \otimes_{\widehat{\mathcal{D}}^\circ_n} \mathrm{Ext}^n_{\widehat{\mathcal{D}}^\circ_n}(V, \widehat{\mathcal{D}}^\circ_n) \to \overline{\mathcal{D}}_n \otimes_{\widehat{\mathcal{D}}^\circ_n} L \to 0,$$

we obtain (c) from part (1) of Lemma 2.3.20 because the right hand side is a quotient of a $\overline{\mathcal{D}}_n$-module which we already know to be of minimal dimension. Therefore the implication is proven.

(2) $\implies$ (3). This is a consequence of Lemma 2.2.14. Indeed, by Lemma 2.3.20 (4) we know that a finitely generated $\overline{\mathcal{D}}_n$-module of minimal dimension is of finite length. It follows by induction from part (1) of the same lemma that a semisimplification of a $\overline{\mathcal{D}}_n$-module of finite length has minimal dimension if and only if the module itself has minimal dimension. Therefore we may use Lemma 2.2.14 to get the desired implication. We also get $A)$ as a byproduct.

(3) $\implies$ (1). Let $L \subset M$ be a lattice such that $\overline{L}$ has minimal dimension. By the very definition we have $\mathrm{Ext}^i_{\overline{\mathcal{D}}}(\overline{L}, \overline{\mathcal{D}}) = 0$ for $0 \le i \le n-1$.Then short exact sequences of Lemma 2.2.13 for $B_0 = \widehat{\mathcal{D}}^\circ_n$ give

$$0 \to \mathrm{Ext}^i_{\widehat{\mathcal{D}}^\circ_n}(L, \widehat{\mathcal{D}}^\circ_n) \otimes_{\widehat{\mathcal{D}}^\circ_n} \overline{\mathcal{D}}_n \to \mathrm{Ext}^i_{\overline{\mathcal{D}}_n}(\overline{L}, \overline{\mathcal{D}}_n) = 0$$

i.e., $\mathrm{Ext}^i_{\widehat{\mathcal{D}}_0}(L, \widehat{\mathcal{D}}_0) \otimes_{\widehat{\mathcal{D}}_0} \overline{\mathcal{D}} = 0$ for $i < n$. By noetherianity of $\widehat{\mathcal{D}}^\circ_n$ (Lemma 3.2.2) we know that the right $\widehat{\mathcal{D}}^\circ_n$-modules $\mathrm{Ext}^i_{\widehat{\mathcal{D}}^\circ_n}(L, \widehat{\mathcal{D}}^\circ_n)$ are finitely generated and therefore must be zero by the Nakayama lemma (cf. Lemma 3.2.3). As we have already explained while proving that (1) $\implies$ 2, we always have isomorphisms $\mathrm{Ext}^i_{\widehat{\mathcal{D}}}(M, \widehat{\mathcal{D}}) = \mathrm{Ext}^i_{\widehat{\mathcal{D}}_0}(L, \widehat{\mathcal{D}}_0) \otimes_{\mathfrak{o}_K} K$. We conclude that $\mathrm{Ext}^i_{\widehat{\mathcal{D}}}(M, \widehat{\mathcal{D}})$ must vanish for $i < n$, i.e., M has minimal dimension. This closes the circle of implications.

To prove $B)$ we use Lemma 2.2.6. Assume that equivalent conditions of Theorem 3.1.1 hold for M and let $L \subset M$ be a lattice which has a reduction of minimal dimension. Consider the complex

$$\mathbf{DR}^\bullet(L) = L \to \bigoplus_{i=1}^{d} L dx_i \to \bigoplus_{i<j} L dx_i \wedge dx_j \to \dots$$

with differentials as in (2.31). This is a bounded complex of complete (by Lemma 3.2.3) and torsion-free (since lattices are ϖ-torsion free) $\mathfrak{o}_K$-modules. Note that by construction we have

$$\mathbf{DR}^\bullet(L) \otimes_{\mathfrak{o}_K} K = \mathbf{DR}^\bullet_{\widehat{\mathcal{D}}_n}(M),$$

and

$$\mathbf{DR}^\bullet(L) \otimes_{\mathfrak{o}_K} k = \mathbf{DR}^\bullet_{\overline{\mathcal{D}}_n}(\overline{L}).$$

The latter has finitely-dimensional cohomology over k by Bernstein's theorem 2.3.22. We may now apply Lemma 2.2.6 and conclude that $\dim_K H^i_{dR}(M) < \infty$ for all i and moreover $\chi_{dR}(M) = \chi_{dR}(\overline{L})$. □

3.3 Globally presented $\mathscr{D}$-modules on Tate polydiscs

The goal of this section is to verify Theorem 1.2.2 in the case when $X = \mathbb{B}^n$ and $\mathscr{M}$ is a globally presented holonomic $\mathscr{D}_X$-module. This is done by reducing the problem to the de Rham cohomology of modules of minimal dimensions over $\mathscr{D}_X(X) = \mathcal{D}_n$ and further to the modules of minimal dimension over completed Weyl algebras and using Theorem 3.1.1. We prove the following:

Proposition 3.3.1. *Let $X = \mathbb{B}^n$ and let $\mathscr{M}$ be a globally presented holonomic $\mathscr{D}_X$-module. Then $\dim_K H^i_{\mathrm{dR}}(X, \mathscr{M}) < \infty$ for all i.*

3.3.1 The base change $\mathcal{D}_n \to \widehat{\mathcal{D}}_n$

We have to study which properties of $\mathcal{D}_n$-modules are preserved after tensoring with $\widehat{\mathcal{D}}_n$. If M is a left $\mathcal{D}_n$-module we write $\widehat{M} = \widehat{\mathcal{D}}_n \otimes_{\mathcal{D}_n} M$.

Lemma 3.3.2. *$\widehat{\mathcal{D}}_n$ is flat as a left and right $\mathcal{D}_n$-module.*

Proof. We first check that $\mathcal{D}_n^\circ$ is left and right noetherian. By [HTT08, D.1.4] it is sufficient to find a filtration on $\mathcal{D}_n^\circ$ such that the associated graded is noetherian. Setting

$$F_t\mathcal{D}_n^\circ = \bigoplus_{|\alpha| \le t} \mathfrak{o}_K\langle x_1, \ldots, x_n\rangle \partial^\alpha$$

it is easy to see that this is indeed a filtration in the sense of [HTT08, Appendix D] and that the associated graded is a (commutative) polynomial ring in $\xi_1, \ldots, \xi_n$, i.e.,

$$\mathrm{gr}^{F_\bullet}\mathcal{D}_n^\circ = \mathfrak{o}_K\langle x_1, \ldots, x_n\rangle[\xi_1, \ldots, \xi_n].$$

It is noetherian by Hilbert's basis theorem because $\mathfrak{o}_K\langle x_1,\dots,x_n\rangle$ is noetherian if $\mathfrak{o}_K$ is a discrete valuation ring.

Now it is well known that if R is a commutative noetherian ring then the I-adic completion $\widehat{R}^I$ is R-flat for all ideals $I \subset R$. The proof presented in [AM16, Chapter 10] is easy to generalize to the case when R is noncommutative assuming that Artin–Rees lemma holds for the I-adic topology on R. This assumption is satisfied if I is generated by a central element by [Row88, p. 413]. Taking $I = (\varpi) \subset \mathcal{D}_n^\circ$ we see that $\widehat{\mathcal{D}}_n^\circ$ is flat over $\mathcal{D}_n^\circ$. Then $\widehat{\mathcal{D}}_n = \widehat{\mathcal{D}}_n^\circ[\varpi^{-1}]$ is flat over $\mathcal{D}_n = \mathcal{D}_n^\circ[\varpi^{-1}]$ because flatness is preserved under localization by [Wei94, Prop 3.2.9]. □

Lemma 3.3.3. *Let M be a finitely generated left $\mathcal{D}_n$-module.*

(1) *If M is of minimal dimension then so is $\widehat{M}$.*

(2) *The complexes $\mathbf{DR}^\bullet{}_{\mathcal{D}_n}(M)$ and $\mathbf{DR}^\bullet{}_{\widehat{\mathcal{D}}_n}(\widehat{M})$ are quasi-isomorphic.*

Proof. Since $\widehat{\mathcal{D}}_n$ is flat over $\mathcal{D}_n$ we have

$$\operatorname{Ext}^i_{\widehat{\mathcal{D}}_n}(\widehat{M},\widehat{\mathcal{D}}_n) = \operatorname{Ext}^i_{\mathcal{D}_n}(M,\mathcal{D}_n)\otimes_{\mathcal{D}_n}\widehat{\mathcal{D}}_n$$

by the Lemma 2.2.4 (4) and since $\mathcal{D}_n$ and $\widehat{\mathcal{D}}_n$ both have homological dimension n, assertion (1) holds. It is less obvious why (2) holds. To prove it we use multiple times equality

$$\mathbb{D}_{\widehat{\mathcal{D}}_n}(\widehat{M}) = \mathbb{D}_{\mathcal{D}_n}(M)\otimes_{\mathcal{D}_n}\widehat{\mathcal{D}}_n, \tag{3.4}$$

which holds by (4) of Lemma 2.2.4. First, let us consider consider the left $\mathcal{D}_n$-module

$$\mathscr{O}_X = \frac{\mathcal{D}_n}{\mathcal{D}_n(\partial_1,\dots,\partial_n)}$$

and the left $\widehat{\mathcal{D}}_n$-module

$$\widehat{\mathscr{O}}_X = \frac{\widehat{\mathcal{D}}_n}{\widehat{\mathcal{D}}_n(\partial_1,\dots,\partial_n)} = \widehat{\mathcal{D}}_n\otimes_{\mathcal{D}_n}\mathscr{O}_X.$$

Note that $\mathscr{O}_X = \widehat{\mathscr{O}}_X$ as left $\mathcal{D}_n$-modules as the former is constructed by forgetting the $\widehat{\mathcal{D}}_n$-module structure on the latter. We prefer to distinguish between these two objects for the clarity of the proof.

It follows from (2.36) that we have an equality

$$\mathbf{DR}^\bullet{}_{\mathcal{D}_n}(M) = \mathbf{R}\mathrm{Hom}_{\mathcal{D}_n}(\mathscr{O}_X, M). \tag{3.5}$$

Then we have

$$\mathbf{DR}^\bullet{}_{\widehat{\mathcal{D}}_n}(\widehat{M}) = \mathbf{DR}^\bullet{}_{\mathcal{D}_n}(\widehat{M}) = \mathbf{R}\mathrm{Hom}_{\mathcal{D}_n}(\mathscr{O}_X, \widehat{M}) = \mathbf{R}\mathrm{Hom}_{\widehat{\mathcal{D}}_n}(\widehat{\mathscr{O}}_X, \widehat{M}), \tag{3.6}$$

where the first equality is the definition, the second is (3.5), and the last one follows from the left adjointness of the tensor product to the restriction of scalars. By a standard computation with the Spencer complex (cf. Lemma 2.3.25, [HTT08, Proof of Proposition 2.6.12]) we have equality

$$\mathbb{D}_{\mathcal{D}_n}(\mathscr{O}_X) = \frac{\mathcal{D}_n}{(\partial_1, \dots, \partial_n)\mathcal{D}_n}[-n]$$

and thus also

$$\mathbb{D}_{\widehat{\mathcal{D}}_n}(\widehat{\mathscr{O}}_X) = \mathbb{D}_{\mathcal{D}_n}(\mathscr{O}_X) \otimes_{\mathcal{D}_n} \widehat{\mathcal{D}}_n = \frac{\widehat{\mathcal{D}}_n}{(\partial_1, \dots, \partial_n)\widehat{\mathcal{D}}_n}[-n].$$

Clearly, the natural map

$$\frac{\mathcal{D}_n}{(\partial_1, \dots, \partial_n)\mathcal{D}_n} \to \frac{\widehat{\mathcal{D}}_n}{(\partial_1, \dots, \partial_n)\widehat{\mathcal{D}}_n}$$

is an isomorphism of right $\mathcal{D}_n$-modules and therefore

$$\mathbb{D}_{\mathcal{D}_n}(\mathscr{O}_X) = \mathbb{D}_{\widehat{\mathcal{D}}_n}(\widehat{\mathscr{O}}_X). \tag{3.7}$$

We now explain why the following chain of equalities is true.

$$\begin{aligned}
\mathbf{DR}^\bullet{}_{\widehat{\mathcal{D}}_n}(\widehat{M}) &\overset{3.6}{=} \mathbf{R}\mathrm{Hom}_{\widehat{\mathcal{D}}_n}(\widehat{\mathscr{O}}_X, \widehat{M}) \\
&= \mathbf{R}\mathrm{Hom}_{\widehat{\mathcal{D}}_n^{op}}(\mathbb{D}_{\widehat{\mathcal{D}}_n}(\widehat{M}), \mathbb{D}_{\widehat{\mathcal{D}}_n}(\widehat{\mathscr{O}}_n)) \\
&\overset{3.4}{=} \mathbf{R}\mathrm{Hom}_{\widehat{\mathcal{D}}_n^{op}}(\mathbb{D}_{\mathcal{D}_n}(M) \otimes_{\mathcal{D}_n} \widehat{\mathcal{D}}_n, \mathbb{D}_{\widehat{\mathcal{D}}_n}(\widehat{\mathscr{O}}_X)) \\
&= \mathbf{R}\mathrm{Hom}_{\mathcal{D}_n^{op}}(\mathbb{D}_{\mathcal{D}_n}(M), \mathbb{D}_{\widehat{\mathcal{D}}_n}(\widehat{\mathscr{O}}_X)) \\
&\overset{3.7}{=} \mathbf{R}\mathrm{Hom}_{\mathcal{D}_n^{op}}(\mathbb{D}_{\mathcal{D}_n}(M), \mathbb{D}_{\mathcal{D}_n}(\mathscr{O}_X)) \\
&= \mathbf{R}\mathrm{Hom}_{\mathcal{D}_n}(\mathscr{O}_X, M) \\
&\overset{3.5}{=} \mathbf{DR}^\bullet{}_{\mathcal{D}_n}(M)
\end{aligned} \tag{3.8}$$

For the first and the last equality without a subscript we use part (3) of Lemma 2.2.4. For the remaining equality we use that $(-) \otimes_{\mathcal{D}_n} \widehat{\mathcal{D}}_n$ is left adjoint to forgetting the structure from $\widehat{\mathcal{D}}_n$ to $\mathcal{D}_n$. This ends the proof of the lemma. □

3.3.2 Proof of Proposition 3.3.1

We finish this section with the proof of Proposition 3.3.1.

Proof. Let $\mathscr{M}$ be a globally finitely presented holonomic $\mathscr{D}_X$ module (in this section $X = \mathbb{B}^n$) and let M be the $\mathcal{D}_n$-module that corresponds to $\mathscr{M}$ by Lemma 2.3.26. Then by the same lemma M is of minimal dimension and we have

$$H^i_{\mathrm{dR}}(X, \mathscr{M}) = H^i_{\mathrm{dR}}(M).$$

On the other hand, by Lemma 3.3.3 we know that $\widehat{M}$ is of minimal dimension and

$$H^i_{\mathrm{dR}}(M) = H^i_{\mathrm{dR}}(\widehat{M}).$$

Therefore the proposition follows from Theorem 3.1.1. □

Example 3.3.4. Consider the left $\mathcal{D}_n$-module $M = \frac{\mathcal{D}_n}{\mathcal{D}_n(1-\varpi\partial_1)}$. It is easy to see that it is nonzero and it is not of minimal dimension for $n \geq 2$. Note that $1 - \varpi\partial_1$ is a unit in $\widehat{\mathcal{D}}_n$ with the inverse $(1-\varpi\partial_1)^{-1} = \sum_{k\geq 0} \varpi^k \partial_1^k$ and therefore $\widehat{M} = \widehat{\mathcal{D}}_n \otimes_{\mathcal{D}_n} M = 0$. This shows that $\widehat{\mathcal{D}}_n$ is not faithfully flat over $\mathcal{D}_n$ and gives an example of a left $\mathcal{D}_n$-module which is not of minimal dimension but (by Lemma 3.3.3) has finite dimensional de Rham cohomology. This example shows also that the converse to the Lemma 3.3.3 (1) cannot hold.

3.4 $\mathscr{D}$-module theoretic direct image along a closed embedding

In this section we investigate how holonomicity and finiteness of dimensions of the de Rham cohomology behaves under direct images along Zariski closed embeddings. This allows us to prove the following proposition.

Proposition 3.4.1. *Let X be a smooth (affinoid) rigid analytic variety that admits a global coordinate system and let $\mathscr{M}$ be a globally finitely presented (left) holonomic $\mathscr{D}_X$-module. Then $\dim_K H^i_{\mathrm{dR}}(X, \mathscr{M}) < \infty$ for all i.*

3.4.1 Properties of the direct image

Most of this subsection is occupied by the proof of the following lemma, which collects properties of i_+ (see Subsection 2.3.7 for the definition) needed for the proof of Theorem 1.2.2.

Lemma 3.4.2. *Let $i : X \hookrightarrow Y$ be a Zariski closed embedding of smooth rigid analytic varieties. Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module. Then*

(1) *If both X and Y admit global coordinate systems and $\mathscr{M}$ is globally finitely presented then so is $i_+\mathscr{M}$.*

(2) *The left $\mathscr{D}_Y$-module $i_+\mathscr{M}$ is coherent.*

(3) *If $\mathscr{M}$ is holonomic then so is $i_+\mathscr{M}$.*

(4) *There exists a natural K-linear quasi-isomorphism of complexes*

$$i_*\mathbf{DR}^\bullet{}_X(\mathscr{M}) \to \mathbf{DR}^\bullet{}_Y(i_+\mathscr{M})[\dim X - \dim Y].$$

Proof. Let $\mathscr{M}$ be a globally finitely presented left $\mathscr{D}_X$-module (resp. globally finitely presented right $\mathscr{D}_Y$-module). Since the functor $\omega_X \otimes_{\mathscr{O}_X} -$ (resp. $- \otimes_{\mathscr{O}_Y} \omega_Y^\vee$) is clearly exact Lemma 2.3.28 implies that it preserves finite global presentation. It follows from the commutativity of the diagram (2.39) and the discussion above that the left $\mathscr{D}_Y$-module $i_+\mathscr{M}$ is globally finitely presented if the right $\mathscr{D}_Y$-module $i_+(\omega_X \otimes_{\mathscr{O}_X} \mathscr{M})$ is globally finitely presented. On the other hand, we have already remarked that the functor i_+ is exact. Therefore it suffices to show that the right $\mathscr{D}_Y$-module $i_+\mathscr{D}_X$ is globally finitely presented. However, if $\mathscr{I}$ is the ideal that cuts out X inside Y then

$$i_+\mathscr{D}_X = i_*\mathscr{D}_{X\to Y} = i_*i^*\mathscr{D}_Y = \mathscr{D}_Y/\mathscr{I}\mathscr{D}_Y.$$

The right-hand side is clearly globally finitely presented as $\mathscr{I}$ is generated by finitely many elements. This establishes (1). (2) follows easily from (1) after passing to the local coordinates for the embedding $i : X \hookrightarrow Y$.

Statement (3) is local so we may assume that the embedding $i : X = \operatorname{Spa} B \hookrightarrow \operatorname{Spa} A = Y$ admits a global coordinate system $y_1, \dots, y_n$ with $X = \{y_{r+1} = \cdots = y_n = 0\}$ and $\mathscr{M}$ is globally finitely presented. Write ∂_i for the derivation dual to dy_i. Then $\mathcal{D}_{A/K} = A[\partial_1, \dots, \partial_n]$, $\mathcal{D}_{B/K} = B[\partial_1, \dots, \partial_r]$ and if $\mathscr{M}$ corresponds to a $\mathcal{D}_{B/K}$-module M (see Lemma 2.3.26) then by formula (2.38) the direct image $i_+\mathscr{M}$ corresponds to $M' = M[\partial_{n+1}, \dots, \partial_m]$. We have to show that if M was of minimal dimension then so is M'. For that we use the characterisation of holonomicity in terms of the dimension of the characteristic variety. By equality (2.32) if $F_\bullet M$ is a good filtration on M then $\dim \operatorname{gr}^F M = r$ and we want to show that for some good filtration $G_\bullet$ on M' we have $\dim \mathrm{gr}^G M' = n$. Set

$$G_t M' = \bigoplus_{0 \le |\alpha| \le t} F_{t-|\alpha|} M.\partial^\alpha,$$

where $\partial^\alpha = \partial_{r+1}^{\alpha_1} \dots \partial_n^{\alpha_{m-r}}$ and $|\alpha| = \sum \alpha_i$. Then

$$\mathrm{gr}_t^G M' = \bigoplus_{0 \le |\alpha| \le t} \mathrm{gr}_{t-|\alpha|}^F M.\xi^\alpha$$

and therefore

$$\mathrm{gr}^G M' = \bigoplus_{t \ge 0} \mathrm{gr}_t^G M = (gr^F M)[\xi_{r+1}, \dots, \xi_n].$$

In particular, we have

$$\dim \mathrm{gr}^G M' = \dim(\mathrm{gr}^F M)[\xi_{r+1}, \dots, \xi_n] = r + (n-r) = n,$$

which proves (3).

The proof of (4) is the most complicated. Write $c = \dim Y - \dim X$. First we describe the natural $\mathscr{O}_Y$-linear maps

$$f^i : i_*(\mathscr{M} \otimes_{\mathscr{O}_X} \Omega_X^i) \to i_+\mathscr{M} \otimes_{\mathscr{O}_Y} \Omega_Y^{c+i}$$

and then we check that these maps give the desired quasi-isomorphism $i_* \mathbf{DR}^\bullet{}_X(\mathscr{M}) \to \mathbf{DR}^\bullet{}_Y(i_+\mathscr{M})[-c]$ by computations in the local coordinates for the closed embedding. Recall that we have the conormal short exact sequence

$$0 \to \mathscr{N}_{X/Y}^\vee \to \Omega_Y^1|_X \to \Omega_X^1 \to 0,$$

which induces exact sequences

$$\Omega_Y^{i-1}|_X \otimes_{\mathscr{O}_X} \mathscr{N}_{X/Y}^\vee \to \Omega_Y^i|_X \to \Omega_X^i \to 0 \tag{3.9}$$

and the natural isomorphisms

$$\bigwedge^c \mathscr{N}_{X/Y} = \det \mathscr{N}_{X/Y} = \omega_X \otimes_{\mathscr{O}_X} \omega_Y^\vee|_X. \tag{3.10}$$

It follows from (3.9) that the natural pairing $\Omega_Y^i|_X \otimes_{\mathscr{O}_X} \det \mathscr{N}_{X/Y}^\vee \to \Omega_Y|_X^{c+i}$ induces natural pairing

$$\Omega_X^i \otimes_{\mathscr{O}_X} \det \mathscr{N}_{X/Y}^\vee \to \Omega_Y^{c+i}|_X. \tag{3.11}$$

Now recall from subsection 2.3.7 that we have an isomorphism of $\mathscr{O}_X$-modules

$$\mathscr{D}_{Y\leftarrow X} = \omega_X \otimes_{\mathscr{O}_X} (\mathscr{O}_X \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}\mathscr{D}_Y \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}\omega_Y^\vee).$$

From this description we easily construct a natural $\mathscr{O}_X$-linear map

$$\det \mathscr{N}_{X/Y} = \omega_X \otimes_{\mathscr{O}_X} (\mathscr{O}_X \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}\omega_Y^\vee) \to \mathscr{D}_{Y\leftarrow X}$$

given locally (in a coordinate system) as

$$\partial_{r+1}\wedge\cdots\wedge\partial_n \mapsto dx_1\wedge\cdots\wedge dx_r \otimes 1\otimes 1\otimes \partial_1\wedge\cdots\wedge\partial_n.$$

After tensoring with $\mathscr{M}$ we obtain an $\mathscr{O}_X$-linear map

$$\mathscr{M}\otimes_{\mathscr{O}_X} \det\mathscr{N}_{X/Y} \to \mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}_X}\mathscr{M} \tag{3.12}$$

given locally as

$$m\otimes\alpha \mapsto \alpha\otimes 1\otimes m.$$

Now (3.11) and (3.12) give natural maps

$$\mathscr{M}\otimes_{\mathscr{O}_X}\Omega_X^i \to (\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}_X}\mathscr{M})\otimes_{\mathscr{O}_X}\Omega_X^i\otimes_{\mathscr{O}_X}\det\mathscr{N}_{X/Y}^\vee \to (\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}_X}\mathscr{M})\otimes_{\mathscr{O}_X}\Omega_{Y|X}^{c+i}$$

Finally, by applying i_* and using the projection formula we obtain morphisms

$$f^i : i_*(\mathscr{M}\otimes_{\mathscr{O}_X}\Omega_X^i) \to i_+\mathscr{M}\otimes_{\mathscr{O}_Y}\Omega_Y^{c+i}.$$

We now check that $f^\bullet$ defines the desired quasi-isomorphism of de Rham complexes. We write δ^i for the differential in the de Rham complex. First, we need to verify that $f^{i+1}\delta^i = \delta^i f^{i+1}$ (i.e., that $f^\bullet$ is a morphism of complexes). Then we show that $f^\bullet$ is injective and the cokernel of $f^\bullet$ is acyclic. All these questions are local so we may work under the assumptions and using the notation made in the proof of (3). A choice of local coordinates $y_1,\dots,y_n$ for the embedding $i : X\hookrightarrow Y$ induces bases $\{dy_1,\dots,dy_n\}$, $\{dy_1,\dots,dy_r\}$, and $\eta = dy_{r+1}\wedge\cdots\wedge dy_n$ of $\Omega^1_{Y|X}$, Ω^1_X and $\mathscr{N}_{X/Y}^\vee$ respectively. Under the identifications $\mathscr{M} = \widetilde{M}$ and $i_+\mathscr{M} = i_*\mathscr{M}[\partial_{r+1},\dots,\partial_n]$, the maps f^i correspond to the module homomorphism

$$\bigoplus_{|I|=i} M.dy_I \to \bigoplus_{|J|=c+i} M[\partial_{r+1},\dots,\partial_n].dy_J; \qquad \alpha\mapsto\alpha\wedge\eta,$$

where $I \subset \{1,\ldots,r\}$ and $J \subset \{1,\ldots,n\}$. From that we easily see that

$$\begin{aligned}
\left(f^{i+1}\delta^i - \delta^i f^i\right)(m.dy_I) &= f^{i+1}\left(\sum_{j=1}^{r} \partial^j m.dy_j \wedge dy_I\right) - \delta^i(m.dy_I \wedge \eta) \\
= \sum_{j=1}^{r} \partial^j m.dy_j \wedge dy_I \wedge \eta - &\sum_{j=1}^{r} \partial^j m.dy_j \wedge dy_I \wedge \eta - \sum_{j=r+1}^{n} \partial^j m.dy_j \wedge dy_I \wedge \eta \\
&= -\sum_{j=r+1}^{n} \partial^j m.dy_j \wedge dy_I \wedge \eta = 0,
\end{aligned}$$

where the last equality follows simply from the fact that $dy_j \wedge \eta = 0$ for $j \geq r+1$. This shows that $f^\bullet$ is in fact a morphism of complexes.

We now show that $f^\bullet$ is a quasi-isomorphism. Clearly, it is injective. It is also clear from the local descriptions of $i_+\mathscr{M}$ and f^i that we can prove our statement by induction on c. So we can assume that $c = 1$. Let $K^\bullet = \operatorname{coker} f^\bullet$. Let us show that the identity map $K^\bullet \to K^\bullet$ is chain-homotopic to zero. We have

$$K^t = \bigoplus_{I=(1\leq i_1<\cdots<i_t\leq n-1)} M[\partial_n].dy_I \oplus \bigoplus_{J=(1\leq j_1\leq\cdots\leq j_{t-1}\leq n-1)} \partial_n M[\partial_n].dy_J$$

so that every element in K^t can be uniquely represented as $\alpha + \partial_n\beta \wedge dy_n$ where α (resp. β) is a $M[\partial_n]$-valued t-form (resp. $(t-1)$-form) that does not contain dy_n. We define the homotopy operators $h^t : K^t \to K^{t-1}$ by

$$h^t : \alpha + \partial_n\beta \wedge dy_n \mapsto (-1)^{t+1}\beta.$$

Note that these maps are well-defined because ∂_n is not a zero-divisor on $M[\partial_n]$. We have to verify the identity

$$\delta^{t-1}h^t + h^{t+1}\delta^t = \mathrm{Id}.$$

We have

$$\begin{aligned}
&(\delta^{t-1}h^t + h^{t+1}\delta^t)(\alpha + \partial_n\beta \wedge dy_n) = \\
&\quad (-1)^{t+1}\delta^{t-1}(\beta) + h^{t+1}\left(\sum_{j=1}^{n} dy_j \wedge \partial^j\alpha + \sum_{j=1}^{n-1} dy_i \wedge \partial_j\partial_n\beta \wedge dy_n\right) = \\
&\quad (-1)^{t+1}\sum_{j=1}^{n} dy_j \wedge \partial_j\beta + \alpha + (-1)^{t+2}\sum_{j=1}^{n-1} dy_j \wedge \partial_j\beta = \\
&\quad \alpha + (-1)^{t+1} dy_n \wedge \partial_n\beta = \\
&\quad \alpha + \partial_n\beta \wedge dy_n.
\end{aligned}$$

This concludes the proof of (4). $\square$

Remark 3.4.3. It is well known that an analogue of the above lemma holds also for algebraic $\mathscr{D}$-modules. While it is possible that the usual proof carries over to our situation, it would require a lot of work to honestly verify that and therefore we prefer to give an elementary argument. While we are working in the rigid analytic setting, it is clear that our argument works also for $\mathscr{D}$-modules in any reasonable geometric situation. The direct description of the quasi-isomorphism $i_* \mathbf{DR}^\bullet{}_X(\mathscr{M}) \to \mathbf{DR}^\bullet{}_Y(i_+\mathscr{M})[\dim X - \dim Y]$ given above does not seem to appear in the standard literature on the subject.

Remark 3.4.4. In the first part of Lemma 3.4.2 we do not assume that the embedding $X \hookrightarrow Y$ admits a global coordinate system but only that both X and Y do. If $X = \operatorname{Spa} A$ then we may (by the definition of an affinoid variety) write $A = K\langle y_1, \ldots, y_n\rangle/I$. Such a choice of a presentation induces a closed embedding $X \hookrightarrow \mathbb{B}^n$. If X admits a global coordinate system then the lemma applies to this embedding although X does not need to be globally cut out in $\mathbb{B}^n$ by some of the coordinates $y_1, \ldots, y_n$.

Remark 3.4.5. The above proof implies that for a *right* globally finitely presented $\mathscr{D}_X$-module $\mathscr{M}$, the direct image $i_+\mathscr{M}$ is always globally finitely presented. It is not clear if this is also true for left $\mathscr{D}_X$-modules when we drop the assumptions about the coordinate systems. The problem is that in general the side-changing operations used in the proof do not preserve global generation. For example, on the projective space the left $\mathscr{D}_{\mathbb{P}^n}$-module $\mathscr{O}_{\mathbb{P}^n}$ is globally finitely presented but the corresponding right $\mathscr{D}_{\mathbb{P}^n}$-module $\omega_{\mathbb{P}^n} = \mathscr{O}_{\mathbb{P}^n}(-n-1)$ has no nonzero global sections.

In this thesis we study direct image only in two special cases. The first one is the case of the Zariski closed embedding discussed in this section. The second one is studied more implicitly. The de Rham cohomology of a $\mathscr{D}$-module is the $\mathscr{D}$-module theoretic direct image to the point along the structure morphism. In the theory of algebraic $\mathscr{D}$-modules one studies direct images in greater generality but we do not expect similar theory to work in the rigid analytic case. We illustrate this with an example.

Example 3.4.6. Let $j : U = \operatorname{Spa} K\langle x, x^{-1}\rangle \to \operatorname{Spa} K\langle x\rangle = X$ be an open embedding. If we consider the $\mathscr{D}_U$-module $\mathscr{O}_U$, then $j_+\mathscr{O}_U$ is the $\mathscr{D}_X$-module corresponding to the $\mathcal{D}_1$-module $K\langle x, x^{-1}\rangle$. This module is not even finitely generated, and in particular not holonomic. Thus the holonomicity is not preserved under arbitrary direct images, contrary to the algebraic case.

3.4.2 Proof of Proposition 3.4.1

Now we deduce Proposition 3.4.1 from Proposition 3.3.1 and Lemma 3.4.2.

Proof of Proposition 3.4.1. Let X be a smooth affinoid variety that admits a global coordinate system. There exists a closed embedding $i : X \hookrightarrow \mathbb{B}^n$ for some n. Let $\mathscr{M}$ be a globally presented holonomic left $\mathscr{D}_X$-module. Since both X and $\mathbb{B}^n$ admit global coordinate systems it follows from part (1) of Lemma 3.4.2 that $i_+\mathscr{M}$ is globally finitely presented. It is also holonomic by (3) of the same lemma. From Proposition 3.3.1 we conclude that $i_+\mathscr{M}$ has finite dimensional de Rham cohomology. From (4) of Lemma 3.4.2 we obtain isomorphisms

$$H^i_{\mathrm{dR}}(X,\mathscr{M}) = H^{i+c}_{\mathrm{dR}}(\mathbb{B}^n, i_+\mathscr{M}),$$

where c is the codimension of X in $\mathbb{B}^n$. This finishes the proof. $\square$

3.5 Proof of the finiteness of the de Rham cohomology

We finish this chapter with the proof of Theorem 1.2.2. For the convenience of the reader we also recall some special cases proven in the previous sections.

Proof of Theorem 1.2.2. If $X = \mathbb{B}^n$ and $\mathscr{M}$ is globally finitely presented then the assertion follows from Proposition 3.3.1. If X admits a global coordinate system and $\mathscr{M}$ is still globally finitely presented then we can consider some closed embedding $i : X \hookrightarrow \mathbb{B}^n$ and the claim for $\mathscr{M}$ follows from the previous case applied to $i_+\mathscr{M}$. This is precisely Proposition 3.4.1.

We now let X to be a smooth quasi-compact, quasi-separated rigid analytic variety and $\mathscr{M}$ a (not necessarily globally finitely presented) holonomic $\mathscr{D}_X$-module. We first prove Theorem 1.2.2 under the additional assumption that X is separated. There exists an affinoid open cover $X = \bigcup_{i=1}^N U_i$ such that each U_i admits a global coordinate system and $\mathscr{M}_{|U_i}$ is a globally finitely presented left $\mathscr{D}_{U_i}$-module (it may be taken to be finite because X is quasi-compact). As X is separated and U_i are affinoid, each finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is also an open affinoid in X. Note that such finite intersection admits a global coordinate system and $\mathscr{M}_{|U_{i_1}\cap\cdots\cap U_{i_k}}$ is a globally finitely presented $\mathscr{D}_{U_{i_1}\cap\cdots\cap U_{i_k}}$-module. Now we consider the spectral sequence from Lemma 2.3.32 associated to the cover $\{U_i\}$, i.e.,

$$E_1^{p,q} = \bigoplus_{1\le i_1,\ldots,i_p\le N} H^q_{\mathrm{dR}}(U_{i_1}\cap\cdots\cap U_{i_p}, \mathscr{M}_{|U_{i_1}\cap\cdots\cap U_{i_p}}) \implies H^{p+q}_{\mathrm{dR}}(X,\mathscr{M}).$$

By the first part of the proof we have

$$\dim_K H^q_{\mathrm{dR}}(U_{i_1}\cap\cdots\cap U_{i_p}, \mathscr{M}_{|U_{i_1}\cap\cdots\cap U_{i_p}}) < \infty.$$

Since $E_\infty^{p,q}$ is a subquotient of $E_1^{p,q}$ we conclude that it is a finite dimensional K-vector space. Therefore $H_{\mathrm{dR}}^{p+q}(X,\mathscr{M})$ admits a finite filtration by finitely dimensional K-vector spaces and hence it is of finite dimension.

Finally, we prove Theorem 1.2.2 in full generality. The argument is essentially a repetition of the argument above. Under our assumptions on X there exists a finite open cover $X = \bigcup_{i=1}^N U_i$ such that each U_i is separated. Then every finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is again separated. Therefore we can use the spectral sequence of Lemma 2.3.32 associated to this open cover to deduce finiteness of the de Rham cohomology from the case when X is separated. $\square$

Chapter 4

Differential operators on curves

4.1 Introduction

In this chapter we change the topic slightly and study differential operators on smooth algebraic curves using valuation theory. Our motivation comes from the fact that the formalism of valuations provides a very natural framework for working with rigid analytic varieties, and therefore one may hope to generalize the results of this chapter to smooth nonarchimedean curves. This is partially done in the last section, where we present some ideas on how to compute an index of a differential operator on a smooth affinoid curve and we give some examples. These ideas foreshadow an analogue of Delinge's index formula (discussed below) for smooth affinoid curves over $\mathbb{C}((t))$.

Let U be a smooth affine curve over $\mathbb{C}$, and let $(\mathscr{E},\nabla)$ be an algebraic vector bundle with connection on U (this connection is necessarily integrable since $\dim U = 1$). It is known that there exists a unique compactification of U, i.e., a smooth projective algebraic curve X that contains U as an open subset. Since the topology on X is cofinite, the complement of U in X consists of finitely many closed points. To each closed point $x \in X$ we can associate a local invariant of $(\mathscr{E},\nabla)$, called its *irregularity* at x, which is usually denoted as $\mathrm{irr}_x(\mathscr{E},\nabla)$. This local invariant has the following properties (cf. [Del70, Page 110]):

(1) $\mathrm{irr}_x(\mathscr{E},\nabla) = 0$ for all $x \in U$,

(2) $\mathrm{irr}_x(\mathscr{E},\nabla) \geq 0$ for all $x \in X$.

We say that $(\mathscr{E},\nabla)$ is *regular* if its irregularity is zero for all $x \in X$. The *Index Theorem* of Deligne [Del70, Formula 6.21.1] compares the Euler characteristic for the de Rham cohomology of $(\mathscr{E},\nabla)$ and its analytification $(\mathscr{E}^{an},\nabla^{an})$ in terms of the irregularity at infinity.

$$\chi_{DR}(U^{an},(\mathscr{E}^{an},\nabla^{an})) - \chi_{DR}(U,(\mathscr{E},\nabla)) = \sum_{x\in X\setminus U} \mathrm{irr}_x(U,\nabla). \tag{4.1}$$

By [Del70, 6.19] and the comparison theorem of Grothendieck [Gro66] we have

$$\chi_{DR}(U^{an},(\mathscr{E}^{an},\nabla^{an})) = (\mathrm{rk}\ \mathscr{E})\chi(X^{an}) = (\mathrm{rk}\ \mathscr{E})\chi_{DR}(U,(\mathscr{O}_U,d)),$$

and therefore formula (4.1) can be rewritten as

$$(\mathrm{rk}\ \mathscr{E})\chi_{DR}(U,(\mathscr{O}_U,d)) - \chi_{DR}(U,(\mathscr{E},\nabla)) = \sum_{x\in X\setminus U} irr_x(U,\nabla). \tag{4.2}$$

It has been shown in [ABC20, Theorem 24.1.3] that one can give a purely algebraic proof of formula (4.2), and this proof is valid whenever U is a smooth algebraic curve over an algebraically closed field of characteristic zero.

In the first part of this chapter we rewrite the algebraic theory of differential operators in the language of valuations. We show that with some elementary algebraic geometry one can give a nice valuation-theoretic formula for the index of a differential operator acting on the affine algebra of a smooth affine curve (Theorem 4.2.8). Then we show how to prove Deligne's theorem with our tools. The second part is devoted to examples. We explain how the index of a differential operator is related to the Euler characteristic (for the de Rham cohomology) of a holonomic $\mathscr{D}$-module. We use Lemma 2.2.6 to show how in some cases one can compute an index of a differential operator on a smooth affinoid curve using Theorem 4.2.8. Then we present some elementary formulas for indices in both affine and affinoid cases. In the former case these examples are due to N. Katz.

4.2 Algebraic Curves

In this section we study differential operators on smooth affine curves and we show how to compute their indices using valuations on the function field of a smooth curve.

4.2.1 Preliminaries on algebraic curves

We start by recalling some basic facts about smooth algebraic curves. Since they are well known and mostly elementary, we simply refer the reader to Hartshorne's book [Har77, Chapter 1.6, Chapter 4] for the proofs of the statements below and we do not give references in the text.

Let k be an algebraically closed field. A *curve* over k is a reduced, irreducible k-scheme X of dimension one. Assume that X is smooth. Then for every closed point $x \in X$ the local ring $\mathscr{O}_{X,x}$ is a discrete valuation ring on the function field $K = \mathscr{O}_{X,\eta}$ ($\eta \in X$ being the generic point). We write ν_x for the corresponding valuation.

Remark 4.2.1. For our purposes it is better to change our notation from multiplicative to the additive one, so in this chapter by a discrete valuation on K we understand a function $\nu : K \to \mathbb{Z} \cup \{\infty\}$ such that $\nu(xy) = \nu(x) + \nu(y)$, $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$ and $\nu(x) = \infty$ if and only if $x = 0$. We also assume that all considered valuations are k-valuations, i.e., that $\nu(k^\times) = 0$.

If $X = \operatorname{Spec} A$ then there is a natural identification

$$X(k) = \{\nu : K \to \mathbb{Z} \cup \{\infty\} : \nu(x) \geq 0 \text{ for all } x \in A\},$$

given by $x \mapsto \mathscr{O}_{X,x}$. Similarly, if X is projective then

$$X(k) = \{\nu : K \to \mathbb{Z} \cup \{\infty\}\}.$$

Every smooth affine curve U has a unique *compactification* X, i.e., there exists a unique smooth projective curve X that contains U as an open subset. In this case $X \setminus U$ is a finite number of closed points. We call them *points at infinity* and we call the corresponding valuations *valuations at infinity*. From now on we try to write X for a smooth projective curve and U for a smooth affine curve. On a projective curve every nonzero rational function has only finitely many zeroes and poles that sum up to zero, i.e., the following lemma holds.

Lemma 4.2.2. *Let X be a smooth projective curve, and let $f \in K^\times = \mathscr{O}_{X,\eta}^\times$. Then:*

(1) $\nu_x(f) = 0$ *for all but finitely many $x \in X(k)$,*

(2) $\sum_{x \in X} \nu_x(f) = 0$.

For future reference we now recall a version of the Riemann-Roch theorem for curves. Let X be a smooth projective curve. A *Weil divisor* on X is a finite formal sum

$$D = \sum_{x \in X(k)} n_x[x]$$

with integral coefficients. We write $D \geq 0$, and say that D is *effective* if $n_x \geq 0$ for all x. If $f \in K^\times$ then we can associate with f the *principal divisor*

$$\mathrm{div}(f) = \sum_{x \in X(k)} \nu_x(f)[x].$$

The *degree* of a divisor is defined as

$$\deg D = \sum_{x \in X} n_x,$$

and its *support* is defined as

$$\mathrm{supp}\, D = \{x \in X : n_x \neq 0\}.$$

Let us denote

$$\mathscr{L}(D) = \{f \in K : \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

This is clearly a k-vector space and we denote its dimension by

$$\ell(D) = \dim_k \mathscr{L}(D).$$

Recall that the *genus* of X is defined by

$$g = \dim_k H^1(X, \mathscr{O}_X).$$

The following is an easy consequence of the usual Riemann-Roch theorem for curves.

Theorem 4.2.3 (Riemann-Roch)**.** *Let X be a smooth projective genus g curve over an algebraically closed field k. Then*

(1) $\ell(D) < \infty$ *for all Weil divisors D.*

(2) *If* $\deg D > 2g - 2$ *then*
$$\ell(D) = \deg D + 1 - g. \tag{4.3}$$

One of the consequences of Theorem 4.2.3 is the following Lemma.

Lemma 4.2.4. *Let U be a smooth affine curve over k and let X be its compactification. Then there exists a rational function on X that is regular on U and has a pole of order at least one at every point of $X \setminus U$.*

Proof. Let $x_1, \ldots, x_n$ be the points at infinity of U. Then for $n \gg 0$ we have $\deg n[x_i] > 2g - 2$ and

$$\dim_k \mathscr{L}((n+1)[x_i]) / \mathscr{L}(n[x_i]) = 1,$$

so there exists a function $f_i \in K$ that has a pole at x_i and is regular at $X \setminus \{x_i\}$. Then $f = \sum_i f_i$ is the desired function. □

4.2.2 Valuations on the ring of differential operators

Let k be an algebraically closed field of characteristic zero and let $k \subset K$ be a field extension of transcendence degree one. In other words K is a field of rational functions on some smooth projective curve over k. In this section we show that a discrete valuation on K can be extended to a valuation on the ring of differential operators $\mathcal{D}_K$ of k-linear differential operators on K in a natural way.

Let $k \subset R_\nu \subset K$ be the discrete valuation ring corresponding to a discrete valuation $\nu : K^\times \to \mathbb{Z}$, $\mathfrak{m} \subset R_\nu$ the maximal ideal and let $t \in \mathfrak{m}$ be a fixed uniformizer. There exist unique k-linear derivations $\partial_t, \delta_t : R_\nu \to R_\nu$ such that $\partial_t(t) = 1$ and $\delta_t(t) = t$. Then $\delta_t = t\partial_t$ and thus $\delta_t(R_\nu) \subset \mathfrak{m}$. These derivations extend uniquely to derivations $\partial_t, \delta_t : K \to K$ and one can consider the ring of *twisted polynomials* (cf. [Ked06] p. 85) $K\langle\delta_t\rangle$, which by definition is the k-subalgebra of $\mathrm{End}_k(K)$ generated by left multiplication by elements from K and by δ_t. By [ABC20, Proposition 3.1.6] (or by Lemma 2.3.7) the obvious map

$$K\langle\delta_t\rangle \to \mathcal{D}_K$$

is an isomorphism. In particular, any operator $P \in \mathcal{D}_K$ can be written uniquely as

$$P = \sum_{i \geq 0} f_i \delta_t^i$$

and the *Gauss valuation*

$$\nu(P) = \min \nu(f_i) \tag{4.4}$$

is well-defined, although at the first sight it is not clear that it does not depend on the choice of t. Valuations on rings of twisted polynomials has been studied in [Ked06, 6.4], and are very useful in understanding differential equations. In particular it follows from loc. cit. that these are really valuations in the sense of the following lemma.

Lemma 4.2.5. *Let $\nu : \mathcal{D}_K \to \mathbb{Z} \cup \{\infty\}$ be the function defined by formula (4.4). Then for any $P, P_1, P_2 \in \mathcal{D}_K$*

(1) $\nu(P) \in \mathbb{Z}$ *for* $P \neq 0$.

(2) $\nu(P_1P_2) = \nu(P_1) + \nu(P_2)$.

(3) $\nu(P_1 + P_2) \geq \min\{\nu(P_1), \nu(P_2)\}$.

(4) *If P has degree zero then P is a left multiplication by some $f \in K$ and we have $\nu(P) = \nu(f)$.*

We now prove the following interesting property of Gauss valuations on $\mathcal{D}_K$.

Proposition 4.2.6. *With the above notation:*

(1) *For all $a \in K$ we have*
$$\nu(P(a)) \geq \nu(P) + \nu(a). \tag{4.5}$$

(2) *There exists a finite subset $S \subset \mathbb{Z}$ (depending on P) such that the equality*
$$\nu(P(a)) = \nu(P) + \nu(a) \tag{4.6}$$
holds whenever $\nu(a) \in \mathbb{Z} \setminus S$.

(3) *The number $\nu(P)$ is independent of the choice of a uniformizer of R_ν.*

Proof. For simplicity we write $R = R_\nu$ and $\delta = \delta_t$. Let $a = ut^n$ with $u \in R^\times$. Then
$$\delta(a) = \delta(u)t^n + nut^n = \nu(a)a + t^{\nu(a)}\delta(u).$$
Since $\delta(R) \subset \mathfrak{m}$ we have $\delta(u) = tr$ for some $r \in R$ and we can rewrite the equality above as
$$\delta(a) = \nu(a)a + t^{\nu(a)+1}r.$$
Iterating this construction we see that for any $k \geq 0$ there exists $r_k \in R$ such that
$$\delta^k(a) = \nu(a)^k a + t^{\nu(a)+1}r_k.$$
In particular, we have
$$\nu(\delta^k(a)) \geq \nu(a).$$
It is now easy to verify that desired inequality (4.5) holds for all $a \in K$. We have
$$P(a) = \sum_{i \geq 0} f_i \delta^i(a),$$
and therefore
$$\nu(P(a)) \geq \min\{\nu(f_i\delta^i(a))\} = \min\{\nu(f_i) + \nu(\delta^i(a))\} \geq \min\{\nu(f_i)\} + \nu(a) = \nu(P) + \nu(a).$$

We now prove that (4.6) holds for all but finitely many values of ν. Write $f_i = u_i t^{m_i}$ with $u_i \in R^\times$ and let $m = \min m_i$. We have

$$P(a) = \sum_{i\geq 0} u_i t^{m_i}(\nu(a)^i a + t^{\nu(a)+1} r_i) = t^m a \sum_{\{i: m_i = m\}} u_i \nu(a)^i + \rho,$$

where $\nu(\rho) > m + \nu(a)$. Consider the polynomial

$$p(y) = \sum_{\{i: m_i = m\}} u_i y^i \in R[y],$$

and its reduction

$$\overline{p}(y) = \sum_{\{i: m_i = m\}} \overline{u_i} y^i \in k[y].$$

Note that polynomials p and $\overline{p}$ depend only on P and not on a. If $\overline{p}(\nu(a)) \neq 0$ then $p(\nu(a))$ is a unit in R and

$$\nu(P(a)) = \nu(t^m a p(\nu(x))) = m + \nu(a).$$

Since char $k = 0$ we see that $\overline{p}(\nu(a)) = 0$ for only finitely many values $\nu(a)$ and the equality (4.6) holds. Finally, this implies that $\nu(P)$ is independent of the choice of a uniformizer t. We have

$$\nu(P) = \lim_{\nu(a)\to\infty} (\nu(P(a)) - \nu(a)), \tag{4.7}$$

and the right-hand side of this equality is independent of the choice of t because the valuation ν on K is independent of the choice of t. □

Example 4.2.7 (Irregularity of a differential operator). Consider the field $\mathbb{C}((z))$ and a differential operator

$$P = \sum_{i=0}^{d} a_i \left(z\frac{d}{dz}\right)^i. \tag{4.8}$$

One classically defines the *irregularity* of P (cf. [Mal72, Page XX.9]) as

$$i(P) = \sup\{-\nu(a_i) + \nu(a_d)\},$$

where ν is the discrete valuation given by the order of a zero/pole. Assume that $k = \mathbb{C}$. A choice of a valuation ν on K and a uniformizer $t \in R_\nu$ gives an injective map

$$(K, \nu, \delta_t) \to \left(\mathbb{C}((z)), \nu, z\frac{d}{dz}\right),$$

that sends t to z. If P is in the image of this map we can compute its irregularity using valuation on $\mathcal{D}_K$ discussed above. From Proposition 4.2.6 this valuation depends only on ν and not on a uniformizer t. For

$$P = \sum_{i=0}^{d} a_i \delta_t^i$$

we have

$$i(P) = \nu(a_d) - \nu(P). \tag{4.9}$$

This formula simplifies when P is a monic differential polynomial with respect to some nonzero derivation $\theta \in \mathrm{Der}_k(K,K)$. If we let

$$P = \theta^d + \sum_{i=0}^{d-1} a_i \theta^i,$$

then $\theta = f\delta_t$ for some $f \in K^\times$ and $\nu(\theta) = \nu(f)$. Since

$$P = f^d \delta_t^d + \text{lower order terms},$$

we conclude from (4.9) that

$$i(P) = d\nu(\theta) - \nu(P). \tag{4.10}$$

4.2.3 Index formula for differential operators

Let $U = \mathrm{Spec}\, A$ be a smooth affine curve over an algebraically closed field k of characteristic zero. We denote by X the compactification of U and by K the field of rational functions on X. Then $X \setminus U = \{x_1, \dots, x_r\}$. We write ν_i for the discrete valuation on K corresponding to x_i.

We have a natural injective map $\mathcal{D}_A \to \mathcal{D}_K$. To be more precise, given $P \in \mathcal{D}_A$ of order n, we extend P to a k-linear endomorphism of K by setting

$$P\left(\frac{a}{b}\right) = b^{-1}\left(P(a) - [P,b]\left(\frac{a}{b}\right)\right). \tag{4.11}$$

The second summand is a differential operator of order $(n-1)$, so we can proceed by induction. An elementary computation shows that formula (4.11) does not depend on the choice of presentation of the fraction $\frac{a}{b}$ and that it indeed defines a differential operator of order n. If $P \in \mathcal{D}_A$ then we understand $\nu_i(P)$ as the valuation of the image of P under the map above. We prove the following theorem.

Theorem 4.2.8. *With the above notation the index of P as a k-linear endomorphism of A satisfies*

$$\chi(P;A) = \sum_{i=1}^{r} \nu_i(P).$$

In particular, this index exists, i.e., the kernel and the cokernel of P have finite dimensions over k.

Remark 4.2.9. If either $X = \mathbb{A}^1$, or $X = \mathbb{G}_m$ then a similar formula appears in Katz's book [Kat90, Lemma 2.9.12, Lemma 2.9.13]. In those cases the proof is much easier, since one can use elementary properties of the degree of a (Laurent) polynomial and there is no need to refer to the Riemann-Roch theorem. We show how to recover Katz's results from Theorem 4.2.8 in Propositions 4.3.1 and 4.3.2.

Example 4.2.10 (Operators of order zero). Assume that P is a nonzero differential operator of order zero. Then $P : A \to A$ is a left multiplication by some element $f \in A$. Since U is smooth, A has no zero-divisors and therefore

$$\ker P = \ker(A \ni x \mapsto fx \in A) = \{0\}.$$

We also have an equality of k-vector spaces

$$\text{coker } P = A/(f) = \bigoplus_{x \in U} k^{\oplus \nu_x(f)},$$

where ν_x is the discrete valuation corresponding to a closed point $x \in U$. Therefore from Lemma 4.2.2 we get that

$$\chi(P,A) = -\dim_k A/(f) = -\sum_{x \in U} \nu_x(f) = \sum_{x \in X \setminus U} \nu_x(f) = \sum_{i=1}^{r} \nu_i(P).$$

This shows that Theorem 4.2.8 for operators of order zero is just a reformulation of Lemma 4.2.2.

Proof of Theorem 4.2.8. The idea is to find a filtration on A that satisfies the assumptions of Lemma 2.2.9. It is not clear whether such filtration always exists (and if it has any geometric interpretation) so we first do a reduction to a certain special case in which its existence is very natural. By Subsection 2.2.3 for any two linear maps L_1, L_2 we have

$$\chi(L_1 L_2) = \chi(L_1) + \chi(L_2)$$

whenever the composition makes sense and both L_1 and L_2 have indices. By Lemma 4.2.5 we have
$$\sum_{i=1}^r \nu_i(P_1P_2) = \sum_{i=1}^r \nu_i(P_1) + \sum_{i=1}^r \nu_i(P_2),$$
and by Example 4.2.10 we know that Theorem 4.2.8 holds for left multiplication by nonzero elements $f \in A$. If f is such an element and P is any nonzero differential operator then $\ker fP = \ker P$ and we have an injective map $\operatorname{coker} P \to \operatorname{coker} fP$.Therefore if fP has an index then so does P and the following equality holds:

$$\chi(P;A) - \sum_{i=1}^r \nu_i(P) = \chi(fP;A) - \sum_{i=1}^r \nu_i(fP).$$

In other words, to prove the desired equality for a differential operator P it suffices to prove it for fP, where $f \in A$ is some nonzero element. By Lemma 4.2.4 there exists a rational function on X that is regular on U and which has a pole of order at least one at every point of $X \setminus U$. Taking f to be an appropriate power of such a function and replacing P by fP we may assume that
$$\nu_i(P) < 0 \qquad \text{for all} \qquad i = 1, \ldots, r.$$
With this assumption we construct a filtration on A that will allow us to compute the index of P. Consider the Weil divisor

$$D = -\sum_{i=1}^r \nu_i(P)[x_i].$$

By assumption $D \geq 0$ and $\operatorname{supp} D = X \setminus U$. Therefore

$$A = \bigcup_{n \geq 0} \mathscr{L}(nD).$$

We use Proposition 4.2.6 to show that the triple $(A, \mathscr{L}(\bullet D), P)$ satisfies the assumptions of Lemma 2.2.9. By definition

$$\mathscr{L}(D) = \{x \in A : \nu_i(x) \geq n\nu_i(P) \text{ for all } i = 1, \ldots, r\}.$$

By inequality (4.5) in the first part of Proposition 4.2.6 we have

$$\nu_i(P(x)) \geq \nu_i(P) + \nu_i(x) \geq (n+1)\nu_i(P)$$

for $x \in \mathscr{L}(nD)$. Therefore

$$P(\mathscr{L}(nD))) \subset \mathscr{L}((n+1)D).$$

Since $\deg D > 0$ by assumption, it follows from the Riemann–Roch formula (4.3) that for $n \gg 0$ the equality

$$\ell(nD) = 1 - g + \deg(nD) = 1 - g + n\deg D \tag{4.12}$$

holds. Let n_0 be a positive integer such that the above equality holds for $n \geq n_0 - 1$ and equalities (4.6) in Proposition 4.2.6 hold for $\nu_i(x) \leq (n_0 - 1)\nu_i(P)$ and $i = 1, \ldots, r$. We claim that for $n \geq n_0$ the induced maps

$$P : \mathrm{gr}^n A = \frac{\mathscr{L}(nD)}{\mathscr{L}((n-1)D)} \to \frac{\mathscr{L}((n+1)D)}{\mathscr{L}(nD)} = \mathrm{gr}^{n+1} A$$

are isomorphisms. Since by (4.12) we have

$$\dim_k \mathrm{gr}^n A = \ell(nD) - \ell((n-1)D) = \deg D \tag{4.13}$$

for all $n \gg 0$, it suffices to show that the induced maps are injective. Take a nonzero $\bar{x} \in \mathrm{gr}^n A$ and let x be its lift to $\mathscr{L}(nD)$. Then for some i we have

$$\nu_i(x) < (n-1)\nu_i(P) \leq (n_0 - 1)\nu_i(P),$$

because otherwise $x \in \mathscr{L}((n-1)D)$. By our assumptions on n_0 equality (4.6) holds, so we have

$$\nu_i(P(x)) = \nu_i(x) + \nu_i(P) < n\nu_i(P),$$

and $\overline{P(x)}$ is a nonzero element of $\mathrm{gr}^{n+1} A$. This shows that the maps $P : \mathrm{gr}^n A \to \mathrm{gr}^{n+1} A$ are isomorphisms and assumptions of Lemma 2.2.9 are satisfied. Using corollary 2.2.10, equality (4.12), and definition of D we compute that

$$\chi(P; A) = -\deg D = \sum_{i=1}^{r} \nu_i(P)$$

to conclude the proof. □

4.2.4 Deligne's index formula

We now prove Deligne's formula (4.2) using results of the previous subsection. Among many equivalent definitions of the *irregularity* $\mathrm{irr}_x(\mathscr{E}, \nabla)$ of $(\mathscr{E}, \nabla)$ we recall the one that is best suited for our purposes. As in the previous subsections, we let U be a smooth affine curve with smooth compactification X. Let $x \in X$ be a closed point, $t \in \mathscr{O}_{X,x}$ a fixed uniformizer, and $\delta : k(X) \to k(X)$ the unique derivation with $\delta(t) = t$. We also write $\eta \in X$ for the generic point. Then ∇ induces a k-linear map

$$\nabla_\delta : \mathscr{E}_\eta \to \mathscr{E}_\eta; \quad m \mapsto \langle \delta, \nabla(m) \rangle$$

which satisfies the Leibniz rule. By the *cyclic vector theorem* [ABC20, Chapter 1.3.2] there exists a *cyclic vector* $v \in \mathscr{E}_\eta$, i.e., a vector v such that $\{v, \nabla_\delta v, \ldots, \nabla_\delta^{n-1} v\}$ is a $k(X)$-basis of $\mathscr{E}_\eta$. In this situation there exist unique $a_0, \ldots, a_{n-1} \in k(X)$ such that

$$\nabla_\delta^n v = \sum_{i=0}^{n-1} a_i \nabla_\delta^i v.$$

Repeating this construction for the dual connection $(\mathscr{E}^\vee, \nabla^\vee)$ we find (locally) a basis for $(\mathscr{E}, \nabla)$ in which the connection matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ a_0 & a_1 & a_3 & \ldots & a_{n-2} & a_{n-1} \end{pmatrix}. \tag{4.14}$$

The *associated differential operator* is then defined as

$$P_{\delta,v} = \delta^n - \sum_{i=0}^{n-1} a_i \delta^i,$$

and the *irregularity* of ∇ at x is defined as

$$\mathrm{irr}_x(\mathscr{E}, \nabla) = i_x(P_{\delta,v}) = \sup\{0, -\nu_x(a_i)\}.$$

Since the choice of a cyclic vector is highly non-canonical it is not clear that this definition is well posed. We refer to [Mal72, Définition 5.3] for the proof that this is indeed the case. Now let $\theta : k(X) \to k(X)$ be any nonzero k-derivation. Then similarly we have an action

$$\nabla_\theta : \mathscr{E}_\eta \to \mathscr{E}_\eta; \quad m \mapsto \langle \theta, \nabla(m) \rangle.$$

Since $\dim X = 1$, there exists unique $f \in k(X)$ such that $\theta = f\delta$. Therefore we have

$$f \nabla_\delta = \nabla_\theta,$$

and, more generally
$$f^i\nabla_\delta^i = \nabla_\theta^i + \text{lower degree terms.}$$

This is easily verified by induction on i using formulas given in the proof of formula (2.27). It follows that v is also a cyclic vector for ∇_θ and we have

$$P_{\theta,v} = f^n P_{\delta,v}.$$

So by the definition of the irregularity of a differential operator we have

$$i(P_{\theta,v}) = i(f^n P_{\delta,v}) = i(P_{\delta,v}).$$

The punchline is that the irregularity of ∇ can be computed with respect to any nonzero derivation of $k(X)$. We now prove the following theorem:

Theorem 4.2.11 (Deligne)**.** *Formula (4.2) holds for any* $(\mathscr{E},\nabla) \in \mathrm{MIC}(U)$.

Remark 4.2.12. We remark that the proof below is considerably longer than the one given in [ABC20, Theorem 24.1.3]. Still, we believe that our proof has some value, mainly because it illustrates the general philosophy that (1) the study of vector bundles with connections on curves can be reduced to the study of differential operators via the cyclic vector theorem, and (2) the latter can be successfully studied with tools from valuation theory.

Proof. We first deal with a special case that follows from Theorem 4.2.8. Then we use "additivity in topological spaces" of both sides of equation (4.2) to prove the general case. For the special case we assume that:

(1) Ω_U^1 is globally free and generated by, say, dx,

(2) $\mathscr{E}$ is globally free of rk $\mathscr{E} = n$,

(3) There exists a basis $e_1,\dots,e_d$ of $\mathscr{E}$ such that $\nabla : \mathscr{E} \to \Omega_U^1(\mathscr{E}) = \mathscr{E}dx$ can be written in this basis as $\nabla = d - Adx$, where A is as in (4.14).

Note that these assumptions are always satisfied Zariski locally on U, because of the local existence of a cyclic vector. Let ∂ be a derivation satisfying $\langle dx, \partial\rangle = 1$. The operator associated to ∇_∂ is then
$$P = \partial^n + \sum_{i=0}^{n-1} a_i \partial^i.$$

Since U is affine the de Rham cohomology of $(\mathscr{E},\nabla)$ is just the cohomology of the complex of global sections

$$\mathscr{E}(U) \xrightarrow{\nabla_\partial} \mathscr{E}(U).$$

It follows from basic and well-known computations (cf. [ABC20, Lemma 3.2.14]) that under our assumptions

$$\chi_{dR}(U,(\mathscr{E},\nabla)) = \chi(\nabla_\partial;\mathscr{E}(U)) = \chi(P;\mathscr{O}_U(U)).$$

By Theorem 4.2.8 we have

$$\chi_{dR}(\mathscr{O}_U,d) = \chi(\partial;\Gamma(U,\mathscr{O}_U)) = \sum_{x\in X\setminus U} \nu_x(\partial).$$

From Theorem 4.2.8 and formula (4.10) we obtain

$$\chi_{dR}(U,(\mathscr{E},\nabla)) = \sum_{x\in X\setminus U} \nu_x(P) = -\sum_{x\in X\setminus U} i_x(P) + n\sum_{x\in X\setminus U} \nu_x(\partial).$$

Therefore

$$\begin{aligned}(\operatorname{rk}\mathscr{E})\chi_{DR}(U,(\mathscr{O}_U,d)) - \chi_{DR}(U,(\mathscr{E},\nabla)) &= n\sum_{x\in X\setminus U} \nu_x(\partial) + \sum_{x\in X\setminus U} i_x(P) - n\sum_{x\in X\setminus U} \nu_x(\partial)\\ &= \sum_{x\in X\setminus U} i_x(P) = \sum_{x\in X\setminus U} \operatorname{irr}_x(U,\nabla).\end{aligned}$$

Thus Theorem 4.2.11 holds in our special case.

We now prove the Theorem under the weaker assumption that there exists an open cover $U = U_1\cup U_2$ such that on each U_i the assumptions of the previous case are satisfied. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E}(U) & \longrightarrow & \mathscr{E}(U_1)\oplus\mathscr{E}(U_2) & \longrightarrow & \mathscr{E}(U_1\cap U_2) & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1_U\otimes\mathscr{E}(U) & \longrightarrow & \Omega^1_U\otimes\mathscr{E}(U_1)\oplus\Omega^1_U\otimes\mathscr{E}(U_2) & \longrightarrow & \Omega^1_U\otimes\mathscr{E}(U_1\cap U_2) & \longrightarrow & 0\end{array}$$

The rows of this diagram are exact since they are Čech complexes for the affine covering of the affine variety U. This gives the *Mayer–Vietoris formula*

$$\chi_{dR}(U,(\mathscr{E},\nabla)) = \chi_{dR}(U_1,(\mathscr{E},\nabla)) + \chi_{dR}(U_2,(\mathscr{E},\nabla)) - \chi_{dR}(U_1\cap U_2,(\mathscr{E},\nabla)). \tag{4.15}$$

Clearly, X is also the compactification of U_1, U_2, and $U_1 \cap U_2$. Therefore from the inclusion-exclusion formula we obtain

$$\sum_{x \in X \setminus (U_1 \cup U_2)} \mathrm{irr}_x(\mathscr{E}, \nabla) = \sum_{x \in X \setminus U_1} \mathrm{irr}_x(\mathscr{E}, \nabla) + \sum_{x \in X \setminus U_2} \mathrm{irr}_x(\mathscr{E}, \nabla) - \sum_{x \in X \setminus (U_1 \cap U_2)} \mathrm{irr}_x(\mathscr{E}, \nabla). \tag{4.16}$$

Substracting equation (4.15) from (4.16) we conclude Theorem 4.2.11 in this case.

Finally, we prove theorem in full generality. Let $U = \bigcup_{i=1}^{s} U_i$ be a finite open cover such that Deligne's formula holds for all $(U_\alpha, (\mathscr{E}, \nabla)_{|U_\alpha})$, where

$$U_\alpha = U_{a_1} \cap \cdots \cap U_{a_k}.$$

Note that such a cover always exists because assumptions of the special case are satisfied locally and are preserved under restricting to smaller open subsets. The case $s = 2$ has already been dealt with in the second special case. The case when $s > 2$ follows formally by induction from that one. Let $U' = \bigcup_{i=1}^{s-1} U_i$ and $U'' = U_s$. Then Deligne's formula holds for U' by inductive assumption and for U'' as well. Moreover, we have $U' \cap U'' = \bigcup_{i=1}^{s-1} (U_i \cap U_s)$ and thus Deligne's formula holds for $U' \cap U''$ again by inductive assumption because it holds for any finite intersection of sets of form $U_i \cap U_s$. □

4.3 Examples

We finish this chapter and the whole thesis by giving some elementary examples of index formulas for holonomic $\mathscr{D}$-modules. If either $X = \operatorname{Spec} A$ is a smooth affine curve, or $X = \operatorname{Spa} A$ is a smooth affinoid curve then it is very easy to give examples of holonomic $\mathscr{D}_X$-modules. Indeed, let M be a left $\mathcal{D}_A$-module and let $\mathscr{M} = \widetilde{M}$ be the corresponding $\mathscr{D}_X$-module. Since $\dim X = 1$, by Lemma 2.3.26 for $\mathscr{M}$ to be holonomic it is necessary and sufficient that

$$\mathrm{Ext}^0_{\mathcal{D}_A}(M, \mathcal{D}_A) = \mathrm{Hom}_{\mathcal{D}_A}(M, \mathcal{D}_A) = 0,$$

because this is precisely the case when M is of minimal dimension. Now let $0 \neq P \in \mathcal{D}_A$ and let us denote

$$\mathscr{M}_P = \mathscr{D}_X / \mathscr{D}_X P.$$

Then we have a short exact sequence

$$0 \to \mathscr{D}_X \xrightarrow{\times P} \mathscr{D}_X \to \mathscr{M}_P \to 0. \tag{4.17}$$

After applying $\mathscr{H}om_{\mathscr{D}_X}(-,\mathscr{D}_X)$ to this sequence we easily conclude that

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}_P,\mathscr{D}_X)=0, \qquad \mathscr{E}xt^1_{\mathscr{D}_X}(\mathscr{M}_P,\mathscr{D}_X)=\mathscr{D}_X/P\mathscr{D}_X,$$

and in particular $\mathscr{M}_P$ is holonomic. Let us further assume that X is étale over $\mathbb{A}^1$ (resp. $\mathbb{B}^1$) and let dx be a global generator of Ω^1_X. Then

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{D}_X)=0, \qquad \mathrm{Ext}^1_{\mathscr{D}_X},(\mathscr{O}_X,\mathscr{D}_X)=\mathscr{O}_X(X),$$

and therefore we have an exact sequence

$$0\to H^0_{\mathrm{dR}}(X,\mathscr{M})\to\mathscr{O}_X(X)\to\mathscr{O}_X(X)\to H^1_{\mathrm{dR}}(X,\mathscr{M})\to 0 \tag{4.18}$$

obtained by applying to (4.17) the functor $\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{O}_X,-)$. The arrow $\mathscr{O}_X(X)\to\mathscr{O}_X(X)$ corresponds to the right multiplication by P on $\mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{D}_X)$, and thus it is simply given by $f\mapsto P^t(f)$ (cf. Formula (2.37)). We obtain the formula

$$\chi_{dR}(X,\mathscr{M}_P)=\chi(P^t;A). \tag{4.19}$$

4.3.1 Examples in the affine case

Let us give some concrete examples in the affine case using formula (4.19) and Theorem 4.2.8.

Proposition 4.3.1 (Katz, [Kat90, Lemma 2.9.12])**.** *Let $X=\mathbb{A}^1$ with global coordinate x.*

(1) *If $P=\sum_{i=0}^n f_i(x)(\frac{xd}{dx})^i$, then $\chi(P;K[x])=-\max\{\deg f_i\}$.*

(2) *If $P=\sum_{i=0}^n g_i(x)(\frac{d}{dx})^i$, then $\chi(P;K[x])=-\max\{\deg g_i-i\}$.*

Proof. Let $y=x^{-1}$. Then $\frac{d}{dx}(y)=-y^2$, and therefore

$$\frac{d}{dx}=-y^2\frac{d}{dy}, \qquad x\frac{d}{dx}=-y\frac{d}{dy}. \tag{4.20}$$

If we write ∞ for the unique point in $\mathbb{P}^1\setminus\mathbb{A}^1$, then $v_\infty(f)=-\deg f$. Using Theorem 4.2.8 we compute that

$$\chi(P;K[x])=v_\infty(P)=\min\{-\deg f_i\}=-\max\{\deg f_i\},$$

so (1) holds. This formula is nice but (2) is a more standard way of presenting a differential operator on $\mathbb{A}^1$. One verifies by induction on i the formula

$$x^i\left(\frac{d}{dx}\right)^i=\prod_{j=0}^{i-1}\left(x\frac{d}{dx}-j\right),$$

which implies that

$$x^i\left(\frac{d}{dx}\right)^i = \left(x\frac{d}{dx}\right)^i + \sum_{j=0}^{i-1} a_{ij}(\frac{xd}{dx})^j,$$

with $a_{ij} \in \mathbb{Z}$. We conclude that

$$f_i(x) = x^{-i}g_i(x) + \sum_{j=i+1}^{n} x^{-j}a_{ji}g_j(x).$$

In particular, we have

$$\deg f_i \leq \max_{j\geq i}\{\deg g_j - j\} \leq \max_{j}\{\deg g_j - j\}.$$

Now let i_0 be the maximal index among indices i satisfying $\deg g_i - i = \max\{\deg g_j - j\}$. Then $\deg f_{i_0} = \max_j\{\deg g_j - j\}$. Therefore (2) follows from (1). □

We now move to the case $X = \mathbb{G}_m$. Let $\mathbb{P}^1 \setminus \mathbb{G}_m = \{0, \infty\}$, and let v_0, v_∞ be corresponding valuations. For a Laurent polynomial

$$f(x) = a_k x^k + \cdots + a_m x^m \in K[x, x^{-1}], \quad k \leq m, \quad a_{-k}a_m \neq 0,$$

we have $v_0(f) = k$ and $v_\infty(f) = k$. We prove the following.

Proposition 4.3.2 (Katz, [Kat90, Lemma 2.9.13]). *With the above notation let $P = \sum_{i=0}^n f_i(x)(\frac{xd}{dx})^i$, where $f_i \in K[x, x^{-1}]$. Then*

(1) $\chi(P; K[x, x^{-1}]) = \min_i v_0(f_i) + \min_i v_\infty(f_i)$.

(2) *If we write*

$$P = \sum_j x^j P_j(\frac{xd}{dx}), \qquad P_i \in K[x],$$

then

$$\chi(P; K[x, x^{-1}]) = \max\{i : P_i \neq 0\} - \min\{i : P_i \neq 0\}.$$

Proof. (1) follows from Theorem 4.2.8, and (2) follows from (1). □

4.3.2 Examples in the affinoid case

We now prove formulas analogous to those given in the previous subsection when $X = \operatorname{Spa} K\langle x\rangle$ is Tate's disc and when $X = \operatorname{Spa} K\langle x, x^{-1}\rangle$ is the annulus $\{|x| = 1\}$. Recall that by definition $K\langle x, x^{-1}\rangle = K\langle x, y\rangle/(xy - 1)$.

Assume more generally that $X = \operatorname{Spa} A$ is a smooth affinoid curve and that the reduction $\widetilde{A}$ is again smooth. Assume moreover that $|A|_{\mathrm{Sup}} \subset |K|$ (affinoid algebras with this property are called *distinguished*, cf. [BGR84, Definition 2, Page 254]). Let $P : A \to A$ be a nonzero differential operator. Then P is continuous by Theorem 2.3.13 and since the valuation on K is discrete we have $|P| \in |K|$. Therefore there exists an integer k such that $|\varpi^k P| = 1$. Of course, we have $\chi(P;A) = \chi(\varpi^k P;A)$. This shows that to compute the index of P we may assume that $|P| = 1$. We are now in the situation of Lemma 2.2.6. The $\mathfrak{o}_K$-module A° is complete and torsion free, and the induced map (which is easily seen to be a nonzero differential operator) $\overline{P} : \widetilde{A} \to \widetilde{A}$ has an index by Theorem 4.2.8. Since under our assumptions $\widetilde{A} = A^\circ \otimes_{\mathfrak{o}_K} k$, we conclude that

$$\chi(P;A) = \chi(\overline{P};\widetilde{A}). \tag{4.21}$$

If we fix a closed point x in the compactification of $\operatorname{Spec} \widetilde{A}$, and define

$$v_x^{\mathrm{res}}(P) = v_x\left(\overline{\varpi^{-\log|P|}P}\right),$$

then formula (4.21) together with Theorem 4.2.8 implies the equality

$$\chi(P;A) = \sum v_x^{\mathrm{res}}(P), \tag{4.22}$$

where the sum on the right hand side runs over all point at the infinity of $\operatorname{Spec} \widetilde{A}$.

Let $f = \sum_{n\geq 0} a_n x^n \in K\langle x\rangle$. We define the *residual degree* of f to be

$$\operatorname{degres} f = \max\{d : |a_d| = |f|\}.$$

In other words, if we rescale f by some power of a uniformizer, so that $\varpi^k f \in \mathfrak{o}_K\langle x\rangle \setminus (\varpi)\mathfrak{o}_K\langle x\rangle$ then

$$\operatorname{degres} f = \deg \overline{\varpi^k f}.$$

Proposition 4.3.3. *Let $P = \sum_{i=0}^n f_i(x)\partial^i : K\langle x\rangle \to K\langle x\rangle$ be a nonzero differential operator. Then*

$$\chi(P;K\langle x\rangle) = -\max\{\operatorname{degres} f_i - i : |f_i| = |P|\}.$$

Proof. This follows from equality (4.21), Proposition 4.3.1 and the easy observation that $|P| = \max\{|f_i|\}$. □

As a very special case we can return to the differential operator $P = \partial - \varpi^{-1}$ discussed in the introduction. It is straightforward that $\chi(P; K\langle x\rangle) = 0$.

We now move to the case of the annulus $\{|x| = 1\}$. Write $\delta = \frac{xd}{dx}$. Given a differential operator

$$P = \sum_{i=1^n} f_i(x)\delta^i, \qquad f_i \in K\langle x, x^{-1}\rangle$$

we can rewrite it as

$$P = \sum_j x^j P_j(\delta), \qquad P_j \in K[x].$$

Proposition 4.3.4. *Let $P = \sum_j x^j P_j(\delta)$ be a nonzero differential operator on $K\langle x, x^{-1}\rangle$. Then*

$$\chi(P; A) = \max\{i : |P_i| = |P|\} - \min\{i : |P_i| = |P|\}.$$

Proof. The claim follows immediately from Propostion 4.3.2 and formula (4.21). $\square$

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