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Report on the Ph.D. thesis of Feliks Rączka

The Ph.D. thesis of Feliks Rączka deals with linear differential equations over the field of Laurent series $\mathbf{C}(\!(t)\!)$. This field is equipped with a valuation, giving rise to a distance, and one may look for solutions of such differential equations in the form of power series, possibly with some prescribed convergence conditions, or investigate the obstructions forbidding solutions to exist.

The interest and relevance of this topic appears clearly when one investigates meromorphic families of complex differential equations, or complex differential systems. In the simplest case, the parameter space is the punctured disc $D(0,1)^*$, with coordinate t, and, in order to understand the degenerating behavior in the neighborhood of the singularity 0, one is led to consider the corresponding differential system over the field $\mathbf{C}(\!(t)\!)$. In this setting, the celebrated Turrittin–Levelt–Hukuhara decomposition theorem asserts that, up to some ramification of the coordinate and base change, the differential system may be put in a particularly simple form (roughly analogous to the Jordan decomposition for matrices).

A strategy that is very similar in spirit was carried out by Kedlaya in order to settle a conjecture by Sabbah proposing a local description of modules with connections (algebraic objects encapsulating the data of systems of differential equations in a coordinate-free way) on complex surfaces. In this case, starting with a smooth complex surface and an isolated singularity, one no longer works directly over the field $\mathbf{C}(\!(t)\!)$, but on an analytic curve over it.

In a different direction, differential equations over nonarchimedean valued fields have been intensively studied in the p-adic setting, where p is a prime number. This time, the motivation comes from the study of varieties over finite fields of characteristic p. For such a variety, the de Rham complex does not give rise to a cohomology theory with the expected properties. One way to bypass this issue is to begin by lifting the variety to characteristic 0 (over some p-adic field), and to compute the de Rham cohomology there. When done properly, one ends up computing the cohomology of a module with connection on a p-adic analytic space, the so-called rigid cohomology of the original variety (of positive characteristic), as defined by Berthelot. This theory has been carefully studied, also in a version with coefficients, and Kedlaya proved its finite-dimensionality in full generality. His proof heavily relies on some specific features of the situation, such as the existence of Frobenius structures.

The different points mentioned above contribute to explain why the subject of differential equations in the nonarchimedean analytic setting is very active nowadays, with various applications in view. The associated cohomology theories are meaningful, and understanding them better is an important issue.

Feliks Rączka's thesis contributes a new major step in this story, by proving finite-dimensionality of de Rham cohomology with coefficients for analytic varieties over $\mathbf{C}((t))$. His main result precisely reads as follows. *Let X be a quasi-compact, quasi-separated, smooth rigid analytic space over $\mathbf{C}((t))$. Then, all the de Rham cohomology groups of X with coefficients in any holonomic $\mathscr{D}_X$-module are finite-dimensional over $\mathbf{C}((t))$.*

This result is striking in several regards. First, it is extremely general. In particular, no assumption at all is made about the class of "differential equations" under consideration, and the result holds for every module with integrable connection (and even more). This comes in stark contrast with the p-adic case, where the corresponding statement is false, already for the trivial module with connection $(\mathcal{O}, d)$ of the unit disk of dimension 1 (since the radius of convergence of a power series may drop after integration). Second, as far as I know, only very few instances of the result were known before this thesis, and in quite specific settings (such as the case of curves in work by Pulita and myself). As a consequence, Feliks Rączka's theorem goes well beyond the state of the art, and provides a definitive answer to the question under consideration. Let me add that, even if it is well-known that strong differences exist between the mixed and equal characteristic cases, the fact that the finite-dimensionality of de Rham cohomology holds in full generality over $\mathbf{C}((t))$ remains somewhat surprising, even if not completely unexpected.

Without going fully into details, it may be interesting to add a word about the notion of $\mathscr{D}$-module, which appears in the statement of the theorem. Recall that the symbol $\mathscr{D}$ stand for the sheaf of differential operators on the given space X, which is a sheaf of rings, and that any differential operator P acting on the functions on X gives rise to a $\mathscr{D}$-module by considering the quotient $\mathscr{D}/\mathscr{D}P$. It follows that $\mathscr{D}$-modules provide a further generalization of differential equations. All modules with integrable connections actually naturally give rise to $\mathscr{D}$-modules, and those arising this way satisfy an additional condition called *holonomicity*.

In his thesis, Feliks Rączka works directly in the setting of $\mathscr{D}$-modules, which seems the right choice to make. Even if the theory is technically more involved, he thus obtains a more general result, and has access to a more flexible formalism to reach is goal. For instance, it turns out useful in a series of reductions made to prove the main result, all of them quite standard: from arbitrary smooth quasi-compact quasi-separated smooth space to smooth affinoid spaces (by Mayer–Vietoris type arguments), and then to closed disks (by push-forward by a closed immersion). The latter step relies on the functorial behavior of $\mathscr{D}$-modules, and only makes sense in this setting. As a consequence, most of Feliks Rączka's work consists in understanding the case of $\mathscr{D}$-modules over closed disks.

Let us now focus on the case of the closed n-dimensional disk $\mathbf{B}^n$. Its ring of functions is the Tate ring $\mathbf{C}((t))\langle x_1, \dots, x_n\rangle$ of power series whose general term tends to 0. In this setting, $\mathscr{D}$-modules correspond bijectively to $\mathcal{D}$-modules, where $\mathcal{D}$ denotes the ring of differential operators on $\mathbf{C}((t))\langle x_1, \dots, x_n\rangle$, that is to say the very concrete ring

$$\mathcal{D} = \mathbf{C}((t))\langle x_1, \dots, x_n\rangle[\partial_1, \dots, \partial_n],$$

where ∂_i is the derivation with respect to x_i.

Although the question asked (finite-dimentionality of de Rham cohomology) is analytic in nature, Feliks Rączka aims at attacking it with algebraic tools. The general principle is to find a model of the whole situation over $\mathbf{C}[\![t]\!]$, and compare its "generic fiber", which is analytic over $\mathbf{C}(\!(t)\!)$, to its "special fiber" obtained by reduction modulo t, which is algebraic over $\mathbf{C}$. Such a strategy is classical and it is not difficult to find a model of the space : here the formal spectrum of $\mathbf{C}[\![t]\!]\langle x_1, \ldots, x_n\rangle$ is a model of the disk $\mathbf{B}^n$, and its special fiber is the affine space $\mathbf{A}^n_{\mathbf{C}}$ with ring of functions $\mathbf{C}[x_1, \ldots, x_n]$. As usual, the hard part lies in finding a model for the additional structure in play, the $\mathcal{D}$-module in our case. This may not be possible in general, but Feliks Rączka rescues the whole strategy by a brilliant remark: everything works as expected when $\mathcal{D}$ is replaced by its t-adic completion $\widehat{\mathcal{D}}$.

Two comments are in order. The first one is that the completion operation preserves the de Rham cohomology. The proof relies on the flatness of $\widehat{\mathcal{D}}$ over $\mathcal{D}$ and concrete computations with the Spencer complex. This is a crucial point since the main theorem deals with the de Rham cohomology of the original $\mathcal{D}$-module and not its completion. As noted in the thesis, this result alone has interesting applications. For instance, the de Rham cohomology of a $\mathcal{D}$-module whose completion is 0 vanishes.

The second comment is that proving the expected results (existence of models and finiteness of cohomology) for modules over $\widehat{\mathcal{D}}$ is still a very non-trivial task. The existence part is where the holonomicity condition comes into play. In this setting, it translates into the module being of *minimal dimension*, a technical condition of homological algebra. For such a module, a nice theory of duality exists, which is the key tool to show it contains a nice lattice (that is to say a model), more precisely a lattice whose reduction modulo t is a module of minimal dimension over the reduction $\overline{\mathcal{D}}$ of $\widehat{\mathcal{D}}$. Note that a $\overline{\mathcal{D}}$-module is nothing but an algebraic $\mathscr{D}$-module over the affine line $\mathbf{A}^n_{\mathbf{C}}$. When it is of minimal dimension, a theorem of Bernstein ensures that its de Rham cohomology is finite-dimensional.

It remains to show that the original $\widehat{\mathcal{D}}$-module has finite-dimensional cohomology. To achieve this, Feliks Rączka proves a technical lemma of independent interest, which allows to deduce finite generation of modules over discrete valuation rings from that of their reductions. (It looks like a sort of Nakayama's lemma, where finite generation is not an assumption but a conclusion.) The lemma works under some assumptions, which are satisfied for cokernels of complete modules, hence is well-suited for cohomology computations.

The thesis concludes with a chapter that is dedicated to a rather different topic, namely valuations on rings of differential operators. Let k be an algebraically closed field of characteristic 0 and let K be the function field of a smooth projective curve C over k.

Let v be a discrete valuation on K that is trivial on k. Picking a uniformizer t in the maximal ideal of v, one may consider a k-linear derivation δ_t such that $\delta_t(t) = t$. Since every k-linear differential operator P of K may be written as a polynomial in δ_t, one may extend v to P by the Gauss valuation formula. The extension of v to the ring of differential operators $\mathcal{D}_K$ is well-defined and satisfies the expected properties.

Feliks Rączka claims that such objects are worth studying and gives some examples where they can be used. The first one is the computation of the index $\chi(P)$ of a differential operator P acting on the ring of functions of an affine curve C' with function field K. Such a curve may be obtained from C by removing finitely many points $x_1, \ldots, x_r$. For each i, the point x_i determines a valuation v_i

on K, which extends to $\mathcal{D}_K$ by the procedure above. We then have

$$\chi(P) = \sum_{i=1}^{r} v_i(P).$$

The proof relies on Riemann–Roch theorem for curves. This equality may be used to effectively compute the index of some differential operators, which allows Feliks Rączka to recover some results by Katz.

As another application, Feliks Rączka gives a new proof of a formula of Deligne expressing the difference between the Euler characteristics of the algebraic and analytic de Rham cohomologies of a module with connection on a curve as a sum of irregularities.

The results presented in this last chapter only contain the first steps of a full theory of valuations on differential rings, and the given applications are mostly variations of existing results. However, the point of view is no doubt original and the proofs are elegant. The underlying theory is certainly worth investigating further, beyond the case of curves, and appears quite promising.

In his thesis, Feliks Rączka investigates the finite-dimensionality of the de Rham cohomology with coefficients, in the setting of analytic varieties over $\mathbf{C}(\!(t)\!)$. This is a fundamental question, which had surprisingly been left almost completely untouched in this context, and to which he gives a definitive answer (always positive) for a very large category of coefficients, namely holonomic $\mathscr{D}$-modules. It is quite rare to see an open question being completely solved in a Ph.D. thesis, and this is no doubt a very remarkable achievement.

The proof of the result is long and difficult and Feliks Rączka shows that he fully masters his subject, refining and combining various results in nonarchimedean funtional analysis, homological algebra (in a commutative or non-commutative context), $\mathscr{D}$-modules, algebraic geometry, etc. Beyond those technical challenges, Feliks Rączka needed to come up with several original ideas, such as the striking fact that the cohomology over the Weyl algebra is controlled by that over its completion. I expect this elegant and deep result to become a classical tool in the theory soon.

In the last chapter of the thesis, Feliks Rączka shows that he is ready to continue his mathematical journey by sharing new perspectives. In a few pages, he lays the foundations of a new theory of valuations on rings of differential operators, leading to convincing examples and promising applications.

For those reasons, I believe Feliks Rączka's thesis manuscript to be outstanding and I strongly recommend that he be awarded the degree of doctor in mathematics.

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