Pierre-Emmanuel Jabin
Professor, Department of Mathematics and Huck Institutes
Pennsylvania State University

State College, March 30th 2023

Referee report on the PhD thesis manuscript by Jakub Skrzeczkowski

The results presented by J. Skrzeczkowski correspond to an impressively extensive set of high-level works, with already 5 published articles, 2 preprints and 2 articles in preparation. The manuscript deals with a wide range of singular problems in PDE, especially around various important type of non-linear parabolic equations and systems. The potential applications of those contributions are just as wide-ranging.

The manuscript is divided in two parts with the first part being devoted to so-called singular limits. The main goal is to derive a system from another through some sort of asymptotic limit in the physical parameters. The second part focuses on “rough behaviors” : Parabolic equations or systems that can exhibit some singular behavior in non-homogeneous settings. The guiding example in the second part is the so-called $p(t,x)$-Laplace equations which extend porous medium or non-linear heat equations by letting the non-linear exponent depend on the time and space.

The introduction to the manuscript is very clear. It of course briefly presents each problem in turn and explain some of the difficulties together with the biological or physical motivation and the potential impact of J. Skrzeczkowski's results. Moreover this introduction showcases the common difficulty and shared ideas within the many works described here and it provides a real unity to the thesis. The introduction also insists on the context of the J. Skrzeczkowski's contributions in the existing literature. This first chapter hence testifies to J. Skrzeczkowski's exceptional scientific culture and maturity.

To delve now more into the manuscript itself, the first part contains 5 chapters logically organized around the various limits that are considered with the addition of a first chapter that summarizes several of the classical compactness approaches. While those are indeed now well-known techniques, they are by no means elementary, which makes this chapter precious. It also exemplifies again J. Skrzeczkowski's mastery of a wide range of tools.

Chapter 3 deals with the so-called fast reaction limit. It corresponds to an important and common question in systems of chemical or biochemical reactions, which are usually modeled through reaction-diffusion systems. One often finds that some chemical reactions are much faster than others. The speed of diffusion may also change considerably from one chemical species to another. While there exists an extensive literature on those issues, most of the results require some sort of monotone regularity on the nonlinearity. This is not the case here and it leads to major differences with the classical approaches as the underlying steady state for the fast reaction is not stable anymore.

Instead J. Skrzeczkowski proposes two somewhat related ideas which are to use Young measures or build an appropriate kinetic formulation. Those are implemented on a simple two-species model but I expect that those ideas are much more broadly applicable. Those new approaches allow to prove the compactness of the concentration of diffusing species and provide a representation for the limit of the concentration of the other species. I will not go into the technical details of the approach but I want to emphasize that the proofs rely on several delicate and careful estimates to take advantage of the energy estimates of the system.

The next chapter begins a thorough investigation of the famous Cahn-Hilliard system with degenerate diffusion. The first step performed in this chapter concerns the rigorous derivation of the system from a Vlasov equation. The corresponding Vlasov model includes a BGK term together with long-range attractive and short-range repulsive interactions. It is then possible to formally obtain Cahn-Hilliard through a so-called hydrodynamic limit, using the well-known parabolic scaling of the kinetic equation. It is an important observation that allows to derive Cahn-Hilliard from first principles and it also necessarily leads to a degenerate diffusion (other models do not seem to be attainable from Vlasov systems).

Various hydrodynamic limits of kinetic equations have already been extensively studied and the present proof follows the same general strategy. But it also relies on a masterful and highly technical combination of a priori estimates. Making rigorous the formal derivation is very challenging and is hence quite an achievement, even if it only yields the degenerate non-local Cahn-Hilliard.

Chapter 5 continues the study of the Cahn-Hilliard equation by proving the rigorous derivation of the local Cahn-Hilliard from the non-local version that had been obtained before. Much of the challenge is again here due to the degenerate diffusion, as the corresponding a priori estimates also degenerate. This issue can be partially resolved through the use of precise compactness estimates from Bourgain-Brézis-Mironescu and Ponce. But the lack of stronger regularity continues to play a role in the proof. Passing to the limit in the weak formulation now requires a smart and delicate analysis to mimic the algebra that is expected from the limit on the non-local terms.

Part I concludes with yet another derivation of the degenerate Cahn-Hilliard equation, this time with a high friction limit from the co-called non-local Euler-Korteweg system that is well-known in Fluid mechanics. The problem looks initially daunting as we do not have a good notion of solution for the Euler-Korteweg system and neither do we have classical solutions to the degenerate Cahn-Hilliard system (which would allow to rely on some weak-strong uniqueness principle). The second issue can be partially resolved by using the non-local Cahn-Hilliard which has smoother solutions and which was already shown by J. Skrzeczkowski to converge to the local Cahn-Hilliard. This then allows the use of a relative entropy approach, starting from the general concept of dissipative measure-valued solutions for the Euler-Korteweg system. This is a very clever strategy that leads to a broad, impactful result, but also involves several complex technical parts.

Part II begins, just like part I, by a chapter that recalls some necessary concepts, here around the so-called Musielak-Orlicz spaces. Many of the comments already made on the role that chapter 2 plays for part I also applies here. In particular it is again a remarkable summary of difficult mathematical tools, that also provides unifying concepts for the many works in this part. The discussion of variable exponent spaces is for example very useful in all subsequent chapters.

Chapter 8 deals with a first example of the type of non-homogeneous problems that are treated. The model under consideration is a “simple” non-linear heat equation where the non-linearity coefficients changes according to time and position. Those equations are commonly called $p(t,x)$-Laplace equation. Formally speaking, the logic behind well-posedness for those models is straightforward as the dissipation term from the L^2 estimate should control the non-linearity. When trying for rigorous proofs however, many issues quickly follow from the changes in exponent. In particular all previous results required some form of continuity in time in the exponent $p(t,x)$. And while one can easily check in some simplified setting that it should be possible to handle exponent $p(t,x)$ that are discontinuous in time, proving it in a general case is another matter.

The whole point revolves around delicate questions in functional analysis, when trying in particular to show that smooth functions are dense in the natural function space. Provided $p(t,x)$ has the right spatial regularity, it is possible to use convolution in space but of course convolution in time would not be working. The heart of J. Skrzeczkowskiś argument is that the equation itself provides the required time regularity once spatial regularity is obtained. This is a very powerful idea that allows to bypass the major technical objection.

Chapter 9 turns to the analysis of some models of non-Newtonian fluids, obtained by considering the incompressible Navier-Stokes system with a modified stress tensor. A typical example is the so-called Cauchy stress tensor that has been extensively studied. J. Skrzeczkowski considers instead another well-known variant, the non-standard growth, that exhibits a time and spatial dependence in the exponent as in the previous chapter. The corresponding inhomogeneities can for example model electrorheological fluids, including the effect of an external, non constant electric field. Many of the challenges faced here are reminiscent of the previous chapter with the main goal being again to allow exponents that are discontinuous in time. But while the approach is somewhat similar, several additional difficulties also appear : handling the non-linear convective terms for example (which do lead to additional restrictions on the exponent) or how to deal with the pressure terms.

The manuscript finishes with a very nice illustration of the ideas developed in this second part. Through a direct argument, J. Skrzeczkowski is indeed able to considerably extend the range of exponents which ensure the absence of the so-called Lavrentiev phenomenon. The Lavrentiev phenomenon appears in some calculus of variation problems, involving for example (p,q) functionals, and leads to the lack of smoothness in the minimizers.

To conclude this review of J. Skrzeczkowski's manuscript, I first want to emphasize the large diversity of problems, applications and techniques that are presented. Furthermore, those contributions demonstrate the outstanding mathematical skills of J. Skrzeczkowski, combined with an extensive scientific culture. The thesis introduces critical new ideas that brought a new vision to several difficult problems and have allowed J. Skrzeczkowski to obtain many breakthrough results. I believe that the approaches developed in the manuscript will have a strong impact in our community and also have the potential to prove extremely useful in a wide range of applications from Physics to the Life Sciences.