REPORT ON *THE COMBINATORIAL STRUCTURE OF CUMULANTS OF SYMMETRIC FUNCTIONS* BY M. KOWALSKI

PER ALEXANDERSSON

1. Summary

The introductory chapters provide a comprehensive overview of the subject, setting a solid foundation for the subsequent chapters. Chapters 3 and 4, which are based on the two included papers, present significant contributions to the field, with the first paper already published.

The first paper deals with certain LLT polynomials, and the new notion of LLT cumulants. LLT cumulants are shown to fit nicely together with the existing theory on LLT polynomials, and also the more recent notion of Macdonald cumulants. In particular, Thm. 3.13 is perhaps my favourite result in this paper. The Schur positivity conjecture (Conjecture 3) is also very intriguing and deserves further study.

The second paper proves Schur positivity of LLT cumulants (Conjecture 3 above) for a certain class of graphs. This is a natural special case to consider and it refines the previous Schur positivity of LLT polynomials for the same class of graphs.

2. Recommendation

I recommend to admit to the PhD defense.

3. Motivation

The two included papers significantly advance the literature on LLT polynomials, particularly through the innovative exploration of LLT cumulants. The research is well-motivated, with a clear vision of the broader implications and future directions.

The work in total seems comparable to some other PhD theses I have read.

4. Comments

4.1. **Paper I.**
The first paper is well-written and easy to follow. Below are some natural questions and comments that perhaps the author has thought about but not included.

In [4] (and earlier in [8]), it is noted that there is a close connection between chromatic symmetric functions and unicellular LLT polynomials. One can then ask the natural question if LLT cumulants correspond to something natural on the chromatic symmetric function side. This should be easy to check on the computer (just apply the plethystic substitution to the unicellular LLT cumulants) and see if the *chromatic analog* is say monomial positive, Schur positive or even e-positive. I suspect that the authors have tried this and nothing interesting came up, but even negative results should be reported. However, introducing the chromatic (quasi)symmetric functions in this setting would perhaps be a lot of extra work with little reward.

The connection with Tutte polynomials is rather interesting and it begs the question wether one can make a stronger connection with matroid theory in this setting. For example, on the chromatic symmetric function side, there is a conjecture that these are Lorentzian, a property closely connected with matroids.

4.2. **Paper II.** One could improve Chapter 4 a bit, or at least explore some special cases in more detail. For example, it is known that Hall–Littlewood polynomials (the modified ones) are special cases of LLT polynomials. These have a known (combinatorial) Schur expansion so it would be natural to explore if LLT cumulants in this (special) setting has a similar Schur expansion. Adding this would strengthen the paper (but it is perhaps too much to ask for to add to the thesis).

Another family to consider is the case of Abelian Dyck paths.

The proof of Lemma 4.4 in the thesis is not very different from [5] (or [3]), and on page 46, the second item in the proof, I think the statement that it suffices to look at cells i, j and $j+1$ locally is misleading. The last centered equation of the proof relies on a very non-local condition that needs to hold (the presence of two East steps above cells j and $j+1$). The pictorial identity on the bottom of page 45 is definitely misleading as the above condition is not included. To be clear, the proof is correct but could be formulated a bit clearer, or even be omitted as it has been proved (in similar ways) before.

Reference [2], [AN21] is in the bibliography but not cited in the thesis. In [AN21, Proposition 4.1], there is a sum over increasing forests that is similar-looking to Equation (4.5) in the thesis. Is there a relation between (4.1) and the symmetric functions introduced by Abreu and Nigro? The relation between Chapter 4 and this paper should be discussed.

What is the status of the paper on which Chapter 4 is based on?

4.3. **Very minor comments.** Page 10, in the definition of $Inv(T)$, should be corrected. It should be written something like

$$\mathrm{Inv}(\mathbf{T}) \coloneqq \{(\square, \square') : \square, \square' \in \boldsymbol{\lambda\mu} \text{ and } ...\}$$

since $\boldsymbol{\lambda\mu}$ is a set of boxes, and not a set of pairs of boxes.

There are some items in the bibliography that should be updated.

- Reference [1] is published, `https://doi.org/10.1017/prm.2021.61`
- In reference [5], LLT should be capitalized.
- Reference [9] is published, `https://doi.org/10.1016/j.aim.2022.108645`
- Reference [12], LLT should be capitalized, and add full article information (Journal, volume, number).

- In [20], q, t should be in math mode.
- In reference [39], Schubert should be capitalized.
- Reference [47] is published, `https://doi.org/10.1007/s10801-020-00950-7` (also put math in math mode)

References

[AN21] Alex Abreu and Antonio Nigro. A symmetric function of increasing forests. *Forum of Mathematics, Sigma*, 9, 2021. `doi:10.1017/fms.2021.33`.