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Toruń, 26.06.2024

Report on the doctoral dissertation by Maciej Kowalski
The combinatorial structure of cumulants of symmetric functions

General remarks. The dissertation was written under the supervision of Maciej Dołęga (IMPAN). It starts with the English and Polish summaries, and consists of Introduction, Preliminaries, two chapters with main results and the bibliography, with a total of 56 pages. As the author states, the thesis is based on two papers:

[A] *LLT cumulants and graph colorings* (joint with PhD supervisor; published in The Electronic Journal of Combinatorics; position [12] in the Bibliography of the thesis),

[B] *LLT cumulants of unicellular Young diagrams, parking functions and Schur positivity* (available on arXiv:2011.15080, position [42] in the Bibliography of the thesis).

The results of paper [A] are described in Chapter 3 and the results of paper [B] in Chapter 4.

There is another (unpublished) paper by Maciej Kowalski on the same subject, not listed in the bibliography of the thesis, available on arXiv:2301.08933:

[C] *A combinatorial formula for LLT cumulants of melting lollipops in terms of spanning trees.*

Description of the Dissertation. In the Introduction, the author collects basic information related to symmetric functions, including Schur functions and other bases of the algebra of symmetric functions, Macdonald and LLT polynomials, and symmetric function cumulants. This section also provides a historical overview and motivations for the topic of the doctoral dissertation.

The Preliminaries section introduces basic definitions and notation used throughout the thesis.

In Chapter 3, the author discusses the properties of quasi-symmetric functions called LLT cumulants, introduced in paper [A]. One of the motivations for introducing LLT cumulants is to attack the conjecture of Schur positivity of Macdonald cumulants (Conjecture 2 in the PhD thesis). In paper [A], the conjecture of Schur positivity of LLT cumulants is stated (Conjecture 3 in the PhD thesis). It is proved that Macdonald cumulants can be expressed as a positive linear combination of LLT cumulants of ribbon shapes (Theorem 3.2 in the thesis). This generalizes an analogous fact for Macdonald and LLT polynomials. Moreover, it is shown that Conjecture 3 implies Conjecture 2 (Theorem 3.3 in the thesis).

To attack Conjecture 3, an LLT polynomial is associated with an LLT graph (i.e., a finite directed graph with three types of edges). Additionally, a combinatorial interpretation of the LLT cumulant of a colored LLT graph G is described as a positive combination of LLT polynomials of subgraphs of G (Theorem 3.5 in the thesis). This graph-theoretical point of view allows for proving some positivity results (Subsection 3.5). In the last part of Chapter 3, Conjecture 3 is proved for some special cases (e.g., for pairs of skew Young diagrams). Additionally, some open questions and problems are stated.

In Chapter 4 the author presents results of the unpublished papers [B], [C]. The main result of this part is Theorem 4.1, where a combinatorial formula for the LLT cumulant $\kappa(\lambda/\mu)$ of a sequence λ/μ of unicellular diagrams corresponding to a melting lollipop is given as a positive combination of LLT polynomials indexed by spanning trees of this melting lollipop. Applying the result of Grojnowski and Haiman, saying that LLT polynomials are Schur positive, as a consequence of Theorem 4.1 one gets an inedependent proof of Schur positivity of LLT cumulants in this special case (the second proof is given in paper [A]). For the special case when the melting lollipop is a complete graph, the author expresses this formula in terms of parking functions (Corollary 4). To obtain the main results of Chapter 4, the doctoral candidate uses:

- the bijection given in Proposition 4.2 between the set of Schröder paths of length m and the set of LLT polynomials of sequences of vertical strips shapes with m boxes; it is a classical correspondence proved by J. Haglund, and the author of the dissertation provides an alternative proof of this fact;
- the bijection given in Proposition 4.3 between the set of plane rooted trees with m vertices and the set of Schröder paths of length m satisfying additional conditions;
- the relations between Schröder paths shown by Alexandersson and Sulzgruber (Lemma 4.4 in the thesis). In the thesis, the author provides an alternative proof of these relations.

Chapter 4 also contains remarks about directions for further research.

Remarks on mathematical results. The results presented in the thesis are new, very interesting, well-motivated, and expand the knowledge in the field of symmetric functions. Answers to natural questions arising in the theory of symmetric functions are provided. The doctoral candidate attacks the problem of Schur positivity of LLT cumulants in special cases. This problem is generally difficult. I especially like the graph-theoretical interpretation of LLT cumulants. I find Theorem 3.5, which shows how the combinatorial properties of LLT graphs correspond to the properties of LLT cumulants, very interesting. Theorem 3.5 works well in practice; for example, the doctoral candidate used it in the proof of Theorem 4.1. I have not noticed any gaps in the proofs. It is evident that the doctoral candidate is well-educated in the area of the dissertation. He uses combinatorial tools fluently and is able to apply them effectively in proving facts. The doctoral candidate is capable of working independently. The work is written at a good scientific level.

Remarks concerning thesis editing. Introduction is very well-written and provides a good overview of the topic being studied. However, I have some criticisms regarding the editing of the thesis.

Preliminaries and the parts of Chapters 3 and 4 that cover notation and definitions are written too briefly. There is a lack of examples illustrating the main definitions, as well as the notation and definitions used later. In my opinion, in a doctoral dissertation, concepts, definitions, and notation should be explained more carefully. Additionally, the lack of proper explanations makes the dissertation difficult to read. Below, I provide few specific comments and remarks, out of many that I have identified.

1. In my opinion, the paper referenced as [C] is more suitable for Chapter 4 than [B]. It is not clear to me why the author refers only to [B] there.
2. As I mentioned earlier, some concepts are described too briefly. This is illustrated by the following notes.
    (a) It was difficult for me to understand the definition of LLT cumulants given in formulas (3.4), (3.5), and (3.6). Up to this point in the dissertation, cumulants were associated with a set of elements of a certain algebra. The above formulas, instead of a set of algebra elements, use LLT polynomials. These definitions became clear to me only when I read the example presented on page 10 of paper [A]. It would be helpful to have such an example in the thesis.

(b) I couldn't find any definition of an inversion pair between vertical strips. This notion is used in the proof of Proposition 4.2.

3. Parts of the dissertation are written carelessly. There are some examples.

   (a) page 41: It seems to me that, to ensure Remark 7 is true (i.e., $s_i > 0$ and $t_i \geqslant 0$), Definition 4.1 needs to be modified, for example, by defining $s_i = h(T) - d(W_i) + 1$.

   (b) page 42: It is not mentioned that the proof of Proposition 4.2 is an alternative proof of J. Haglund's classical result. This is noted in paper [C].

   (c) page 46: Superscripts in graph relations are related to Lemma 3.2 in paper [C]. This notation is not present in the dissertation.

   (d) Bibliography: There is no information about the status of paper [12], which contains the results of the doctoral dissertation.

4. The proof of Theorem 4.1 is technical. An illustrative example would make it easier to understand.

These critical remarks relate to the style of editing the thesis. They do not affect the positive evaluation of the doctoral dissertation.

Conclusion. The dissertation fulfills all the legal requirements for the doctoral degree in mathematics. I recommend it for the further stages of consideration.