Splitting of globally hyperbolic spacetimes with timelike boundary

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Spacetimes with timelike boundary

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A *spacetime with timelike boundary* is a time-oriented Lorentzian manifold with timelike boundary, i.e., there exists a continuous timelike vector field T.

Spacetimes with timelike boundary

Proposition

The following properties are equivalent for any Lorentzian manifold with timelike boundary $(\overline{V}, g)$*:*

1. $(\overline{V}, g)$ *is time-orientable*
2. (V, g) *and* $(\partial V, g)$ *are time-orientable*
3. *There exists a smooth timelike vector field* T *on* $\overline{V}$ *that is tangent to* ∂V*.*

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- Each connected component of the boundary is also a spacetime (without boundary).

Spacetimes with timelike boundary

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Proposition (Solís,2006)

Let $(\overline{V}, g)$ be a spacetime with timelike boundary, then the following statements hold:

- $I^+(p)$ *and* $I^-(p)$ *are open subsets for all* $p \in \overline{V}$.
- *For any* $p, q, r \in \overline{V}$, $p \ll q \leq r$ *implies* $p \ll r$.
- $J^{\pm}(p) \subset cl(I^{\pm}(p))$ *for all* $p \in \overline{V}$.

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- *Strongly causal* if for all $p \in \overline{V}$ and any neighbourhood $U \ni p$ there exists another neighbourhood $U' \subset U$, $p \in U'$, such that any causal curve with endpoints at U' is entirely contained in U.

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- *Strongly causal* if for all $p \in \overline{V}$ and any neighbourhood $U \ni p$ there exists another neighbourhood $U' \subset U$, $p \in U'$, such that any causal curve with endpoints at U' is entirely contained in U.
- *Stably causal* if there exists a time function, i.e. , a continuous funtion $t : \overline{V} \to \mathbb{R}$ such that is strictly increasing over future-directed causal curves.

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- *Causally continuous* if distinguishing and $I^{\pm} : \overline{V} \to \mathfrak{P}(\overline{V})$ are continuous (for the natural topology in the set of parts $\mathfrak{P}(\overline{V})$ which admits as a basis the sets $\{U_K : K \subset \overline{V} \text{ is compact}\}$, where $U_K = \{A \subset \overline{V} : A \cap K = \emptyset\}$).

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- *Causally simple* if it is causal and $J^{\pm}(p)$ are closed subsets for all $p \in \overline{V}$.

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- *Causally simple* if it is causal and $J^{\pm}(p)$ are closed subsets for all $p \in \overline{V}$.
- *Globally hyperbolic* if it is causal and $J^{+}(p) \cap J^{-}(q)$ are compact subsets for all $p, q \in \overline{V}$.

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Proposition

Let $(\overline{V}, g)$ *be a spacetime with timelike boundary.*

- *If it is globally hyperbolic then it is causally simple.*
- *If it is causally simple then it is causally continuous.*
- *If it is causally continuous then it is stably causal.*
- *Stably causal* $\Rightarrow$ *strongly causal* $\Rightarrow$ *distinguishing* $\Rightarrow$ *causal* $\Rightarrow$ *chronological.*

Spacetimes with timelike boundary

Causality conditions on V and ∂V that are inherited from $\overline{V}$:

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Theorem

Let $(\overline{V}, g)$ be a spacetime with timelike boundary.

1. *If $(\overline{V}, g)$ is causally continuous then $(V, g|_V)$ is causally continuous and $(\partial V, g|_{\partial V})$ is stably causal.*
2. *If $(\overline{V}, g)$ is globally hyperbolic then $(\partial V, g|_{\partial V})$ is globally hyperbolic[a].*

[a] Solís' Ph. D. Thesis, 2006

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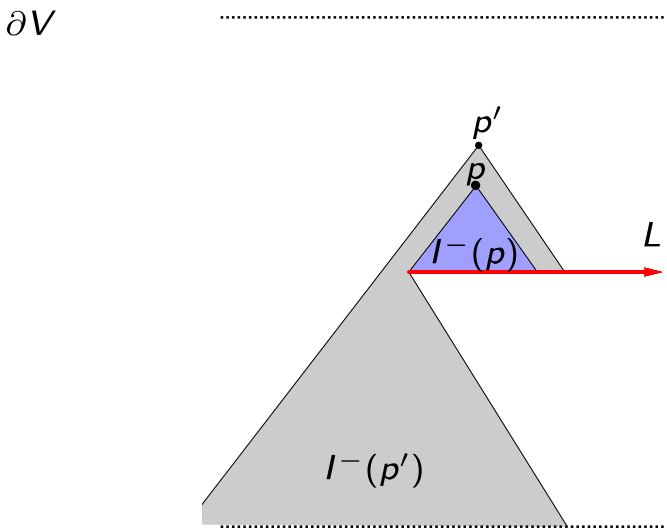

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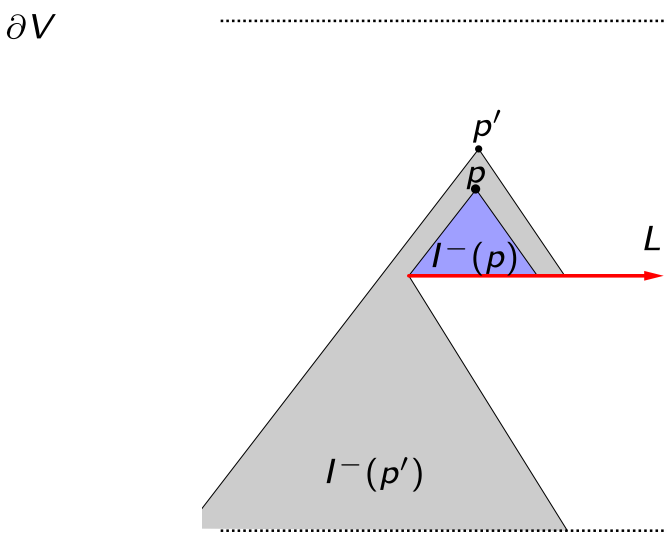

- $\overline{V} = \{(t, x, y) \mid y \geq 0\}$ with $g = -dt^2 + dx^2 + dy^2$.

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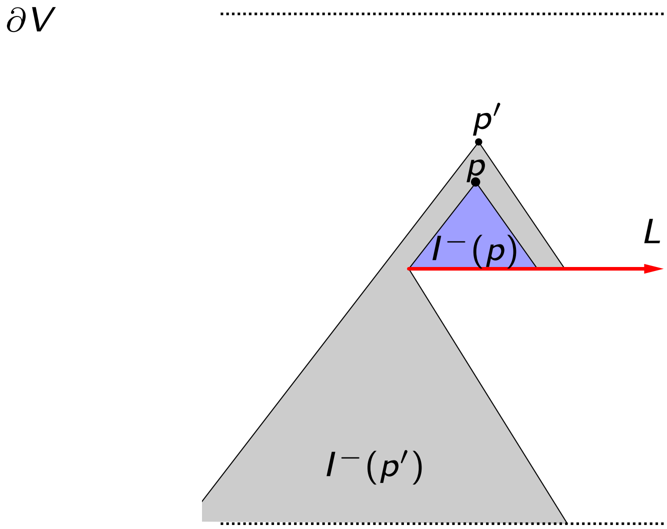

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- Let $\partial V := \{y = 0\} \setminus L$.

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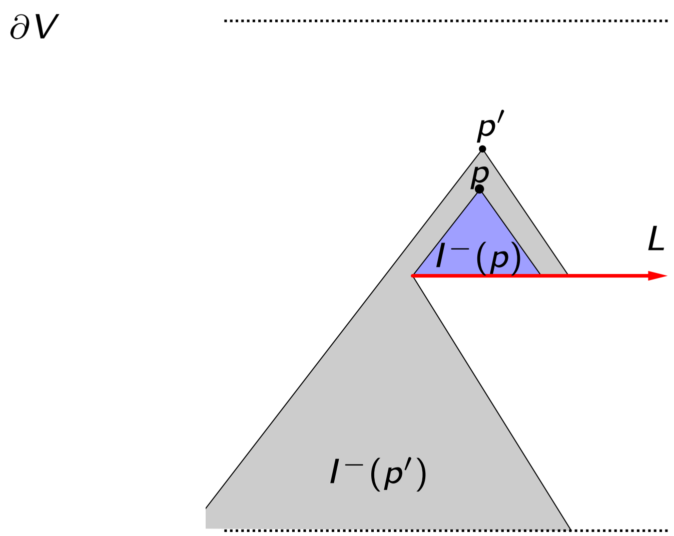

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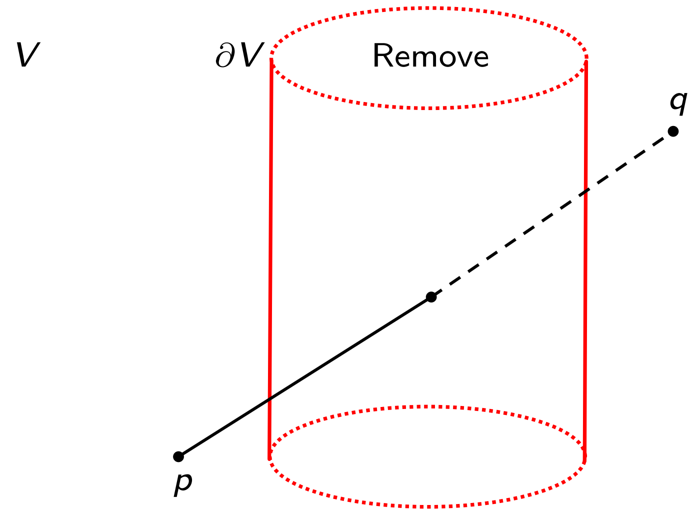
- Let $\partial V := \{y = 0\} \setminus L$.
- $(\partial V, g \mid_{\partial V})$ is not causally continuous.



- $(\overline{V}, g)$ is globally hyperbolic with timelike boundary, but $(V, g|_V)$ is not causally simple.

Spacetimes with timelike boundary

Definition

An achronal set $\bar{\Sigma} \subset \overline{V}$ *is a* Cauchy hypersurface *if it is intersected exactly once by every inextensible timelike curve.*

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Recall classical Geroch's theorem:

Theorem (Geroch,1972)

(V, g) *is globally hyperbolic spacetime if and only if admits a Cauchy hypersurface* Σ*. Even more, in this case, (i) the spacetime admits a Cauchy time function, (ii) all Cauchy hypersurfaces are homeomorphic to* Σ*, and* V *is homeomorphic to* $\mathbb{R} \times \Sigma$*.*

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Theorem

For any globally hyperbolic spacetime with timelike boundary $(\overline{V}, g)$*, Geroch's function* $t = ln(-\frac{t^-}{t^+})$ *is a Cauchy time function, that is,* t *is a time function and all its levels are Cauchy hypersurfaces.*

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Corollary

Let $(\overline{V}, g)$ *be globally hyperbolic with timelike boundary,* t *any Cauchy time function and* $\bar{\Sigma}_0 = t^{-1}(0)$. *Then* $\overline{V}$ *is homeomorphic to* $\mathbb{R} \times \bar{\Sigma}_0$. *Moreover,* $\bar{\Sigma}_0$ *is acausal and any other Cauchy hyp.* $\bar{\Sigma}$ *is homeomorphic to* $\bar{\Sigma}_0$.

Spacetimes with timelike boundary

Aim:

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Theorem (A.-Flores-Sánchez, 2018)

Any $(\overline{V}, g)$ globally hyperbolic spacetime with timelike boundary admits a Cauchy temporal function τ whose gradient is tangent to ∂V. Therefore, $\overline{V}$ splits smoothly as a product $\mathbb{R} \times \bar{\Sigma}$, where $\bar{\Sigma}$ is a $(n-1)$ Cauchy hypersurface with boundary, and the metric can be written as a parametrized ortogonal product:

$$g = -\Lambda d\tau^2 + g_\tau$$

where $\Lambda : \mathbb{R} \times \bar{\Sigma} \to \mathbb{R}$ is a positive function, g_τ is a Riemannian metric on each slice $\{\tau\} \times \bar{\Sigma}$ varying smoothly with τ, and these slices are spacelike Cauchy hypersurfaces with boundary.

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Main idea

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- $(\overline{V}, g^*)$ is globally hyperbolic with timelike boundary.
- ∂V "looks totally geodesic" inside $(\overline{V}, g^*)$.
- Construct a Cauchy temporal function (with tangent gradient to the boundary) τ in $(\overline{V}, g^*)$.

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Theorem (A.-Flores-Sánchez, 2018)

Let $(\overline{V}, g)$ be a globally hyperbolic spacetime with timelike boundary. Then, there exists a Lorentzian metric g' with $g' > g$ and such that $(\overline{V}, g')$ is globally hyperbolic with timelike boundary.

Spacetimes with timelike boundary

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Lemma (Existence of a global tubular neighborhood)

There exists a smooth function $\rho : \partial V \to \mathbb{R}$, $\rho > 0$, such that the orthogonal exponential map

$$\exp^{\perp} : \{(\hat{p}, s) \in \partial V \times [0, \infty) : 0 \leq s < \rho(\hat{p})\} \to \overline{V}, \qquad (\hat{p}, s) \mapsto \exp_{\hat{p}}(tN_p)$$

is a diffeomorphism onto its image E, which will be called tubular neighborhood *of ∂V.*

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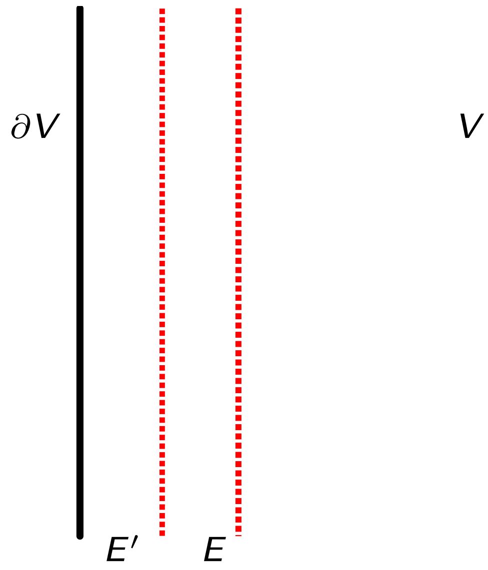

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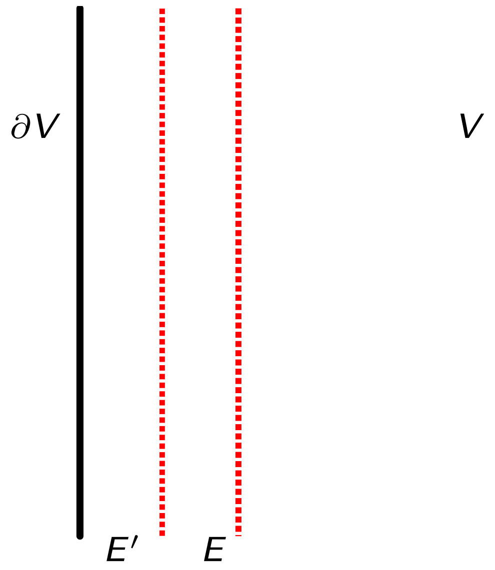

- In E take the Lorentzian metric $g_0 = \hat{g}_0 + ds^2$, where $\hat{g}_0$ globally hyperbolic metric on ∂V, with $g' > g_0 > g$ in E.

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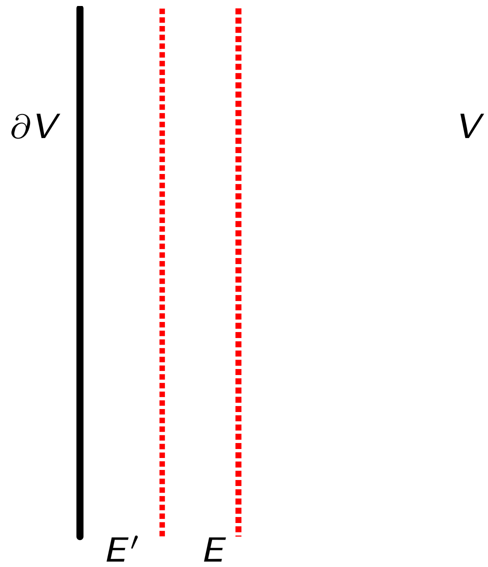

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- Open covering $\{\overline{V} \setminus cl(E'), E\}$ and $\{(1-\mu(\cdot)), \mu(\cdot)\}$ partition of unity.

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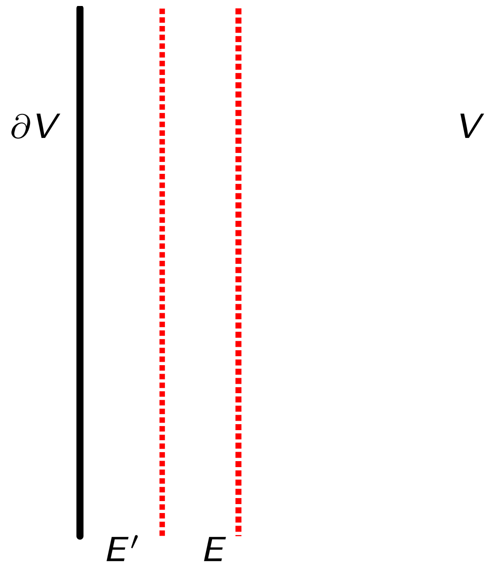

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- Open covering $\{\overline{V} \setminus cl(E'), E\}$ and $\{(1-\mu(\cdot)), \mu(\cdot)\}$ partition of unity.
- Metric $g^* = (1-\mu(\cdot))g + \mu(\cdot)g_0$.

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- ∂V is totally geodesic in $(\overline{V}, g^*)$.
- If $\tau : \overline{V} \to \mathbb{R}$ is a Cauchy temporal function for $(\overline{V}, g^*)$, then τ is a Cauchy temporal function for $(\overline{V}, g)$. Even more, if $\nabla^{g^*}\tau$ is tangent to ∂V, then $\nabla\tau$ is tangent to ∂V as well.

Spacetimes with timelike boundary

Theorem (A.-Flores-Sánchez, 2018)

The globally hyperbolic spacetime with timelike boundary $(\overline{V}, g^*)$ *admits a Cauchy temporal function* $\tau : \overline{V} \to \mathbb{R}$ *with* $\nabla^*\tau$ *tangent to the boundary* ∂V.

Sketch:

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- τ is also a Cauchy temporal function for $(\overline{V}, g)$.

References

1. L. Aké, J. L. Flores, and M. Sánchez. Splitting of globally hyperbolic spacetimes with timelike boundary. In progress, 2018.
2. R. Geroch. Domain of dependence. *J. Math. Phys.*, 11:437–449, 1970.
3. E. Minguzzi and M. Sánchez. The causal hierarchy of spacetimes, *ESI Lectures on Mathematics and Physics*.
4. O. Müller and M. Sánchez. Lorentzian manifolds isometrically embeddable in L^N, *Trans. Amer. Math. Soc.*, 363 (2011), 5367-5379.
5. O. Müller. A note on invariant temporal functions, *Lett. Math. Phys.* 106, no. 7, (2016).
6. D. Solís. global properties of asymptotically de Sitter and Anti de Sitter spacetimes. Ph.D. Thesis, University of Miami, 2006.

Thanks for your attention