CURVES IN LORENTZ-MINKOWSKI PLANE WITH PRESCRIBED CURVATURE

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1. $\kappa(x, y) = \kappa(y)$. Castro I., Castro-Infantes I., *Plane curves with curvature depending on distance to a line,* Diff. Geom. Appl., 2016, 44, 77–97.
2. $\kappa(x, y) = \kappa(\sqrt{x^2 + y^2})$. *Castro I., Castro-Infantes I., Castro-Infantes, J.,* New plane curves with curvature depending on distance from the origin, Mediterr. J. Math., 2017, 14, 108:1–19.

Curves with prescribed curvature

Theorem $\kappa(y)$

Prescribe $\kappa = \kappa(y)$ continuous. The problem of determining a curve $\gamma(s) = (x(s), y(s))$ -s arc length- with curvature $\kappa(y)$ is solvable by:

1. $\int \kappa(y)dy = \mathcal{K}(y)$, *geometric linear momentum*.
2. $s = s(y) = \int \frac{dy}{\sqrt{1-(\mathcal{K}(y))^2}} \dashrightarrow y = y(s) \dashrightarrow \kappa = \kappa(s)$.
3. $x(s) = -\left(\int \mathcal{K}(y(s))ds\right)$.

- γ is uniquely determined, up to translations in the x-direction, by $\mathcal{K}(y)$

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Theorem $\kappa(r)$

Prescribe $\kappa = \kappa(r)$ such that $r\kappa(r)$ continuous. The problem of determining a curve $\gamma(s) = r(s)\, e^{i\theta(s)}$ with curvature $\kappa(r)$ is solvable by:

1. $\int r\kappa(r)dr = \mathcal{K}(r)$, *geometric angular momentum*.
2. $s = s(r) = \int \frac{rdr}{\sqrt{r^2-(\mathcal{K}(r))^2}} \dashrightarrow r = r(s) \dashrightarrow \kappa = \kappa(s)$.
3. $\theta(s) = \int \frac{\mathcal{K}(r(s))}{r(s)^2} ds$.

- γ is uniquely determined, up to rotations, by $\mathcal{K}(r)$

Singer's Problem on the Euclidean plane: Euler elastic curves

Elastica under *tension* $\sigma \in \mathbb{R}$: $2\ddot{\kappa} + \kappa^3 - \sigma\,\kappa = 0$

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- $\kappa(y) = 2\lambda y$, $\lambda > 0 \to \mathcal{K}(y) = \lambda y^2 + c$
  elastica under tension $\sigma = -4\lambda c$

Maximum curvature $k_0 = 2\sqrt{\lambda}\sqrt{1-c}$, $c < 1$

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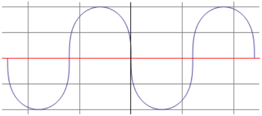
- $c > -1$, *wavelike*:

$\kappa(s) = k_0 \, \mathrm{cn}\left(\frac{k_0 s}{2p}, p\right)$

$p^2 = \frac{1-c}{2}$, $s \in \mathbb{R}$



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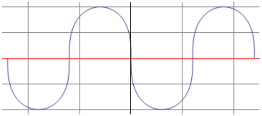

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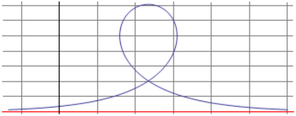

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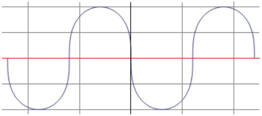

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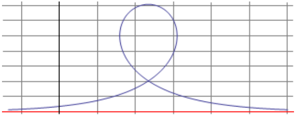
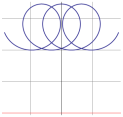


- $c < -1$, *orbitlike*:

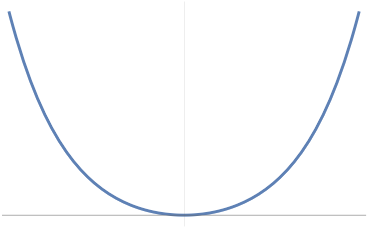
$\kappa(s) = k_0 \text{ dn}\left(\frac{k_0 s}{2}, p\right)$

$p^2 = \frac{2}{1-c},\ s \in \mathbb{R}$

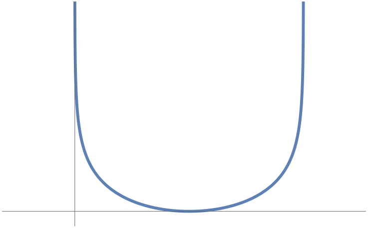


Singer's Problem on the Euclidean plane

The **catenary** $y = \cosh x$, $x \in \mathbb{R}$ is the only plane curve (up to translations in the x-direction) with curvature $\kappa(y) = 1/y^2$ and geometric linear momentum $\mathcal{K}(y) = -1/y$.

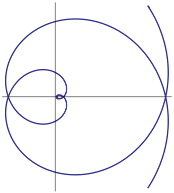


The **grim-reaper** $y = -\log \sin x$, $0 < x < \pi$ is the only plane curve (up to translations in the x-direction) with curvature $\kappa(y) = e^{-y}$ and geometric linear momentum $\mathcal{K}(y) = -e^{-y}$.

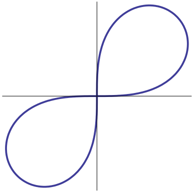


Singer's Problem on the Euclidean plane

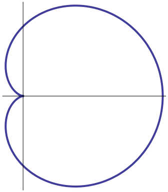
The **Norwich spiral** is the only (non circular) plane curve, up to rotations, with curvature $\kappa(r) = 1/r$ and geometric angular momentum $\mathcal{K}(r) = r$.



The **Bernoulli lemniscate** $r^2 = 3\sin 2\theta$ is the only plane curve, up to rotations, with geometric angular momentum $\mathcal{K}(r) = r^3/3$ and curvature is $\kappa(r) = r$.



The **cardioid** $r = \frac{9}{8\lambda^2}(1 + \cos\theta)$, is the only plane curve (up to rotations) with radial primitive curvature $\mathcal{K}(r) = \frac{2\lambda}{3} r\sqrt{r}$ and curvature is $\kappa(r) = \lambda/\sqrt{r}$.



The Lorentz-Minkowski plane

We denote by $\mathbb{L}^2 := (\mathbb{R}^2, g = -dx^2 + dy^2)$ the Lorentz-Minkowski plane.

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- A non-zero vector $v \in \mathbb{L}^2$ is spacelike if $g(v, v) > 0$, lightlike if $g(v, v) = 0$, and timelike if $g(v, v) < 0$.

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- A curve $\gamma = (x, y) : I \subseteq \mathbb{R} \to \mathbb{R}^2$ is called spacelike (resp. timelike) if the tangent vector $\gamma'(t)$ is spacelike (resp. timelike) for all $t \in I$.

A point $\gamma(t)$ is called a lightlike point if $\gamma'(t)$ is a lightlike vector.

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Classical Existence Theorem

It is possible to obtain a parametrization by arc-length of a curve γ in terms of integrals of its curvature $\kappa = \kappa(s)$. Concretely, any spacelike curve $\alpha(s)$ in $\mathbb{L}^2$ can be represented (up to isometries) by

$$\alpha(s) = \left(\int \sinh \varphi(s) ds, \int \cosh \varphi(s) ds \right) \text{ with } \frac{d\varphi(s)}{ds} = \kappa(s),$$

and any timelike curve $\beta(s)$ can be represented (up to isometries) by

$$\beta(s) = \left(\int \cosh \phi(s) ds, \int \sinh \phi(s) ds \right) \text{ with } \frac{d\phi(s)}{ds} = \kappa(s).$$

Singer's Problem on Lorentz-Minkowski plane

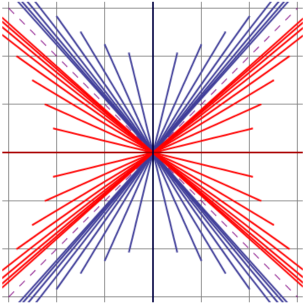
Geodesics

The spacelike geodesics are written as:

$$\alpha_{\varphi_0}(s) = (\sinh \varphi_0\, s, \cosh \varphi_0\, s),\ s \in \mathbb{R},\ \ \varphi_0 \in \mathbb{R},$$

while the timelike geodesics can be written as:

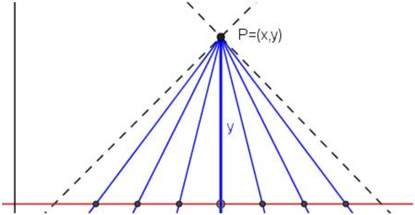
$$\beta_{\phi_0}(s) = (\cosh \phi_0\, s, \sinh \phi_0\, s),\ s \in \mathbb{R},\ \ \phi_0 \in \mathbb{R}.$$



Lorentzian Pseudodistance

We define the *Lorentzian pseudodistance* by

$$\delta : \mathbb{L}^2 \times \mathbb{L}^2 \to [0, +\infty), \ \delta(P, Q) = \sqrt{|g(\overrightarrow{PQ}, \overrightarrow{PQ})|}.$$



Spacelike geodesics in $\mathbb{L}^2$ passing through P and with a point P' in x-axis.

Then:

$$0 < \delta(P, P')^2 = \left(1 - \frac{1}{m^2}\right) y^2 = \frac{y^2}{\cosh^2 \varphi_0} \leq y^2$$

- Equality holds if and only if vertical geodesic.

Thus: $|y|$ is the maximum Lorentzian pseudodistance through spacelike geodesics from $P = (x, y)$, $y \neq 0$, to the x-axis.

Singer's Problem on $\mathbb{L}^2$

Determine those curves $\gamma = (x, y)$ in $\mathbb{L}^2$ whose curvature κ depends on some given function $\kappa = \kappa(x, y)$.

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We focus on spacelike and timelike curves, since the curvature κ is in general not well defined on lightlike points, and because lightlike curves in $\mathbb{L}^2$ are segments parallel to the straight lines determining the light cone.

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1. Pseudodistance to a fixed spacelike geodesic: $\kappa(x, y) = \kappa(x)$.
2. Pseudodistance to a fixed timelike geodesic: $\kappa(x, y) = \kappa(y)$.
3. Pseudodistance to a fixed lightlike geodesic: $\kappa(x, y) = \kappa(v)$, $v = y - x$
4. Pseudodistance to a fixed point: $\kappa(x, y) = \kappa(\rho)$, $\rho = \sqrt{|-x^2 + y^2|}$.

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Duality between spacelike and timelike curves

If $\gamma = (x, y)$ is a spacelike (resp. timelike) curve with $\kappa = \kappa(y)$, then $\hat{\gamma} = (y, x)$ is a timelike (resp. spacelike) curve with $\kappa = \kappa(x)$.

Index

Curvature depending on distance to a timelike geodesic

Theorem

Prescribe $\kappa = \kappa(y)$ continuous.
Then the problem of determining locally a spacelike or timelike curve $(x(s), y(s))$ with geometric linear momentum $\mathcal{K}(y)$ (and curvature $\kappa(y)$ satisfying $d\mathcal{K} = \kappa(y)dy$), is solvable by quadratures by ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

1. $\int \kappa(y)dy = \mathcal{K}(y)$.
2. $s = s(y) = \int \frac{dy}{\sqrt{\mathcal{K}(y)^2 + \epsilon}}$,
   where $\mathcal{K}(y)^2 + \epsilon > 0$, $\dashrightarrow y = y(s) \dashrightarrow \kappa(s)$.
3. $x(s) = \int \mathcal{K}(y(s))ds$.

▶ Such a curve is uniquely determined by $\mathcal{K}(y)$ up to a translation in the x-direction (and a translation of the arc parameter s).
• $\mathcal{K}(y)$ will distinguish geometrically the curves inside a same family by their relative position with respect to the x-axis.

Example: geodesics

Geodesics: $\kappa \equiv 0$

- $\mathcal{K}(y) = c \in \mathbb{R}$. $s = \int \frac{dy}{\sqrt{c^2+\epsilon}} = \frac{y}{\sqrt{c^2+\epsilon}}$, $c^2 + \epsilon > 0$.

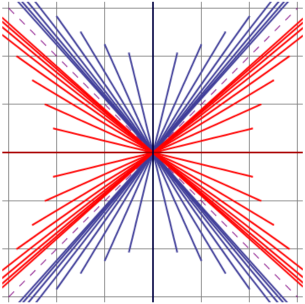
$x(s) = c\, s$ and $y(s) = \sqrt{c^2+\epsilon}\, s, \quad s \in \mathbb{R}$.

$\epsilon = 1$: $K \equiv c := \sinh \varphi_0 \rightarrow$ spacelike geodesics α_{φ_0}.

$c = 0 = \varphi_0$ corresponds to the y-axis.

$\epsilon = -1$: $K \equiv c := \cosh \phi_0 \rightarrow$ timelike geodesics β_{ϕ_0}.

$c = 1 \Leftrightarrow \phi_0 = 0$ corresponds to the x-axis.



Example: circles

Circles: $\kappa \equiv k_0 > 0$

- $\mathcal{K}(y) = k_0 y + c$, $c \in \mathbb{R}$. $\quad s = \int \frac{dy}{\sqrt{(k_0 y + c)^2 + \epsilon}}$.

$\epsilon = 1$: $s = \mathrm{arcsinh}(k_0 y + c)/k_0$.
$x(s) = \cosh(k_0 s)/k_0$ and $y(s) = (\sinh(k_0 s) - c)/k_0$.
$\epsilon = -1$: $s = \mathrm{arccosh}(k_0 y + c)/k_0$
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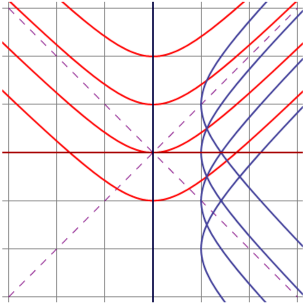

They correspond respectively to spacelike and timelike pseudocircles in $\mathbb{L}^2$ of radius $1/k_0$.

Elasticae on $\mathbb{L}^2$: $\kappa(y) = 2ay + b$ with $a \neq 0$, $b \in \mathbb{R}$.

Definition

A spacelike or timelike curve γ is said to be an *elastica under tension* σ if it satisfies the differential equation $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, for some value of $\sigma \in \mathbb{R}$.
The *energy* $E \in \mathbb{R}$ of an elastica is: $E := \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$.

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Proposition

Let γ be a spacelike or timelike curve in $\mathbb{L}^2$.

- If the curvature of γ is given by $\kappa(y) = 2ay + b$, $a \neq 0$, $b \in \mathbb{R}$, with geometric linear momentum $\mathcal{K}(y) = ay^2 + by + c$, $a \neq 0$, $b, c \in \mathbb{R}$:

  Then γ is an elastica under tension $\sigma = 4ac - b^2$ and energy $E = 4\epsilon a^2 + \sigma^2/4$ (where $\epsilon = 1$ if γ is spacelike and $\epsilon = -1$ if γ is timelike).

Spacelike elasticae $\equiv \kappa(y) = 2y$ and $\epsilon = 1$.

- $\mathcal{K}(y) = y^2 + c$, $c = \sinh\eta \in \mathbb{C}$ $\quad (s_\eta = \sinh\eta$ and $c_\eta = \cosh\eta)$

$$x_\eta(s) = (s_\eta + c_\eta)s + \sqrt{c_\eta}\left(\mathrm{cn}(\sqrt{c_\eta}\,s, k_\eta)\left(k_\eta^2\,\mathrm{sd}(\sqrt{c_\eta}\,s, k_\eta) - \mathrm{ds}(\sqrt{c_\eta}\,s, k_\eta)\right) - 2E(\sqrt{c_\eta}\,s, k_\eta)\right)$$

$$y_\eta(s) = \sqrt{c_\eta}\;\mathrm{cs}(\sqrt{c_\eta}\,s, k_\eta)\,\mathrm{nd}(\sqrt{c_\eta}\,s, k_\eta),\quad k_\eta^2 = \frac{1-\tanh\eta}{2}$$

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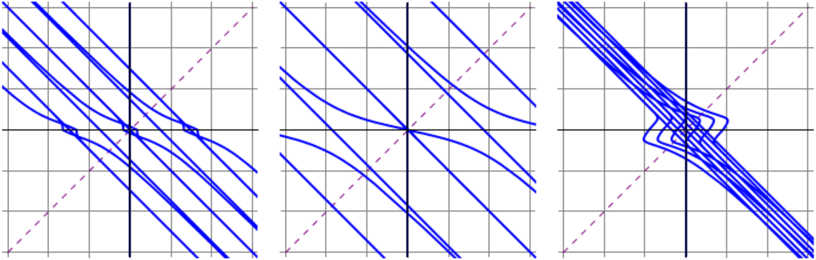


Spacelike elastic curves $\alpha_\eta = (x_\eta, y_\eta)$, $(\eta = 0, 1{,}5, -1{,}5)$.

Timelike elasticae $\equiv \kappa(y) = 2y$ and $\epsilon = -1$.

- $\mathcal{K}(y) = y^2 + 1$ $(c = 1)$.

$x_1(s) = s - \sqrt{2}\coth(\sqrt{2}s)$,

$y_1(s) = -\frac{\sqrt{2}}{\sinh(\sqrt{2}s)}$, $s \neq 0$.

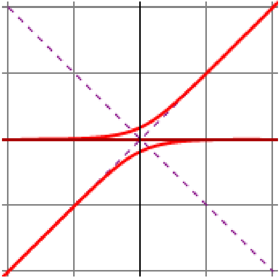
$\kappa_1(s) = -\frac{2\sqrt{2}}{\sinh(\sqrt{2}s)}$.

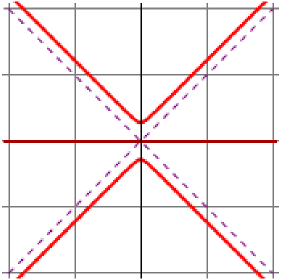
- $\mathcal{K}(y) = y^2 - 1$ $(c = -1)$.

$x_{-1}(s) = \sqrt{2}\tan(\sqrt{2}s) - s$,

$y_{-1}(s) = \pm\frac{\sqrt{2}}{\cos(\sqrt{2}s)}$, $|s| < \frac{\pi}{2\sqrt{2}}$.

$\kappa_{-1}(s) = \frac{\mp 2\sqrt{2}}{\cos(\sqrt{2}s)}$.





Timelike elasticae $\equiv \kappa(y) = 2y$ and $\epsilon = -1$.

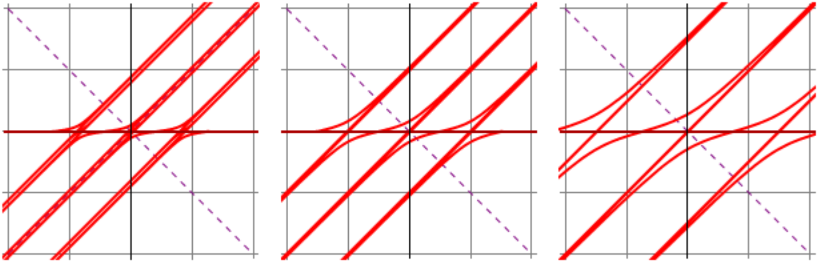
- $\mathcal{K}(y) = y^2 + \cosh^2\delta$, $\delta > 0$, $(c > 1)$.

$$x_\delta(s) = c_\delta^2 s + \sqrt{c_\delta^2+1}\left(\mathrm{dn}(\sqrt{c_\delta^2+1}\,s, k_\delta)\,\mathrm{tn}(\sqrt{c_\delta^2+1}\,s, k_\delta) - E(\sqrt{c_\delta^2+1}\,s, k_\delta)\right),$$

$$y_\delta(s) = s_\delta\ \mathrm{tn}(\sqrt{c_\delta^2+1}\,s, k_\delta),\ \ k_\delta^2 = \frac{2}{1+\cosh^2\delta},$$

$$s \in \left((2m-1)K(k_\delta)/\sqrt{c_\delta^2+1}, (2m+1)K(k_\delta)/\sqrt{c_\delta^2+1}\right),\ m \in \mathbb{N}.$$

$$\kappa_\delta(s) = 2s_\delta\ \mathrm{tn}(\sqrt{c_\delta^2+1}\,s, k_\delta).$$



Timelike elastic curves $\beta_\delta = (x_\delta, y_\delta)$ $(\delta = 0{,}5, 1, 1{,}5)$.

Timelike elasticae $\equiv \kappa(y) = 2y$ and $\epsilon = -1$.

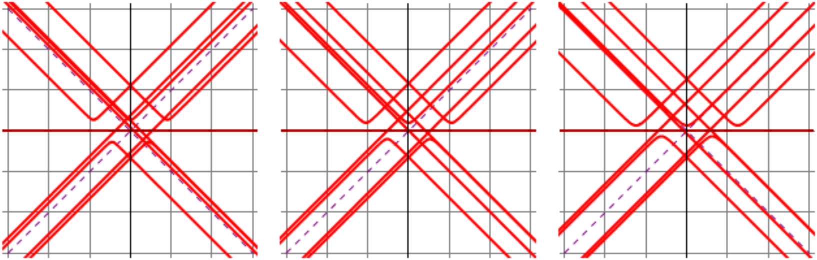
- $\mathcal{K}(y) = y^2 + \sin\psi$, $|\psi| < \pi/2$, $(|c| < 1)$.

$x_\psi(s) = s + \sqrt{2}\left(\mathrm{dn}(\sqrt{2}\,s, k_\psi)\,\mathrm{tn}(\sqrt{2}\,s, k_\psi) - E(\sqrt{2}\,s, k_\psi)\right)$,

$y_\psi(s) = \sqrt{1 - s_\psi}\,\mathrm{nc}(\sqrt{2}\,s, k_\psi)$, $k_\psi^2 = \frac{1+\sin\psi}{2}$,

$s \in \left((2m-1)K(k_\psi)/\sqrt{2}, (2m+1)K(k_\psi)/\sqrt{2}\right)$, $m \in \mathbb{N}$.

$\kappa_\psi(s) = 2\sqrt{1 - s_\psi}\,\mathrm{nc}(\sqrt{2}\,s, k_\psi)$.



Timelike elastic curves $\beta_\psi = (x_\psi, y_\psi)$ $(\psi = -\pi/4, 0, \pi/6)$.

Timelike elasticae $\equiv \kappa(y) = 2y$ and $\epsilon = -1$.

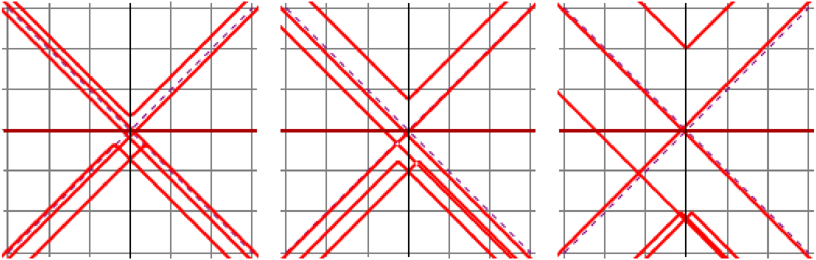
- $\mathcal{K}(y) = y^2 - \cosh^2\tau$, $\tau > 0$, $(c < -1)$.

$x_\tau(s) = = s + \sqrt{1+c_\tau^2}\left(\mathrm{dn}(\sqrt{1+c_\tau^2}\,s, k_\tau)\,\mathrm{tn}(\sqrt{1+c_\tau^2}\,s, k_\tau) - E(\sqrt{1+c_\tau^2}\,s, k_\tau)\right),$

$y_\tau(s) = \sqrt{1+c_\tau^2}\ \mathrm{dc}(\sqrt{1+c_\tau^2}\,s, k_\tau),\ \ k_\tau^2 = \frac{\sinh^2\tau}{1+\cosh^2\tau},$

$s \in \left((2m-1)K(k_\tau)/\sqrt{1+c_\tau^2}, (2m+1)K(k_\tau)/\sqrt{1+c_\tau^2}\right),\ m \in \mathbb{N}.$

$\kappa_\tau(s) = 2\sqrt{1+c_\tau^2}\ \mathrm{dc}(\sqrt{1+c_\tau^2}\,s, k_\tau).$



Timelike elastic curves $\beta_\tau = (x_\tau, y_\tau)$, $(\tau = 1, 2, 3)$.

Curves with $\kappa(y) = \lambda/y^2$, $\lambda > 0 \to \lambda = 1$

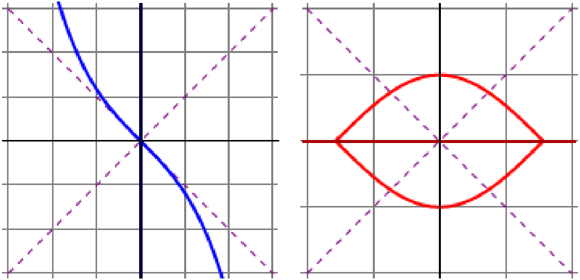
- $\mathcal{K}(y) = -1/y$. **Lorentzian catenaries**

$\epsilon = 1$. Spacelike case:
$x(s) = \mp \operatorname{arccosh} s$, $s > 1$.
$y(s) = \pm\sqrt{s^2 - 1}$, $|s| > 1$.
$\kappa(s) = \frac{1}{s^2-1}$, $s > 1$.

$y = -\sinh x$, $x \in \mathbb{R}$.

$\epsilon = -1$. Timelike case:
$x(s) = \mp \arcsin s$, $|s| < 1$.
$y(s) = \pm\sqrt{1 - s^2}$, $|s| < 1$.
$\kappa(s) = \frac{1}{1-s^2}$, $|s| < 1$.

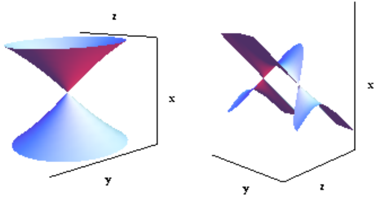
$y = \pm \cos x$, $|x| < \pi/2$.



Curves with $\kappa(y) = \lambda/y^2$, $\lambda > 0 \to \lambda = 1$

Lorentzian catenaries.

Kobayashi introduced, by studying maximal rotation surfaces in $\mathbb{L}^3$, (up to dilations) the catenoid of the first kind with equation $y^2 + z^2 - \sinh^2 x = 0$ and the catenoid of the second kind with equation $x^2 - z^2 = \cos^2 y$.



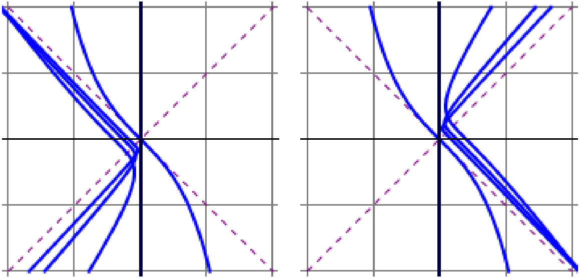
The generatrix curves of both catenoids may be referred as **Lorentzian catenaries** and coincide with the curves described before.

1. The Lorentzian catenary of the first kind $y = -\sinh x$, $x \in \mathbb{R}$, is the only spacelike curve (up to translations in the x-direction) with geometric linear momentum $\mathcal{K}(y) = -1/y$.
2. The Lorentzian catenary of the second kind $x = \pm\cos y$, $|y| < \pi/2$, is the only spacelike curve (up to translations in the y-direction) with geometric linear momentum $\mathcal{K}(x) = -1/x$.

Curves with $\kappa(y) = \lambda/y^2$, $\lambda > 0 \to \lambda = 1$

- $\mathcal{K}(y) = c - 1/y$. $\epsilon = 1$, Spacelike case:

$$x = \frac{1}{c^2+1}\left(c\sqrt{(c^2+1)y^2 - 2cy + 1} - \frac{1}{\sqrt{c^2+1}}\operatorname{arcsinh}((c^2+1)y - c)\right).$$



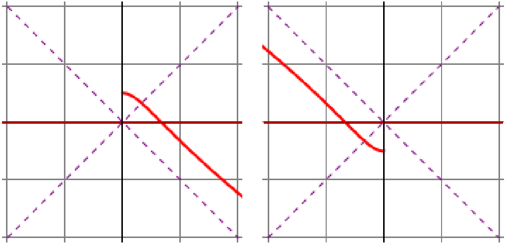
Curves with $\mathcal{K}(y) = c - 1/y$; $c \leq 0$ (left) and $c \geq 0$ (right).

Curves with $\kappa(y) = \lambda/y^2$, $\lambda > 0 \to \lambda = 1$

- $\mathcal{K}(y) = c - 1/y$. $\epsilon = -1$, Timelike case:

· $\mathcal{K}(y) = 1 - 1/y$:
$x = \frac{(2-y)\sqrt{1-2y}}{3}$, $y < 1/2$.

· $\mathcal{K}(y) = -1 - 1/y$:
$x = -\frac{(2+y)\sqrt{1+2y}}{3}$, $y > -1/2$.



· $\mathcal{K}(y) = c - 1/y$, $|c| > 1$:

$$x = \frac{1}{c^2-1}\left(c\sqrt{(c^2-1)y^2 - 2cy + 1} + \frac{\log\left(2(\sqrt{c^2-1}\sqrt{(c^2-1)y^2-2cy+1}+(c^2-1)y-c)\right)}{\sqrt{c^2-1}}\right).$$

· $\mathcal{K}(y) = c - 1/y$, $|c| < 1$:

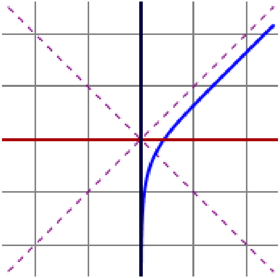
$$x = \frac{1}{c^2-1}\left(c\sqrt{(c^2-1)y^2 - 2cy + 1} - \frac{1}{\sqrt{1-c^2}}\arcsin((c^2-1)y - c)\right)$$

Curves with $\kappa(y) = \lambda e^y$, $\lambda > 0 \to \lambda = 1$

- $\mathcal{K}(y) = e^y$. **Lorentzian grim-reapers**.

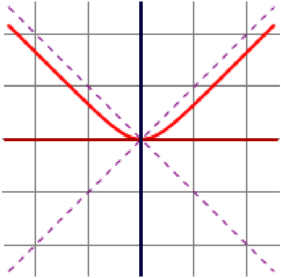
$\epsilon = 1$. Spacelike case:
$x(s) =$
$-\log \tanh(-s/2)$, $s < 0$.
$y(s) = \log(-\operatorname{csch} s)$, $s < 0$.
$\kappa(s) = -\operatorname{csch} s$, $s < 0$.

$y = \log(\sinh x)$, $x > 0$.



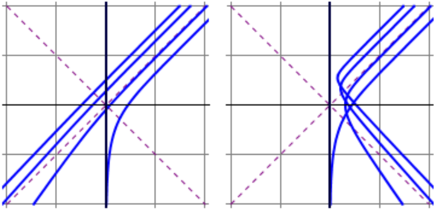
$\epsilon = -1$. Timelike case:
$x(s) =$
$\log(\sec s + \tan s)$, $|s| < \pi/2$.
$y(s) = \log \sec s$, $|s| < \pi/2$..
$\kappa(s) = \sec s$, $|s| < \pi/2$.

$y = \log(\cosh x)$, $x \in \mathbb{R}$.



Curves with $\kappa(y) = \lambda e^y$, $\lambda > 0$

- $\mathcal{K}(y) = e^y + c$, $c \neq 0$.



Spacelike case ($\epsilon = 1$):

$x = \operatorname{arcsinh}(e^y + c) - \frac{c}{\sqrt{c^2+1}} \operatorname{arcsinh}\left(c + (c^2+1)e^{-y}\right).$

Timelike case ($\epsilon = -1$):

$\cdot\mathcal{K}(y) = e^y + 1$:

$x = 2\log(\sqrt{e^y} + \sqrt{e^y + 2}) - \sqrt{1 + 2e^{-y}}.$

$\cdot\mathcal{K}(y) = e^y + c$, $|c| > 1$:

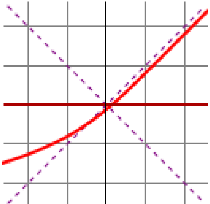
$x = \log\left(2(\sqrt{P(e^y)} + e^y + c)\right) - \frac{c \log\left(2e^{-y}(\sqrt{c^2-1}\sqrt{P(e^y)} + ce^y + c^2 - 1)\right)}{\sqrt{c^2-1}}$

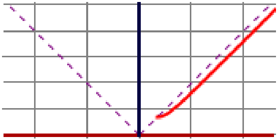
$\cdot\mathcal{K}(y) = e^y - 1$:

$x = 2\log(\sqrt{e^y} + \sqrt{e^y - 2}) - \sqrt{1 - 2e^{-y}}.$

$\cdot\mathcal{K}(y) = e^y + c$, $|c| < 1$:

$x = \log\left(2(\sqrt{P(e^y)} + e^y + c)\right) + \frac{c}{\sqrt{1-c^2}} \arcsin\left(c + (c^2-1)e^{-y}\right).$

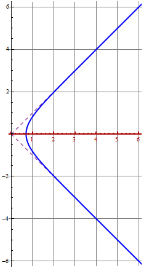




Other curves in $\mathbb{L}^2$

- $\mathcal{K}(y) = -\coth y$. $\quad x(s) = \mp\sqrt{s^2-1}, \quad y(s) = \pm \operatorname{arccosh} s, s > 1$.
  $\kappa(s) = \frac{1}{s^2-1}$. Lorentzian catenary of 1st kind: $x = -\sinh y$, $y \in \mathbb{R}$.
- $\mathcal{K}(y) = \tan y$. $x(s) = \mp\sqrt{1-s^2}, \quad y(s) = \pm \arcsin s, |s| < 1$.
  $\kappa(s) = \frac{1}{1-s^2}$. Lorentzian catenary of 2nd kind $x = \pm \cos y$, $|y| < \pi/2$.
- $\mathcal{K}(y) = \cosh y$. $x(s) = -\log(\sinh(-s))$, $y(s) = 2 \operatorname{arctanh} e^s, s < 0$.
  $\kappa(s) = -\operatorname{csch} s$. Lorentzian grim-reaper $y = \log(\sinh x)$, $x > 0$.

- $\mathcal{K}(y) = \sinh y$.
  $x(s) = \log(2 \csc s)$,
  $y(s) = \log(\tan(s/2))$.
  $\kappa(s) = \csc s$, $|s| < \pi$



Index

Curvature depending on distance to a lightlike geodesic

Theorem

Prescribe $\kappa = \kappa(v)$ continuous. Then the problem of determining locally a spacelike or timelike curve

$$\left(\frac{u(s)-v(s)}{2}, \frac{u(s)+v(s)}{2}\right)$$

with geometric linear momentum $\mathcal{K}(v)$
(and curvature $\kappa(v)$ satisfying $-\epsilon\, d(1/\mathcal{K}) = \kappa(v)dv$)
is solvable by quadratures by $\epsilon = 1$ spacelike, $\epsilon = -1$ timelike.

1. $\displaystyle\int \kappa(v)dv = \frac{-\epsilon}{\mathcal{K}(v)}$,
2. $\displaystyle s = s(v) = \epsilon \int \mathcal{K}(v)dv, \dashrightarrow v = v(s), \dashrightarrow \kappa(s)$
3. $\displaystyle u(s) = \int K(v(s))ds.$

▶ Such a curve is uniquely determined by $\mathcal{K}(v)$ up to a translation in the u-direction (and a translation of the arc parameter s).
• $\mathcal{K}(v)$ will distinguish geometrically the curves inside a same family by their relative position with respect to the u-axis.

Examples: constant curvature

Geodesics: $\kappa \equiv 0$

- $\mathcal{K}(v) = -\epsilon/c$, $c \neq 0$. $u(s) = -\epsilon s/c$, $v(s) = -cs$, $s \in \mathbb{R}$,

(lines passing through the origin with slope $m = \frac{\epsilon + c^2}{\epsilon - c^2}$.)

$\epsilon = 1 \Rightarrow |m| > 1$ spacelike geodesics, $\epsilon = -1 \Rightarrow |m| < 1$ timelike geodesics.

Examples: constant curvature

Geodesics: $\kappa \equiv 0$

- $\mathcal{K}(v) = -\epsilon/c$, $c \neq 0$. $u(s) = -\epsilon s/c$, $v(s) = -cs$, $s \in \mathbb{R}$,

(lines passing through the origin with slope $m = \frac{\epsilon + c^2}{\epsilon - c^2}$.)

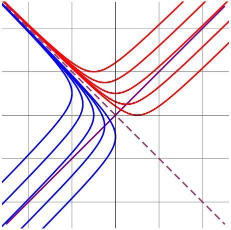
$\epsilon = 1 \Rightarrow |m| > 1$ spacelike geodesics, $\epsilon = -1 \Rightarrow |m| < 1$ timelike geodesics.

Circles: $\kappa \equiv k_0 > 0$

- $\mathcal{K}(v) = \frac{-\epsilon}{(c + k_0 v)}$, $c \in \mathbb{R}$. $u(s) = -\epsilon e^{k_0 s}/k_0$, $v(s) = (e^{-k_0 s} - c)/k_0$.

$\epsilon = 1 \Rightarrow x(s) = (-\cosh(k_0 s) + c/2)/k_0$, $y(s) = -(\sinh(k_0 s) + c/2)/k_0$.

$\epsilon = -1 \Rightarrow x(s) = (\sinh(k_0 s) + c/2)/k_0$, $y(s) = (\cosh(k_0 s) - c/2)/k_0$.



(Spacelike and timelike pseudocircles in $\mathbb{L}^2$ of radius $1/k_0$.)

Curves with $\kappa(v) = av + b$, $a \neq 0$, $b \in \mathbb{R} \rightarrow a = b = 1$

Elastica under tension σ equation: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, with $\sigma \in \mathbb{R}$.

Energy $E \in \mathbb{R}$ of an elastica: $E := \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$.

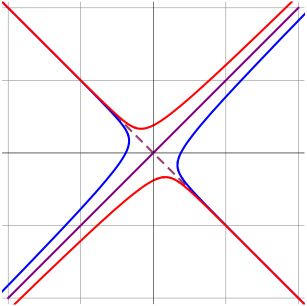
Curves with $\kappa(v) = av + b$, $a \neq 0$, $b \in \mathbb{R} \to a = b = 1$

Elastica under tension σ equation: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, with $\sigma \in \mathbb{R}$.

Energy $E \in \mathbb{R}$ of an elastica: $E := \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$.

- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c}$, $c \in \mathbb{R}$. ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

1. $c = 0$: $u(s) = -\epsilon\frac{s^3}{3}$, $v(s) = 1/s$, $\kappa(s) = 2/s$, $s \neq 0$.



Spacelike (blue) and timelike (red) elastic curve with $\sigma = E = 0$.

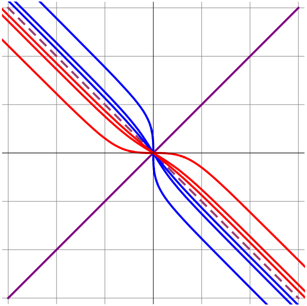
Curves with $\kappa(v) = av + b$, $a \neq 0$, $b \in \mathbb{R} \to a = b = 1$

Elastica under tension σ equation: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, with $\sigma \in \mathbb{R}$.

Energy $E \in \mathbb{R}$ of an elastica: $E := \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$.

- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c}$, $c \in \mathbb{R}$. ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

2. $c > 0$ $\quad u(s) = -\frac{\epsilon}{c}\left(\frac{s}{2} + \frac{\sin(2\sqrt{c}s)}{4\sqrt{c}}\right)$, $v(s) = -\sqrt{c}\tan(\sqrt{c}s)$.
   $\kappa(s) = -2\sqrt{c}\tan(\sqrt{c}s)$, $|s| < \pi/2\sqrt{c}$.



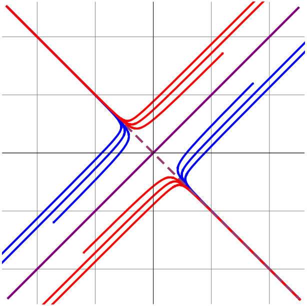
Spacelike (blue) and timelike (red) elastic curves in $\mathbb{L}^2$ with $\sigma = 4c > 0$ and $E = 4c^2$, $c = 1, 2, 3$.

Curves with $\kappa(v) = av + b$, $a \neq 0$, $b \in \mathbb{R} \to a = b = 1$

Elastica under tension σ equation: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, with $\sigma \in \mathbb{R}$.

Energy $E \in \mathbb{R}$ of an elastica: $E := \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$.

- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c}$, $c \in \mathbb{R}$. ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)
  3. $c < 0$ $u(s) = \frac{\epsilon}{c}\left(-\frac{s}{2} + \frac{\sinh(2\sqrt{-c}\,s)}{4\sqrt{-c}}\right)$, $v(s) = \sqrt{-c}\coth(\sqrt{-c}\,s)$.
     $\kappa(s) = 2\sqrt{-c}\coth(\sqrt{-c}s)$, $s \neq 0$.



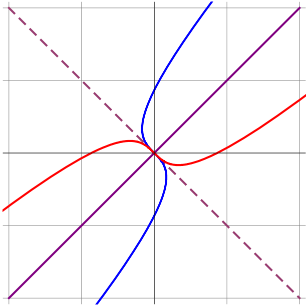
Spacelike (blue) and timelike (red) elastic curves in $\mathbb{L}^2$ with $\sigma = 4c < 0$ and $E = 4c^2$, $c = -1, -2, -3$.

Curves with $\kappa(v) = a/v^2$, $a \neq 0 \to a = 1$

- $\mathcal{K}(v) = \epsilon v$. ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

$u(s) = 2\epsilon\sqrt{2}s\sqrt{s}/3, \quad v(s) = \sqrt{2s}, \quad \kappa(s) = \frac{1}{2s}, \quad s > 0.$

We arrive at the graphs $u = \epsilon\, v^3/3$, $v > 0$ for $\epsilon = \pm 1$.



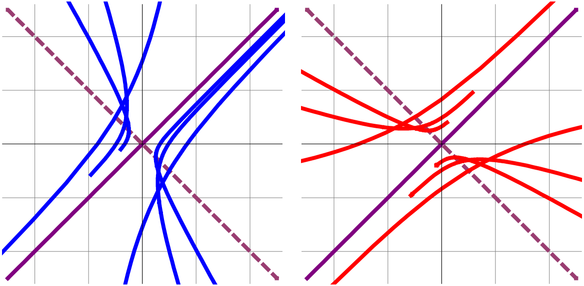
Spacelike (blue) and timelike (red) curve in $\mathbb{L}^2$ with $\mathcal{K}(v) = \epsilon v$, $\epsilon = \pm 1$.

Curves with $\kappa(v) = a/v^2$, $a \neq 0 \to a = 1$

- $\mathcal{K}(v) = \frac{-\epsilon v}{c\,v-1}$, $c \neq 0$. ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

$u(v) = \frac{\epsilon}{c^3}\left(c\,v - 1 - \frac{1}{c\,v-1} + 2\log(c\,v-1)\right)$,

for $v > 1/c$ if $c > 0$ and for $v < 1/c$ if $c < 0$.



Spacelike curves with $\mathcal{K}(v) = -\frac{v}{c\,v-1}$ (left) and timelike curves with $\mathcal{K}(v) = \frac{v}{c\,v-1}$ (right).

Curves with $\kappa(v) = a\,e^{v}$, $a \neq 0 \to a = 1$

- $\mathcal{K}(v) = -\frac{\epsilon}{e^{v}+c}$, $c \in \mathbb{R}$. ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

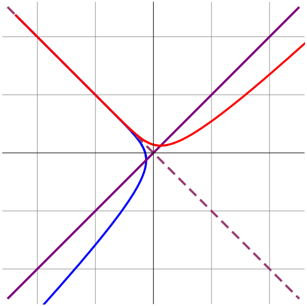
Curves with $\kappa(v) = a\, e^v$, $a \neq 0 \to a = 1$

- $\mathcal{K}(v) = -\frac{\epsilon}{e^v + c}$, $c \in \mathbb{R}$. ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

  1. $c = 0$: $u(s) = -\epsilon s^2/2$, $v(s) = -\log s$, $\kappa(s) = 1/s$, $s > 0$.

Lorentzian grim-reapers

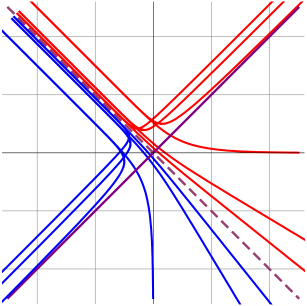
The curves are the graph of $u = -\epsilon\, e^{-2v}/2$, $v \in \mathbb{R}$.
They satisfy the translating-type soliton equation $\kappa = g((1,1), N)$.



Spacelike (blue) and timelike (red) curves with $\mathcal{K}(v) = -\frac{\epsilon}{e^v}$.

Curves with $\kappa(v) = a\, e^{v}$, $a \neq 0 \rightarrow a = 1$

- $\mathcal{K}(v) = -\frac{\epsilon}{e^{v}+c}$, $c \in \mathbb{R}$. ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)
  2. $c \neq 0$: $u(s) = -\frac{\epsilon}{c}\left(s + \frac{1}{c\, e^{cs}}\right)$, $v(s) = \log \frac{c}{e^{cs}-1}$, $s > 0$.
     $\kappa(s) = \frac{c}{e^{cs}-1}$, $s > 0$.



Spacelike curves (blue) and timelike curves (red) with $\mathcal{K}(v) = -\frac{\epsilon}{e^{v}+c}$, $c \neq 0$.

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Curvature depending on pseudodistance from the origin

We study $\gamma = (x, y)$ with $\kappa = \kappa(\rho)$, where ρ is the *Lorentzian pseudodistance from the origin*:

$$\rho := \sqrt{|g(\gamma, \gamma)|} = \sqrt{|-x^2 + y^2|} \geq 0.$$

We use what we can call *pseudopolar coordinates* (ρ, ν), $\rho \geq 0$, $\nu \in \mathbb{R}$ being the *orthochrone angle*.
Since $g(\gamma, \gamma) = -x^2 + y^2 = \pm\rho^2$, we distinguish:

$$\gamma^+ \equiv \begin{cases} x = \rho \sinh \nu,\ y = \rho \cosh \nu, & \text{if } -x^2 + y^2 \geq 0, y \geq 0 \\ x = -\rho \sinh \nu,\ y = -\rho \cosh \nu, & \text{if } -x^2 + y^2 \geq 0, y \leq 0 \end{cases}$$

$$\gamma^- \equiv \begin{cases} x = \rho \cosh \nu,\ y = \rho \sinh \nu, & \text{if } -x^2 + y^2 \leq 0, y \geq 0 \\ x = -\rho \cosh \nu,\ y = -\rho \sinh \nu, & \text{if } -x^2 + y^2 \leq 0, y \leq 0 \end{cases}$$

In fact, it will be enough obviously to consider the first and third cases, since the map $(x, y) \to (-x, -y)$ is an isometry of $\mathbb{L}^2$.

Curvature depending on pseudodistance from the origin

Theorem

Prescribe $\kappa = \kappa(\rho)$ such that $\rho\,\kappa(\rho)$ is continuous.
Then the problem of determining locally a spacelike or timelike curve

$$\gamma_\epsilon^\pm(s) = (\pm\rho_\epsilon^\pm(s)\sinh\nu_\epsilon^\pm(s), \pm\rho_\epsilon^\pm(s)\cosh\nu_\epsilon^\pm(s)),$$

with geometric angular momentum $\mathcal{K}(\rho)$ (and curvature $\kappa(\rho)$ satisfying $d\mathcal{K} = \rho\kappa(\rho)d\rho$ is solvable by ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

1. $\int \rho\,\kappa(\rho)d\rho = \mathcal{K}(\rho)$.
2. $s = s(\rho) = \int \frac{\rho\,d\rho}{\sqrt{\mathcal{K}(\rho)^2 \pm \epsilon\rho^2}}$, where $\mathcal{K}(\rho)^2 \pm \epsilon\rho^2 > 0$ --→ $\rho = \rho_\epsilon^\pm(s) > 0$. --→ $\kappa(s)$
3. $\nu_\epsilon^\pm(s) = \int \frac{\mathcal{K}(\rho_\epsilon^\pm(s))}{\rho_\epsilon^\pm(s)^2} ds$, where $\rho_\epsilon^\pm(s) > 0$.

▶ Such a curve is uniquely determined by $\mathcal{K}(\rho)$ up to a ν-orthochrone Lorentz transformation (and a translation of the arc parameter s).
• $\mathcal{K}(\rho)$ will distinguish geometrically the curves inside a same family by their relative position with respect to the origin.

Constant curvature: pseudocircles

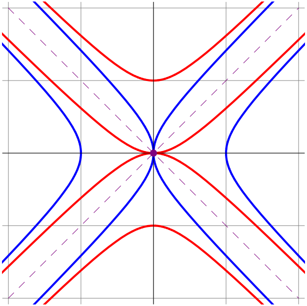
$\mathcal{K}(\rho) = k_0\rho^2 + c$, $c \in \mathbb{R}$. $\quad s = \int \rho \, d\rho / \sqrt{(k_0\rho^2 + c)^2 \pm \rho^2}$.

- $\mathcal{K}(\rho) = k_0\rho^2$.

$\rho^+(s) = \frac{\sinh(k_0 s)}{k_0}$, $\nu^+(s) = k_0 s$.

$\rho^-(s) = \frac{\cosh(k_0 s)}{k_0}$, $\nu^-(s) = k_0 s$.

Pseudocircles of radius $1/2k_0$.



Spacelike (blue) and timelike (red) pseudocircle with $\mathcal{K}(\rho) = \rho^2/2$ ($\kappa \equiv 1$) in $\mathbb{L}^2$.

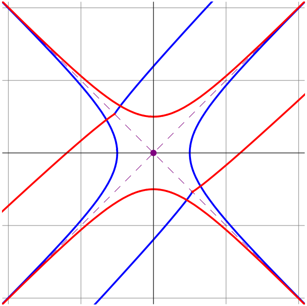
Norwich spiral: $\kappa(\rho) = \frac{1}{\rho}$

- $\mathcal{K}(\rho) = \rho + c, \ \ c \neq 0.$

$\rho^+(t) = \frac{c}{2}\left(\sinh(\sqrt{2}t) - 1\right)$, and

$\nu^+(t) = t + \log\left(\frac{\sinh\left(\frac{\sqrt{2}\,t - \operatorname{arcsinh} 1}{2}\right)}{\cosh\left(\frac{\sqrt{2}\,t + \operatorname{arcsinh} 1}{2}\right)}\right), \ \ t > \frac{1}{\sqrt{2}} \operatorname{arcsinh} 1.$

$\rho^-(t) = \frac{c}{2}(1 - t^2)$, and $\nu^-(t) = t + 2 \operatorname{arctanh} t, \ \ |t| < 1.$



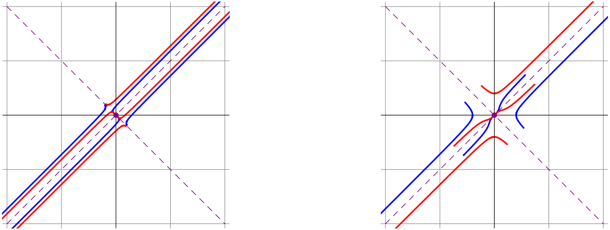
Lorentzian Norwich spiral.

Curves with $\kappa(\rho) = 2\lambda + \mu/\rho$, $\lambda, \mu \neq 0 \to \lambda = 1$

- $\mathcal{K}(\rho) = \rho^2 + \mu\rho$, $\quad (\mu = \sinh\eta,\ \eta \in \mathbb{R})$

$$\rho_\eta^+(s) = \sinh s - \sinh\eta, \qquad \nu_\eta^+(s) = s + \tanh\eta \log\left(\frac{\sinh(\frac{s-\eta}{2})}{\cosh(\frac{s+\eta}{2})}\right),\ \ s > \eta.$$

1. $\mu = \pm 1$.
   $\rho_1^-(s) = \cosh s - 1$ and $\nu_1^-(s) = s - \coth(s/2)$, $s \neq 0$ (left)
   $\rho_{-1}^-(s) = \cosh s + 1$ and $\nu_{-1}^-(s) = s - \tanh(s/2)$, $s \in \mathbb{R}$ (right)

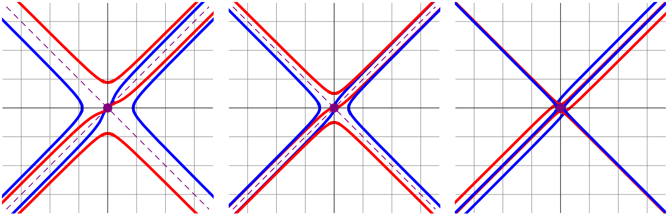


Curves with $\kappa(\rho) = 2\lambda + \mu/\rho$, $\lambda, \mu \neq 0 \to \lambda = 1$

- $\mathcal{K}(\rho) = \rho^2 + \mu\rho$, $\quad (\mu = \sinh\eta,\ \eta \in \mathbb{R})$

$$\rho_\eta^+(s) = \sinh s - \sinh\eta, \qquad \nu_\eta^+(s) = s + \tanh\eta \log\left(\frac{\sinh(\frac{s-\eta}{2})}{\cosh(\frac{s+\eta}{2})}\right), \ s > \eta.$$

2. $|\mu| < 1$. $\quad \mu = \cos\alpha$, with $0 < \alpha < \pi$.
   $\rho_\alpha^-(s) = \cosh s - \cos\alpha$,
   $\nu_\alpha^-(s) = s + 2\cot\alpha\ \arctan\left(\cot(\alpha/2)\tanh(s/2)\right),\ s \in \mathbb{R}$.



Curves with $\kappa(\rho) = 2\lambda + \mu/\rho$, $\lambda, \mu \neq 0 \to \lambda = 1$

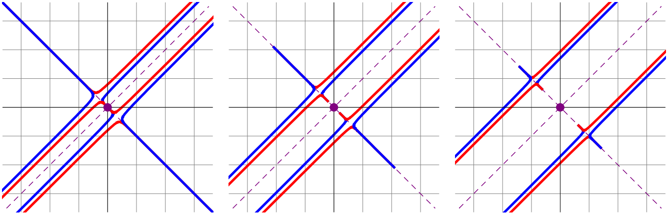
- $\mathcal{K}(\rho) = \rho^2 + \mu\rho$, $\quad (\mu = \sinh\eta,\ \eta \in \mathbb{R})$

$$\rho^+_\eta(s) = \sinh s - \sinh\eta, \qquad \nu^+_\eta(s) = s + \tanh\eta \log\left(\frac{\sinh(\frac{s-\eta}{2})}{\cosh(\frac{s+\eta}{2})}\right),\ \ s > \eta.$$

3. $\mu > 1$. $\quad \mu = \cosh\delta,\ \delta > 0$.

   $\rho^-_\delta(s) = \cosh s - \cosh\delta,$

   $\nu^-_\delta(s) = s + \coth\delta\ \log\left(\frac{\sinh(\frac{s-\delta}{2})}{\sinh(\frac{s+\delta}{2})}\right), |s| > \delta.$

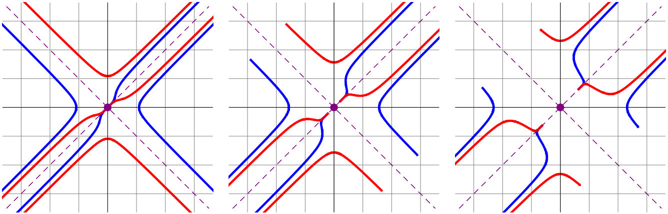


Curves with $\kappa(\rho) = 2\lambda + \mu/\rho$, $\lambda, \mu \neq 0 \to \lambda = 1$

- $\mathcal{K}(\rho) = \rho^2 + \mu\rho$, $\quad (\mu = \sinh\eta,\ \eta \in \mathbb{R})$

$$\rho_\eta^+(s) = \sinh s - \sinh\eta, \qquad \nu_\eta^+(s) = s + \tanh\eta \log\left(\frac{\sinh(\frac{s-\eta}{2})}{\cosh(\frac{s+\eta}{2})}\right), \quad s > \eta.$$

4. $\mu < -1$. $\quad \mu = -\cosh\tau,\ \tau > 0$.
$\rho_\tau^-(s) = \cosh s + \cosh\tau$,
$\nu_\tau^-(s) = s + \coth\tau\ \log\left(\frac{\cosh(\frac{s-\tau}{2})}{\cosh(\frac{s+\tau}{2})}\right), s \in \mathbb{R}$.



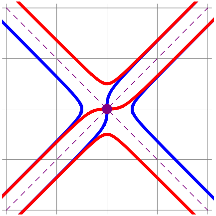
Sinusoidal spirals: $\kappa(\rho) = \lambda\,\rho^{n-1}$, $\lambda > 0$, $n \in \mathbb{R} \setminus \{-1, 0\}$.

- $\mathcal{K}(\rho) = \frac{\lambda}{n+1}\rho^{n+1}$ $\begin{cases} \lambda\,\rho_+^n = (n+1)\sinh\left(n\nu_+\right), \; n \neq 0, \; n \neq -1, \\ \lambda\,\rho_-^n = (n+1)\cosh\left(n\nu_-\right), \; n \neq 0, \; n \neq -1, \end{cases}$

Sinusoidal spirals: $\kappa(\rho) = \lambda\,\rho^{n-1}$, $\lambda > 0$, $n \in \mathbb{R} \setminus \{-1, 0\}$.

- $\mathcal{K}(\rho) = \frac{\lambda}{n+1}\rho^{n+1}$ $\begin{cases} \lambda\,\rho_+^n = (n+1)\sinh(n\nu_+)\,, & n \neq 0,\ n \neq -1, \\ \lambda\,\rho_-^n = (n+1)\cosh(n\nu_-)\,, & n \neq 0,\ n \neq -1, \end{cases}$

1. $n = 2$: the *Lorentzian Bernoulli pseudolemniscate*
   $\rho_+^2 = \sinh 2\nu_+$, $\rho_-^2 = \cosh 2\nu_-$ with $\mathcal{K}(\rho) = \rho^3$.
2. $n = 1/2$: the *Lorentzian pseudocardioid*
   $\sqrt{\rho}_+ = \sinh(\nu_+/2)$, $\sqrt{\rho}_- = \cosh(\nu_-/2)$ with $\mathcal{K}(\rho) = \rho^{3/2}$.

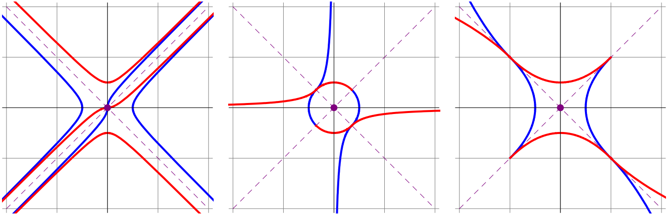


Sinusoidal spirals with $n = 2$ (left) and $n = 1/2$ (right).

Sinusoidal spirals: $\kappa(\rho) = \lambda\,\rho^{n-1}$, $\lambda > 0$, $n \in \mathbb{R} \setminus \{-1, 0\}$.

- $\mathcal{K}(\rho) = \frac{\lambda}{n+1}\rho^{n+1}$ $\begin{cases} \lambda\,\rho_+^n = (n+1)\sinh(n\nu_+)\,, & n \neq 0,\ n \neq -1, \\ \lambda\,\rho_-^n = (n+1)\cosh(n\nu_-)\,, & n \neq 0,\ n \neq -1, \end{cases}$

3. $n = 1$: the pseudocircles $\rho_+ = \sinh\nu_+$, $\rho_- = \cosh\nu_-$ with $\mathcal{K}(\rho) = \rho^2$.
4. $n = -2$: the *Lorentzian equilateral pseudohyperbolas* $\rho_+^2 = -1/\sinh 2\nu_+$, $\rho_-^2 = 1/\cosh 2\nu_-$ with $\mathcal{K}(\rho) = 1/\rho$.
5. $n = -1/2$: the *Lorentzian pseudoparabolas* $\sqrt{\rho}_+ = -1/\sinh(\nu_+/2)$, $\sqrt{\rho}_- = 1/\cosh(\nu_-/2)$ with $\mathcal{K}(\rho) = \sqrt{\rho}$.

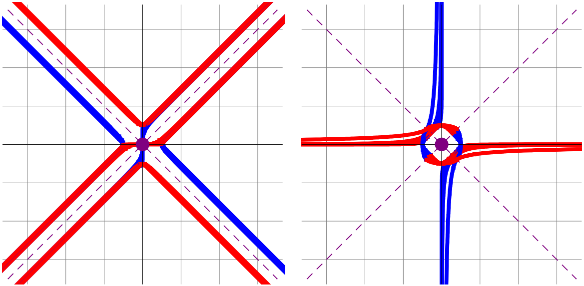


Sinusoidal spirals: $n = 1$ (left), $n = -2$ (center), $n = -1/2$ (right).

Sinusoidal spirals: $\kappa(\rho) = \lambda\,\rho^{n-1}$, $\lambda > 0$, $n \in \mathbb{R} \setminus \{-1, 0\}$.

- $\mathcal{K}(\rho) = \frac{\lambda}{n+1}\rho^{n+1}$ $\begin{cases} \lambda\,\rho_+^n = (n+1)\sinh(n\nu_+)\,, \; n \neq 0,\; n \neq -1, \\ \lambda\,\rho_-^n = (n+1)\cosh(n\nu_-)\,, \; n \neq 0,\; n \neq -1, \end{cases}$

6. Some general examples of $\mathcal{K}(\rho) = \frac{\lambda}{n+1}\rho^{n+1}$



Sinusoidal spirals: $n \geq 5/2$ (left) and $n \leq -3/2$ (right), $n \in \mathbb{Q}$.

Thanks for your attention!