A Natural Metric Topology on the Causal Boundary of Globally Hyperbolic Spacetimes

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Motivations

MOTIVATION:

- The addition of ideal points to spacetimes has long been a prime strategy to analyse singularities and/or the asymptotic geometry, especially with an eye towards physical applications. However, there are many (inequivalent) proposals in the literature on how to implement this in a mathematically precise way. Arguably, none of these proposals is entirely satisfactory.
- By far the best-known, most widely studied among these is the *conformal boundary* of the spacetime M. However, the existence of a conformal boundary imposes an *ad hoc* restriction on M: it relies on the existence of a suitable open conformal embedding into a larger spacetime $\tilde{M}$, inducing a piecewise C^1 boundary ∂M with fairly restrictive properties. It is unclear to us how generic this construction is.

MOTIVATION:

- Another well-known case is the so-called *causal boundary*, introduced by Geroch, Kronheimer and Penrose (GKP) [1972]. Unlike the conformal boundary, such a boundary is a well-defined conformal invariant for *any strongly causal spacetime*. Moreover, it has been shown by many authors to possess nice universal properties for a large class of spacetimes (which includes globally hyperbolic ones).
- The original GKP construction, however, presented a number of undesirable properties even in simple cases. Fortunately, many of these glitches have been addressed and solved by a suitable redefinition of the *c*-boundary by Marolf and Ross [2003], and further streamlined in a quite satisfactory way by Flores, Herrera and Sánchez [2011].

MOTIVATION:

- In spite of its many good features, the topology introduced by the latter authors on the c-completion , the *chronological topology*, fails to be Hausdorff in many cases, including globally hyperbolic examples, although it is often T_1. (As emphasized most forcefully by S. Harris, however, this seems to be an inescapable and largely harmless feature of c-boundaries.)
- Nevertheless, *in the particular case of globally hyperbolic spacetimes*, a quite natural *Hausdorff* (indeed metrizable) topology τ_c can be defined on the *future* (or *past*) c-completion. This topology is strictly finer than the chronological topology, but retains all of its good properties. (This context is the most important one for evolution of initial data in GR.) Some previous Hausdorff topologies for the c-completion do appear in the literature appear in GKP [1972] and O. Müller [2014].

Outline of the Construction

Indecomposable Pasts

A subset $P \subset M$ is a *past set* if $I^-(P) = P$. Given such a past set P:

IP1) P is *indecomposable* (IP) if it is not the union of proper past subsets,

IP2) P is *proper indecomposable* (PIP) if $P = I^-(p)$ for some $p \in M$,

IP3) P is *terminal indecomposable* (TIP) if $P = I^-(\gamma)$ for some future-inextendible timelike curve γ,

We denote by $\hat{M}$ the set of all IPs in M. If M is strongly causal, then there is a natural injection

$$i : p \in M \hookrightarrow I^-(p) \in \hat{M}.$$

The *future c-boundary* of M is

$$\partial_+ M := \hat{M} \setminus i(M).$$

The future c-boundary: chronological relations

On $\hat{M}$ we define a *chronological relation*

$$P \ll Q \iff \exists q \in Q \text{ such that } P \subset I^-(q).$$

$(\hat{M}, \ll)$ is what S. Harris termed a *chronological set*. In particular, due to results by S. Harris [1998], when M is globally hyperbolic we can show that $(\hat{M}, \ll)$ is characterized as a future-completion of M (in the sense of chronological sets) by a universal property, and hence is unique up to isomorphisms in a suitable category.

Outline of the Construction:

- In order to topologize $\hat{M}$ we shall adopt the topology of Hausdorff closed limits. Let X be any set, and let (S_n) be a sequence of subsets of X. Define

$$\text{lím inf}(S_n) := \{x \in X \mid \text{ any neigh. of } x \text{ intersects all but finitely many } S_n\text{'s}\}$$
$$\text{lím sup}(S_n) := \{x \in X \mid \text{ any neigh. of } x \text{ intersects infinitely many } S_n\text{'s }\}.$$

  If $S_\infty = \text{lím inf}(S_n) = \text{lím sup}(S_n)$, this is called the *Hausdorff closed limit* of the sequence.

- The topology in question was introduced in metric spaces by F. Hausdorff himself in the 50's, and is well known in the study of minimal surfaces.

- C. Vega and G. Galloway [2016] have shown that achronal boundaries are preserved by Hausdorff closed limits.

Outline of the Construction (CONT.):

- We fix a topological metric d on M, without loss of generality we can pick d with the *Heine-Borel property*, i.e., any closed d-bounded subset of M is compact. Using d, we wish to topologize the set

$$\mathcal{C}_M := \{S \subseteq M \, : \, S \text{ is closed and non-empty}\}.$$

Denote by $C_c(M)$ the set of continuous real-valued function on M endowed with the topology of uniform convergence in compact subsets. In particular, we know that $C_c(M)$ is metrizable. Consider then the map

$$\Phi : S \in \mathcal{C}_M \mapsto \Phi_S \in C_c(M) \tag{1}$$

given by

$$\Phi_S(x) := d(x, S), \forall x \in M, \forall S \in \mathcal{C}_M.$$

Outline of the Construction (CONT.):

Main properties of Φ

For the mapping Φ, the following properties hold.

i) Φ is one-to-one. In particular, there exists a unique (metrizable) topology τ_c on $\mathcal{C}_M$ for which Φ is a homeomorphism.

ii) For each $S \in \mathcal{C}_M$, Φ_S is Lipschitz. In particular, the image $\Phi(\mathcal{C}_M)$ is an equicontinuous subset of $C_c(M)$.

iii) If a sequence (S_n) in $\mathcal{C}_M$ converges in the topology τ_c to $S \in \mathcal{C}_M$, then S is the Hausdorff closed limit of (S_n).

iv) $\Phi(\mathcal{C}_M)$ is closed in $C_c(M)$.

v) If a sequence $(S_n) \subset \mathcal{C}_M$ has a Hausdorff closed limit $S_\infty \in \mathcal{C}_M$, then $S_n \to S_\infty$ in τ_c.

Outline of the Construction (CONT.):

- In order to topologize $\hat{M}$ we shall adopt the topology on $\mathcal{C}_M$ as described in the previous section.
- Key fact: the mapping $\chi : P \in \hat{M} \mapsto \overline{P} \in \mathcal{C}_M$ is one-to-one. Therefore, there exists a unique (metrizable) topology on $\hat{M}$ (which we also denote by τ_c) for which χ is a homeomorphism.
- Have to make sure it coincides with the conformal boundary in the "most interesting cases." It does coincide with the Flores-Herrera-Sánchez topology when the latter is Hausdorff.

Statement of the Main Results

RESULTS I:

Summary of Properties of τ_c

Suppose M is globally hyperbolic. The following facts hold for the topology τ_c on the future-completion $\hat{M} = M \cup \partial^+ M$.

i) The natural inclusion $i : M \hookrightarrow \hat{M}$ is an open continuous map. In particular, $i(M)$ is an open (dense) set in $\hat{M}$ and the induced topology on M by i coincides with the manifold topology.

iii) $\hat{I}^{\pm}(P)$ is open, for all $P \in \hat{M}$.

iv) Any future-directed chain $(P_n) \subset \hat{M}$ converges in τ_c.

RESULTS II: Comparison with the conformal boundary

Definitions - I

Let $A \subset M$ be any set.

a) A is *causally convex* if any causal curve segment in M with endpoints in A is entirely contained in A,

b) A is *future-precompact* [resp. *past-precompact*] is there exists a compact set $K \subset M$ such that $A \subset I^-(K)$ [resp. $A \subset I^+(K)$].

The *future boundary* [resp. *past boundary* of A is

$$\partial^+ A := I^+(A) \cap \partial A \text{ [resp. } \partial^- A := I^-(A) \cap \partial A].$$

RESULTS II: Comparison with the conformal boundary (CONT.):

Definitions - II

A conformal extension $(\tilde{M}, \tilde{g})$ of a globally hyperbolic $M = (M, g)$ is *future-nesting* if

N1) $(\tilde{M}, \tilde{g})$ is also globally hyperbolic,

N2) $M \subset \tilde{M}$ is causally convex and future-precompact.

RESULTS II: Comparison with the conformal boundary (CONT.):

Theorem

Let $(\tilde{M}, \tilde{g})$ be a future-nesting conformal extension of a globally hyperbolic spacetime (M, g). Then

1) the future conformal boundary $\partial^+ M$ is an achronal C^0 hypersurface in $(\tilde{M}, \tilde{g})$ homeomorphic to a Cauchy hypersurface in (M, g).
2) $M \cup \partial^+ M \subset \tilde{M}$ with the induced topology is homeomorphic to the future c-completion $\hat{M}$ of M with the topology τ_c. The homeomorphism can be chosen so that it maps $\partial^+ M$ onto $\partial_+ M$.

CONCLUSIONS

- We have indicated how to define a metrizable topology with natural and good properties on the future c-boundary of globally hyperbolic spacetimes.
- In particular, the topology is such that the c-boundary coincides in a precise sense with the conformal boundary when the latter is available.
- Some standard structure like null infinity and black holes can be defined in this generalized context, and some of their properties retrieve. This work is part of an ongoing effort to make the c-boundaries amenable to applications.

DZIĘKUJĘ CI!
(Thank you!)