Ehlers-Kundt conjecture about Gravitational Waves and Dynamical Systems

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(Based on joint work with M SÁNCHEZ, arxiv: 1706.03855)

1. INTRODUCTION

HEROIC EPOCH:

Highlights in the theoretical setting of gravitational waves:

- **Einstein'18**: Prediction in the framework of GR (quadruple formula).
- **Einstein double reversal:** Robertson clarified flaws in the use of coordinates by Einstein and Rosen.
- **Bondi'57**: Discovered a singularity-free solution of a plane gravitational wave carrying energy.
- **Pirani'57**: Linked singularities to curvature by appealing to mathematical tools by Synge, Petrov and Lichnerowicz.
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- In 1962 Ehlers and Kundt summarized rigorously the foundations on gravitational waves.
- After proving that plane waves are (geodesically) complete they posed the following problem:

  "Prove the plane wave to be the only complete [gravitational] pp-waves, no matter which topology one chooses."

- According to them, complete and Ricci flat pp-waves would represent a graviton field independent of any matter by which it would be generated.
- So, EK conjecture assigns a role to gravitational plane waves in GR similar to the one played by source-free photons in Electrodynamics.

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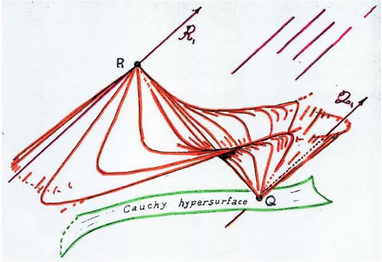
PENROSE'S REMARKABLE PROPERTY:

- In 1965 Penrose pointed out the *remarkable property* that plane waves are not globally hyperbolic. More precisely:

  "No spacelike hypersurface exists in the spacetime which is adequate for the global specification of Cauchy data."



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PRECISE MATHEMATICAL FORMULATION:

- Any *pp-wave* (*parallelly propagated plane-fronted wave*) can be written as $\mathbb{R}^4$ endowed with metric:

$$g = dx^2 + dy^2 + 2du\,dv + H(z,u)du^2, \quad z := (x,y), \quad (x,y,u,v) \in \mathbb{R}^4,$$

where $H : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a smooth function.

- A pp-wave is *gravitational* if g is Ricci flat. This is equivalent to being H harmonic respect to its first pair of variables, i.e.,

$$\Delta_z H(z,u) := (\partial_x^2 H + \partial_y^2 H)(z,u) \equiv 0.$$

- A pp-wave is a *plane wave* when, at each $u \in \mathbb{R}$, the u-constant function $H(\cdot, u)$ is a polynomial in x, y of degree ≤ 2.

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REFORMULATION IN LAGRANGIAN MECHANICS

EK Conjecture

Any gravitational and complete pp-wave must be a plane wave.

- **F,Sánchez'03**: A pp-wave is complete iff all the trajectories solution of the following equation are complete:

$$\ddot{z}(s) = -\nabla_z V(z(s), s), \quad \text{with } V := -H. \tag{1}$$

EK Conjecture (Lagrangian reformulation)

Let $V(z,u)$ non-autonomous potential on $\mathbb{R}^2$, harmonic in z.
The dynamical system (1) is complete $\Leftrightarrow$ $V(z,u)$ is a (at most) quadratic polynomial in z for all $u \in \mathbb{R}$.

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RELEVANT GROWTH BEHAVIORS FOR H

- *H polynomially bounded at finite u-times* or just *polynomially u-bounded* if, for each $u_0 \in \mathbb{R}$, there exists $\epsilon_0 > 0$ such that

$$H(z,u) \leq q_0(z) \qquad \forall (z,u) \in \mathbb{R}^2 \times (u_0 - \epsilon_0, u_0 + \epsilon_0),$$

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- **Ehlers-Kundt'62**: All planes waves, gravitational or not, are complete (equation (1) reduces to a linear ODE system).
- **F-Sánchez'06** + **Candela-Romero-Sánchez'13**: Pp-waves with $H = -V$ quadratically polinomially u-bounded are complete. As a consequence, EK conjecture true in this case.
- **Leistner-Schliebner'16**: *The universal covering of any compact Ricci-flat Brinkmann spacetime is a plane wave.*
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2. MAIN RESULT AND SETUP

MAIN RESULT

Theorem (F-Sánchez, arxiv:1706.03855)

Let $V : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be a polynomially u-bounded C^1-potential which is also C^2 and harmonic (thus, analytic) in its first pair of variables $z = (x, y)$.
Then, all the solutions to $\ddot{z}(s) = -\nabla_z V(z(s), s)$ are complete iff the function $V(\cdot, u)$ is an at most quadratic polynomial for each $u \in \mathbb{R}$.

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DISCUSSION

- Necessity of the polynomial bound:
  - Harmonicity implies analyticity and, then, the possibility to expand $H(\cdot, u)$, $u \in \mathbb{R}$, as an infinite polynomial series.
  - However, our technique crashes for infinite series.
- Significative (even in the autonomous case $V(z, u) \equiv V(z)$):
  - EK conjecture holds at any finite perturbative order.
  - z-harmonicity as a new type of hypotheses for incompleteness.
  - Relation between the autonomous case and the theory on completeness of *holomorphic vector fields* on $\mathbb{C}^2$.
  - Open case of physical and mathematical interest in Relativity, Classical Mechanics and Dynamical systems.

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EXPRESSION OF V IN POLAR COORDINATES:

Autonomous case

Any autonomous polynomially u-bounded harmonic potential V can be written as

$$V(\rho,\theta) = -\lambda\rho^n \cos n(\theta+\alpha) - \sum_{m=0}^{n-1} \lambda_m \rho^m \cos m(\theta+\alpha_m),$$

being $\lambda > 0$ and λ_m, α, α_m constants.

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EXPRESSION OF V IN POLAR COORDINATES:

Non-autonomous case

Any polynomially u-bounded harmonic potential V can be written, in **some** $I_0 = (u_0 - \epsilon_0, u_0 + \epsilon_0)$, *as*

$$V(\rho, \theta, u) = -\lambda(u)\rho^n \cos n(\theta + \alpha(u)) - \sum_{m=0}^{n-1} \lambda_m(u)\rho^m \cos m(\theta + \alpha_m(u)),$$

$\lambda(u) > 0$ *and* C^1*-smooth* $\lambda(u)$, $\lambda_m(u)$, $\alpha(u)$, $\alpha_m(u)$.

Beware: Previous expression fails in the example below for $u_0 = 0$,

$$V(z, u) = \begin{cases} e^{-1/u^2}\rho^n \cos(n\theta + 1/u) & u \neq 0 \\ 0 & u = 0. \end{cases} \rightsquigarrow \text{non-cont. } \alpha(u) = 1/u.$$

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Criterion for incompleteness (lower radial quadratic bound)

For $n > 2$, assume:

$$\ddot{\rho}(s) \geq n\lambda_0 \rho(s)^{n-1}, \quad \rho(0) > 0, \quad \dot{\rho}(0) \geq 0.$$

$\lambda_0 > 0 \Rightarrow$ *all the solutions are incomplete (to the right).*

Idea of the proof. After some manipulations

$$\int_{\rho(0)}^{\rho} \frac{d\overline{\rho}}{\sqrt{2\lambda_0(\overline{\rho}^n - \rho(0)^n) + \dot{\rho}(0)^2}} \geq s(\rho)$$

and the integral is finite for $\rho = \infty$. □

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$$\ddot{\rho}(s) \geq n\lambda_0 \rho(s)^{n-1}, \quad \rho(0) > 0, \quad \dot{\rho}(0) \geq 0.$$

$\lambda_0 > 0 \Rightarrow$ *all the solutions are incomplete (to the right).*

Idea of the proof. After some manipulations

$$\int_{\rho(0)}^{\rho} \frac{d\overline{\rho}}{\sqrt{2\lambda_0(\overline{\rho}^n - \rho(0)^n) + \dot{\rho}(0)^2}} \geq s(\rho)$$

and the integral is finite for $\rho = \infty$. □

PRECISE RESULT

Proposition

For some $0 < \theta_0 \leq \theta_+ \leq \pi/2n$ *and* $\rho_0 > 0$, *any trajectory* γ *with*

$$\gamma(0) = (\rho(0), \theta(0)) \in D[\rho_0, \theta_0] := \{(\rho, \theta) : \rho > \rho_0, |\theta| < \theta_0\}$$

and $\dot{\rho}(0) \geq 0$, $\dot{\theta}(0) = 0$, *satisfies:*

(a) γ *remains in* $D[\rho_0, \theta_+] := \{(\rho, \theta) : \rho > \rho_0, |\theta| < \theta_+\}$

(b) *whenever in this region,* $\ddot{\rho}(s) \geq \lambda_0 n \rho(s)^{n-1}$ $(n > 2)$.

Thus, γ *is incomplete.*

3. OUTLINE OF PROOF (autonomous case)

THE HOMOGENEOUS CASE:

- If V is homogeneous then it reduces to the monomial

$$V(\rho, \theta) = -\rho^n \cos(n\theta), \quad n > 2.$$

- Then, the radial curve

$$\gamma_k(s) = (\rho(s), \theta(s) \equiv \theta_k), \quad \theta_k = 2\pi k/n, \quad k = 0, \ldots, n-1,$$

is a trajectory for V whenever $\ddot{\rho} = n\rho(s)^{n-1}$, and thus, is incomplete.

- There exists a radial region $D_k[\rho_0, \pi/2n]$ around each γ_k such that,
  (a) the trajectories starting in $D_k[\rho_0, \pi/2n]$ remain in $D_k[\rho_0, \pi/2n]$,
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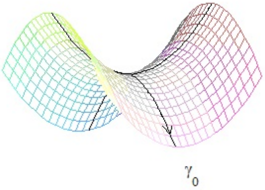
THE NON-HOMOGENEOUS CASE:

- If V is non-homogeneous, terms of lower order appear:

$$V(\rho,\theta) = -\rho^n \cos(n\theta) - \sum_{m=1}^{n-1} \lambda_m \rho^m \cos m(\theta + \alpha_m), \quad n > 2.$$

- Identify some angle $\hat{\theta}_0$ whose associated radial direction γ_0 has steepest decreasing V.



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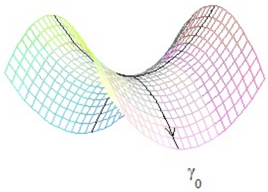

- Consider

$$-\nabla V = \begin{array}{l}(n\rho^{n-1}\cos n\theta + \sum_{m=1}^{n-1} m\lambda_m \rho^{m-1}\cos m(\theta+\alpha_m))\partial_\rho \\ -(n\rho^{n-2}\sin n\theta + \sum_{m=1}^{n-1} m\lambda_m \rho^{m-2}\sin m(\theta+\alpha_m))\partial_\theta.\end{array}$$

- $\partial_\theta V$ vanishes for big ρ at n angles $\vartheta_k(\rho) \in [0, 2\pi)$:

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STEP 0: Preliminar computations to find suitable technical bounds for V, $\partial V/\partial \rho$ on $D[\rho_0, \theta_+]$.

STEP 1: Bounding the increase of angular peaks by the radial distance.

- The increase in amplitude of each oscillation of γ, starting at $D[\rho_0, \theta_+]$, around γ_k is estimated in terms of the radial component: *if $s_1 \in (s_0, b)$ satisfies $|\theta(s_0)| < |\theta(s_1)| < \theta_+$, and $\theta(s)$ is monotonous on (s_0, s_1), then*

$$|\theta(s_1)| - |\theta(s_0)| \leq A/\rho(s_0).$$

- **Tool:** a careful balance of the energies of the trajectories in comparison with their radial projections.

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STEP 2: Fast increase of ρ from peak to peak.

- We derive the formula

$$\rho(s) > \rho(s_0)e^{\overline{\theta}(s)/\Lambda} \ \ \forall s \in (s_0, b) \text{ s.t. } \gamma([s_0, s]) \subset D[\rho_0, \theta_+],$$

where

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4. CONCLUSION

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Open lines of research:

- Non-polynomial case
- Beyond the original motivation: higher dimensions, impulsive waves, Finslerian modifications of the waves...

EK conjecture introduces the pattern

$$\text{Source-free dynamics} \Longrightarrow \begin{cases} \text{Natural (mathematical) vacuum, or} \\ \text{Incompleteness (eventually missed source),} \end{cases}$$

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5. SOME REFERENCES

"Heroic epoch":

- **Einstein**, ber Gravitationswellen, Sitzungsberichte der Kniglich Preussischen Akademie der Wissenschaften Berlin (1918) part 1: 154-167.
- **Pirani**, Invariant formulation of gravitational radiation theory, *Phys. Rev.*, **105** (1957) 1089-1099.
- **Ehlers, Kundt**, Exact solutions of the gravitational field equations, in Gravitation: *an introduction to current research*, L. Witten (ed.), J. Wiley and Sons, NY (1962) 49-101.
- **Penrose**, A remarkable property of plane waves in general relativity, *Rev. Modern Phys.* **37** (1965) 215-220.

EK conjecture:

- **Bicak**, Selected solutions of Einstein's field equations, *Lect. Notes Phys.* 540 (2000).
- **Hubeni, Rangamani**, *J. High Energy Phys.* **12** (2002) (39 pp).
- **Leistner, Schliebner**, *Math. Ann.* 364 (2016), no. 3-4, 1469-1503.
- **Coste e Silva, —-, Herrera**, to appear in *Adv. Theor. Math. Phys.*, arXiv:1605.03619.
- **—-, Sánchez**
  - *Class. Quantum Grav.* **20** (2003) 2275-2291.
  - In: Analytical and numerical approaches to mathematical relativity, 79-98. Lecture Notes in Phys., **692** (2006).
  - Ehlers-Kundt Conjecture about Gravitational Waves and Dynamical Systems, arxiv: 1706.03855.

Background on completeness

- **Abraham, Marsden**, *Foundations of Mechanics*, 2nd Ed. Addison-Wesley Publishing Co., Boston (1987).
- **Candela, —-, Sánchez**, *General Relativity Gravitation* **35** (2003), no. 4, 631–649.
- **Candela, Romero, Sánchez** *Arch. Ration. Mech. Anal.* **208** (2013), no. 1, 255-274.
- **Sánchez**, in: Geometry, mechanics, and dynamics, 343-372, Fields Inst. Commun., 73, Springer, New York (2015).

Polynomial holomorphic v.f. on $\mathbb{C}^2$

- **Forstneric**, *Math. Z.* **223** (1996), no. 1, 123-153.
- **Brunella**, *Topology* **43** (2004), no. 2, 433-445.
- **Bustinduy,** *Adv. Math.* **285** (2015), 1339-1357.
- **López, Mucio-Raymundo**, Complex analysis and related topics (Cuernavaca, 1996), 171-195, *Oper. Theory Adv. Appl.*, 114, Birkhäuser, Basel (2000).

Beyond classical gravitational waves

- **Smann, Steinbauer**, Completeness of general pp-wave spacetimes and their impulsive limit, *Class. Quantum Grav.* **33** (2016), 215006 (27 pp).
- **Fuster, Pabst**, Finsler pp-waves, *Physical Review D*, **94** (2016) 104072 (5 pp).

THANKS FOR YOUR ATTENTION!