TOWARDS AN EXPLORATION OF SPACELIKE SINGULARITIES IN GENERAL SPACETIMES

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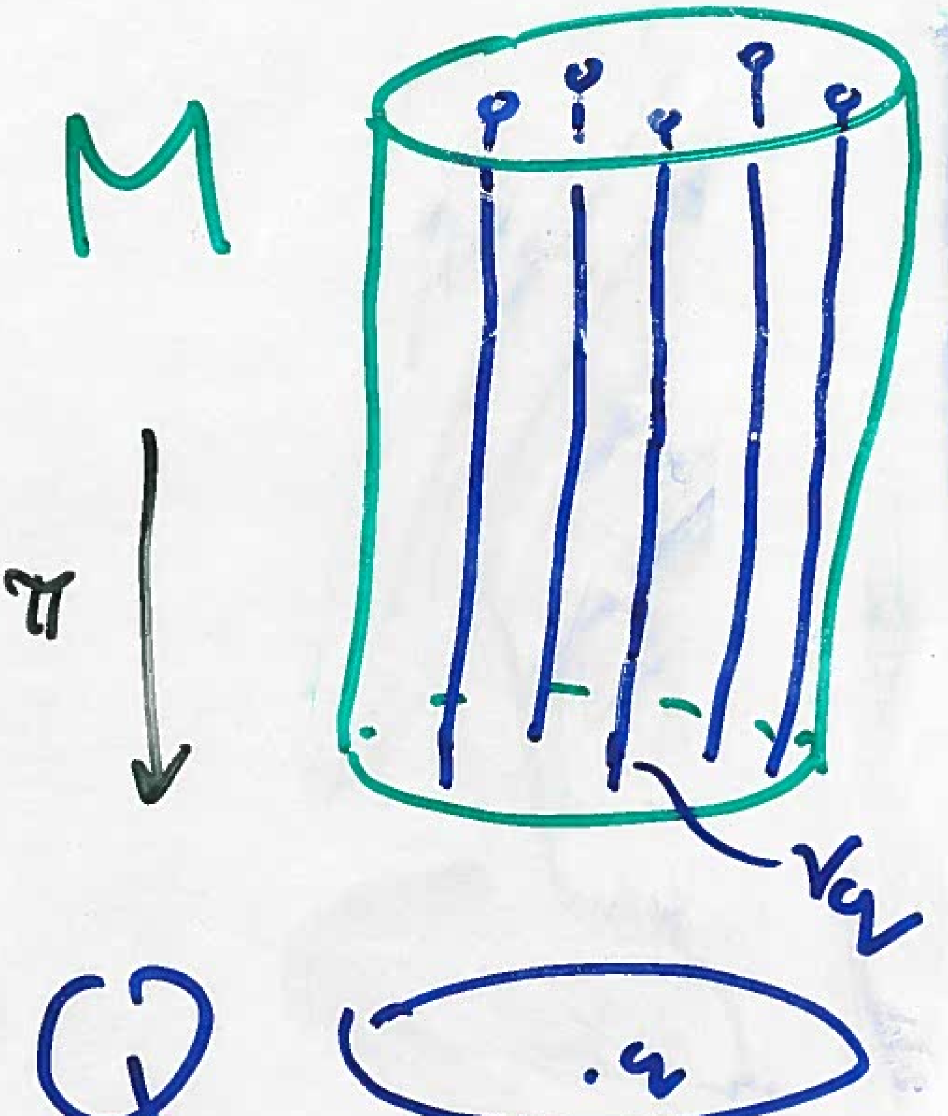
ASSUMED STRUCTURE:



FOLIATION TIMELIKE FUTURE – INCOMPLETE CURVES γ_q

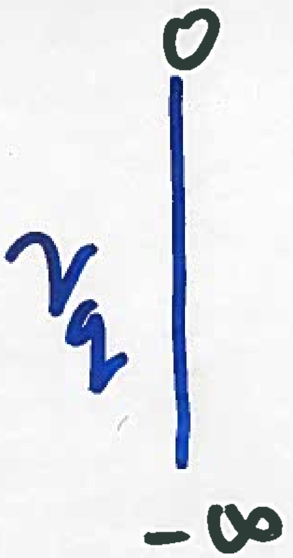
$Q = M/\cong$

$M \cong \mathbb{R} \times Q$

ASSUME FOLIATION HAS NO ANCESRTAL PAIRS:

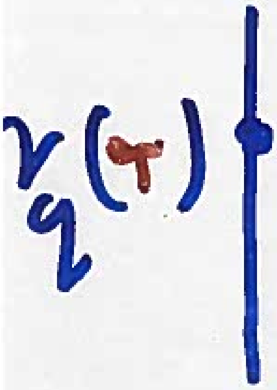
all of γ_p in future of all of γ_q

$\Rightarrow$ Q is Hausdorff manifold



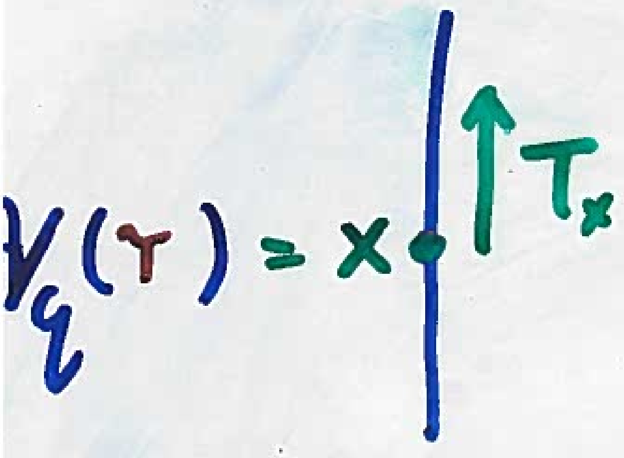
unit-speed parametrized on

$(-\infty, 0)$



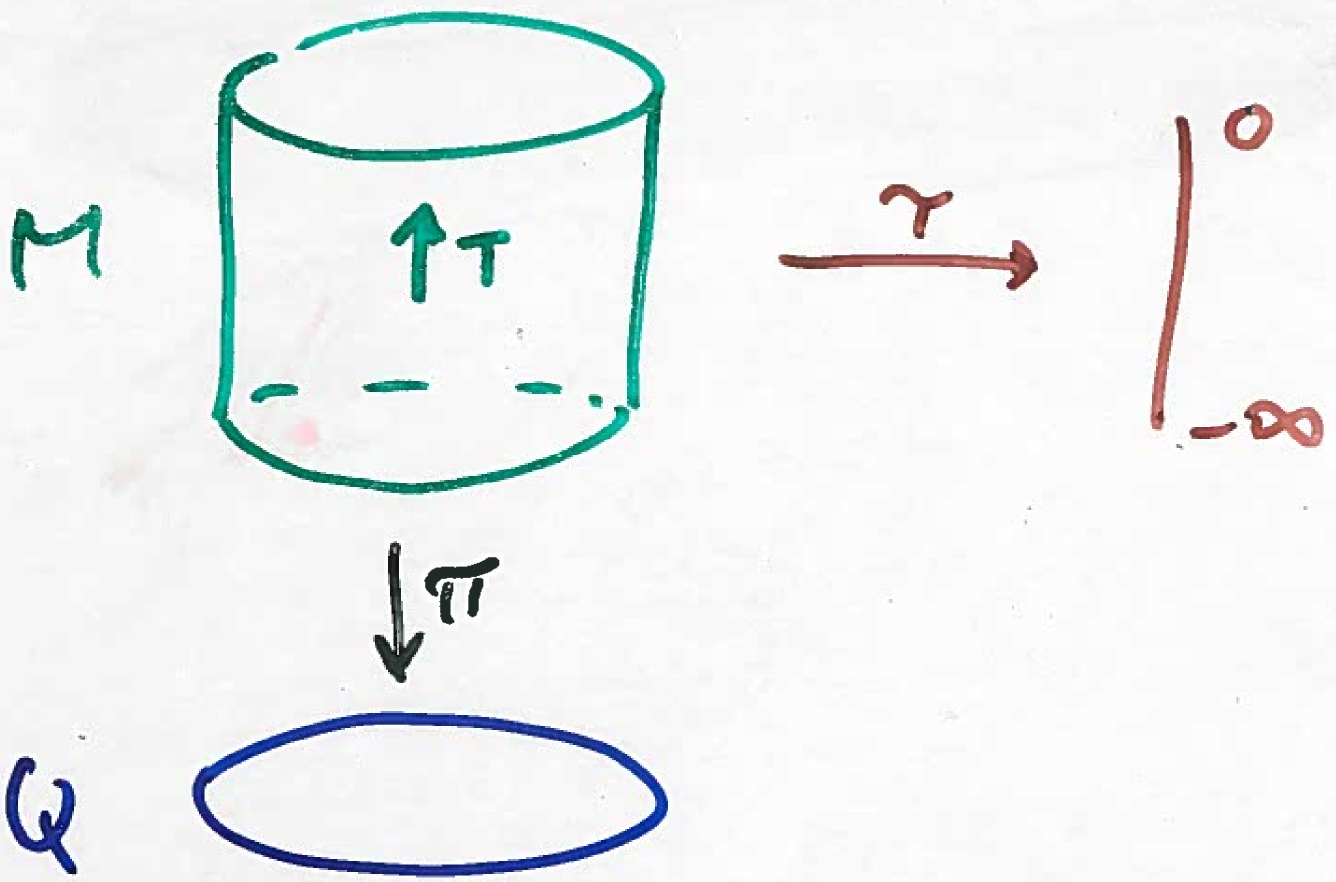
$\tau : M \longrightarrow (-\infty, 0)$

global time coordinate



$T_x = \dot{\gamma}_q(\tau)$

global timelike vector field

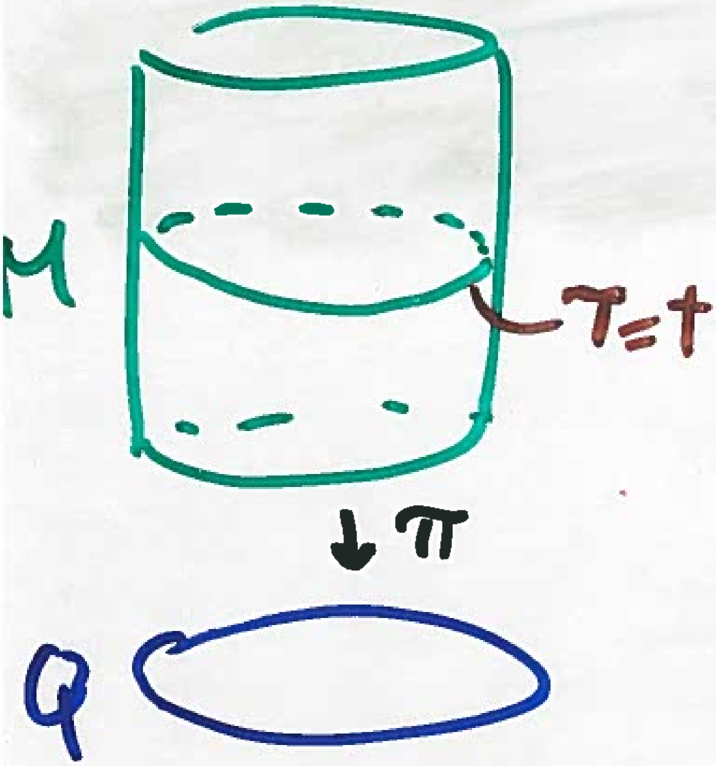


1-form α on M

$$\ker(\alpha) = T^{\perp}$$

$$\alpha(T) = 1 \quad \Big\| \quad d\tau(T) = 1$$

$$(\alpha - d\tau)(T) = 0$$



at each τ-level

$\exists!$ 1-form

η^τ on Q

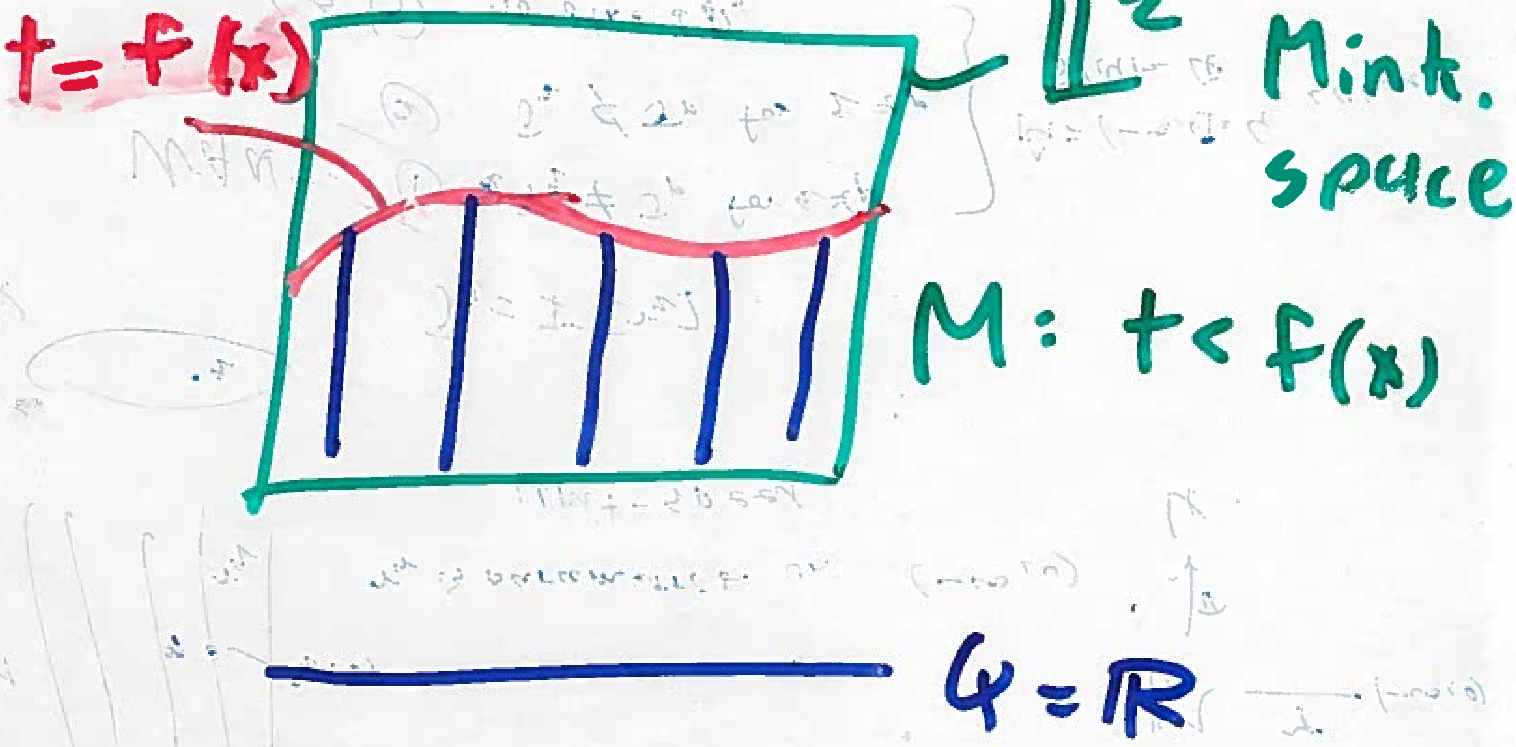
so that

$$-2(\alpha - d\tau) = \pi^* \eta^\tau$$

η^τ : time-dependent

$\boxed{\text{drift form}}$

example of drift-form



$$\eta = -\frac{1}{2} f'(x)\,dx$$

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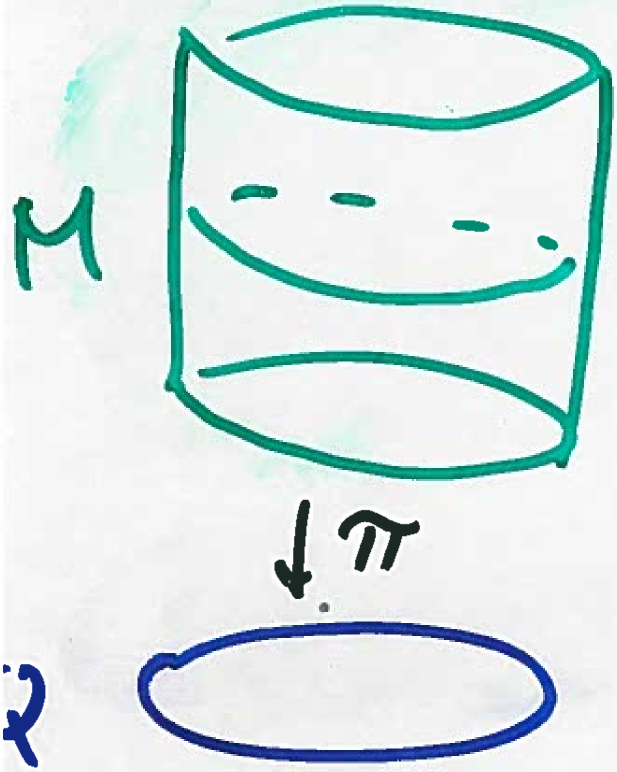
g = spacetime metric

$$(g + \alpha^2)(T, T) = 0$$

$$(g + \alpha^2)(T, T^\perp) = 0$$



at each τ-level $\exists!$ Riemannian metric h^τ on Q so that

$$g = -\alpha^2 + \pi^* h^\tau$$

$$= -\left(d\tau - \tfrac{1}{2}\pi^* \eta^\tau\right)^2 + \pi^* h^\tau$$

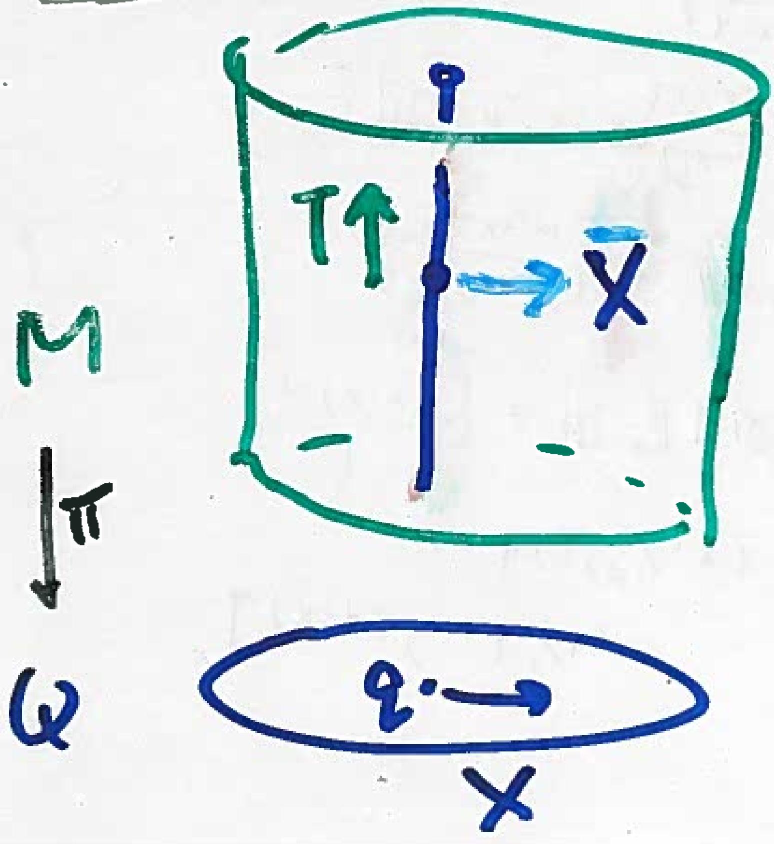
h^τ: time-dependent

$\boxed{Q\text{-metric}}$ Riem.

GEOMETRY:

$\frac{d}{d\tau} h^\tau$ controls growth of h^τ

— whether or not there is singularity at $\tau = 0$



$\pi_* \bar{X} = X$

$\bar{X} \perp T$

time-dependent

stucture forms on Q

$$[T, \bar{X}] = \beta^{\tau}(X)\,T$$

$$[\bar{X}, \bar{Y}] = \phi^{\tau}(X,Y)\,T + \overline{[X,Y]}$$

$$D_T T = \overline{\beta^{\#}}$$ ← raise index (vector)

$\Rrightarrow$ β^{τ} measures geodesicality of $\{\gamma_q\}$

also: $\beta^\tau = \frac{1}{2}\frac{d}{d\tau}\eta^\tau$

in geodesic case:

η^τ is constant in τ

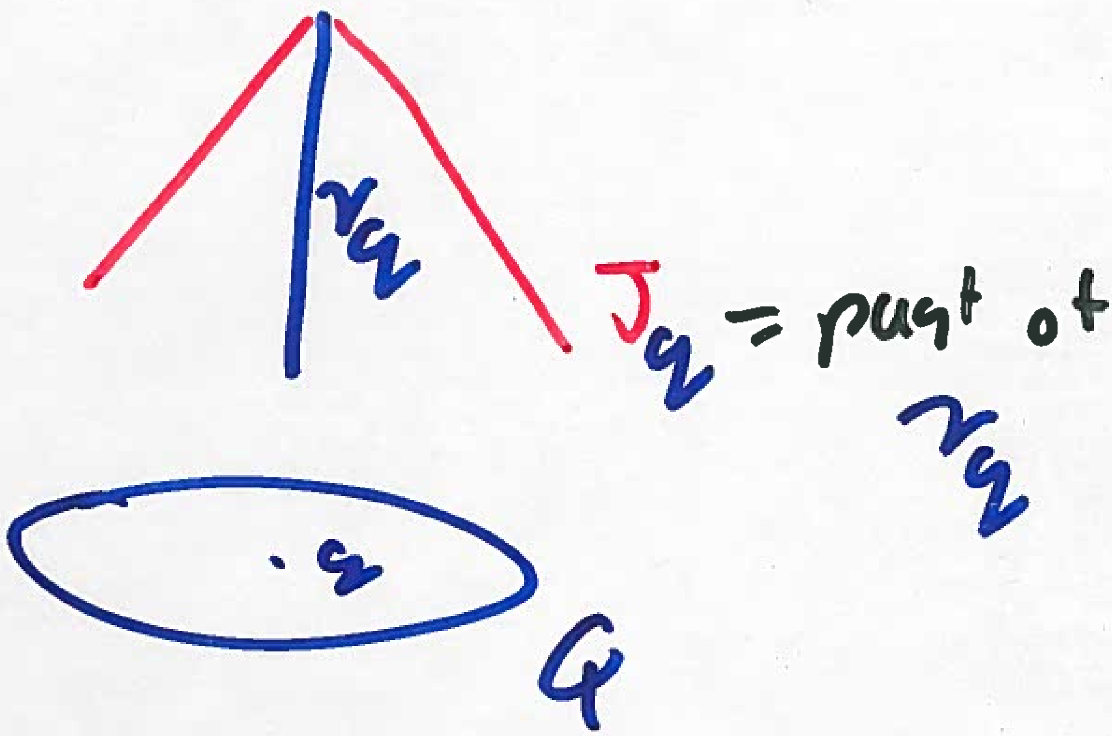
$\phi^\tau = d\eta^\tau$

in geodesic case:

$d\alpha = -\frac{1}{2}\pi^* \phi^\tau$

ϕ^τ measures integrability of $T^\perp$

CAUSAL BOUNDARY



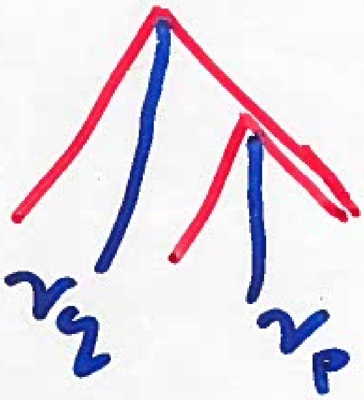
WANT:

Future Causal Boundary is $\{J_q\}$ and is spacelike with topology of Q

WANT:

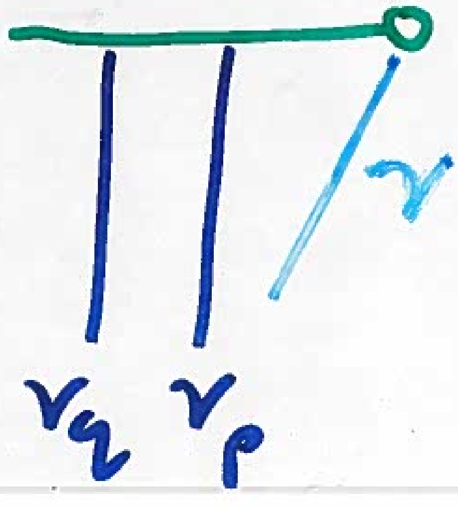
not: 

① $\{ J_q \}$ all distinct

② not: 

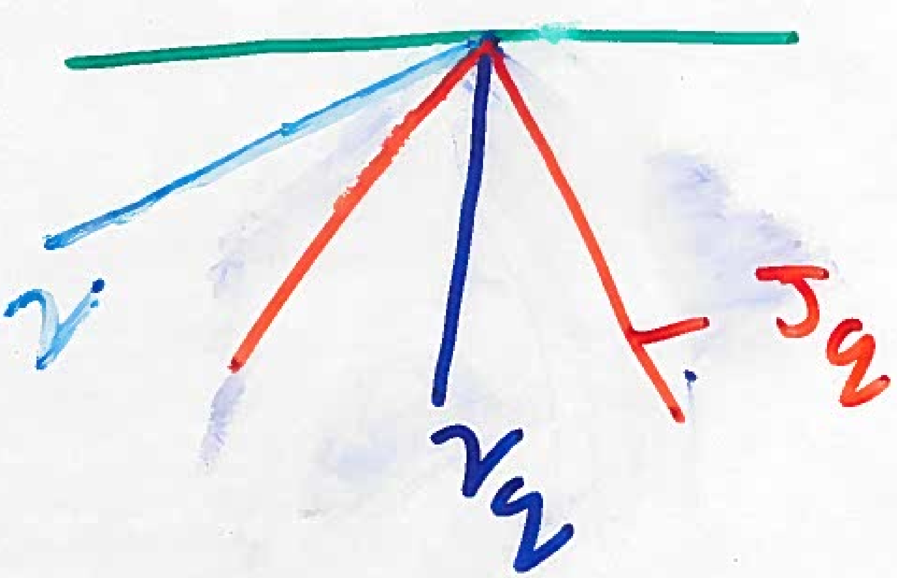
$J_p \subset J_q$

③ any causal curve is contained in some J_q

not: 

(completeness condition on h^{τ})

also not:



(controlled growth in h^{τ}, n^{τ})

WANT topology:

(4) $\{J_{q_n}\} \longrightarrow J_q$

$\Longleftrightarrow$ $\{q_n\} \longrightarrow q$

(5) $\{\gamma_{q_n}(\tau_n)\} \longrightarrow J_q$

$\Longleftrightarrow$ $\{q_n\} \longrightarrow q$

and

$\{\tau_n\} \longrightarrow 0$

sufficient for all
except no other
boundary points:

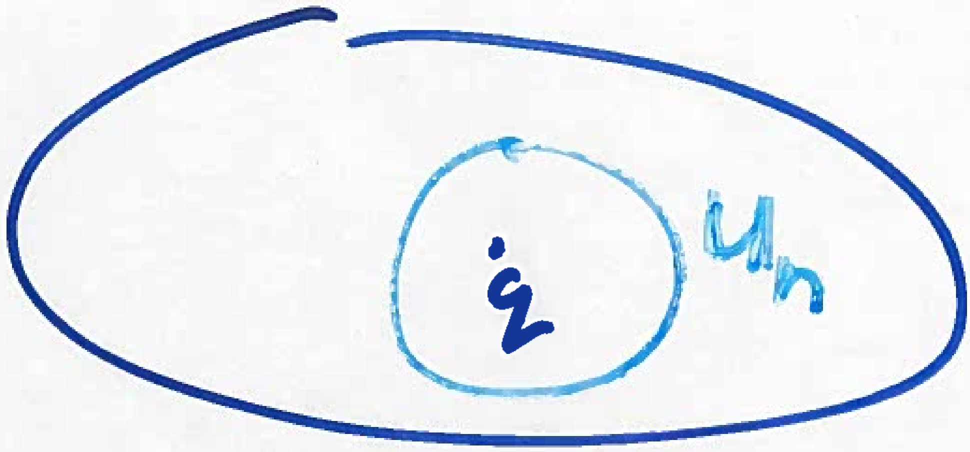
① uniformly small η^τ

$|\eta^\tau|_{h^\tau} \leq 2(1-a) \qquad a<1$

② uniformly bounded
control of h^τ

a) $\displaystyle\int_{T_0}^{0} \frac{1}{|X|_{h^\tau}}\, d\tau < \infty$

b)



each q has fundamental neighborhood system $\{U_n\}$ and numbers $\{B_n\}$ so that ...

for any curve c in $\mathcal{U}_n$

there is reparamertization

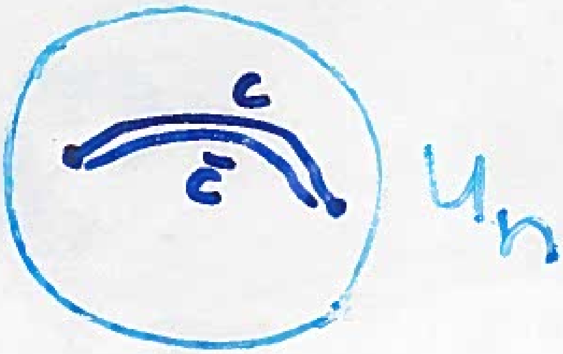
$\bar{c}$ of c

so that for all s

for all τ

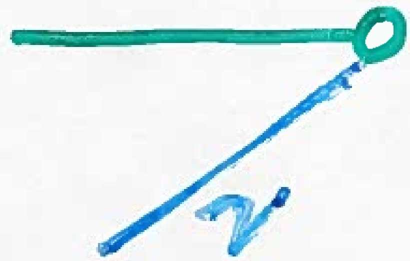
$$\frac{1}{B_n} |\dot{\bar{c}}(0)|_{h^\tau} \leq |\dot{\bar{c}}(s)|_{h^\tau} \leq B_n |\dot{\bar{c}}(0)|_{h^\tau}$$

and $\{B_n\} \to 1$



$\boxed{\text{quasi-constant-speed parametrizati}}$

sufficient to prevent



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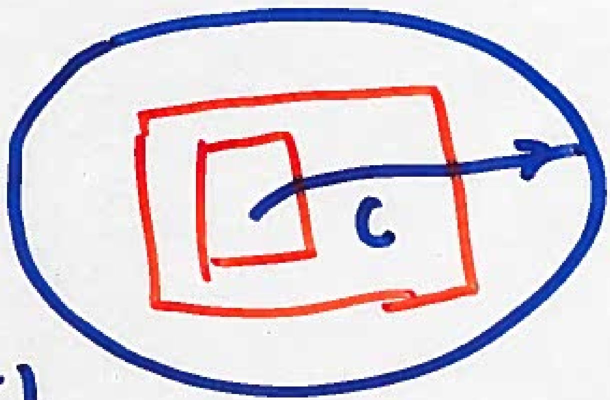
completeness of h^T means



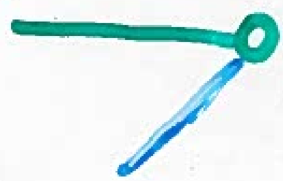
for any c escaping all compact sets

Q

$$\int_c |\dot{c}|_{h^T} = \infty$$

OR

$$\int_c \frac{|\dot{c}|_{h^T}}{|\dot{c}(0)|_{h^T}} = \infty$$

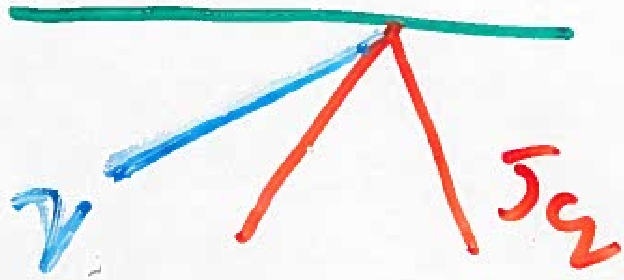
to prevent 

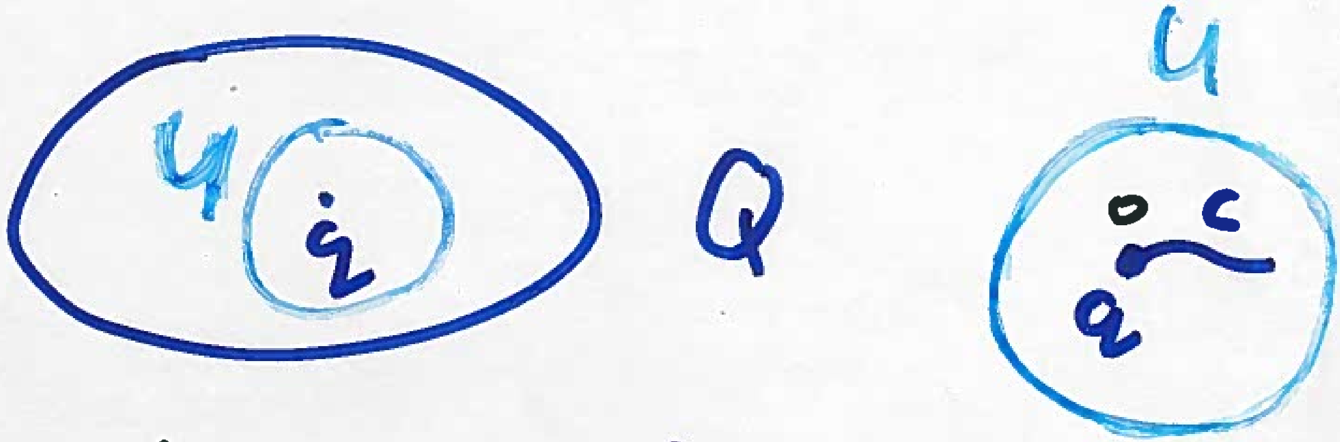
uniform completeness:

$$\int_c \inf_T \frac{|\dot c|_{h^T}}{|\dot c(0)|_{h^T}} = \infty$$

for any c in Q
esaping all compact sets

sufficient to prevent





each q in Q has nbhd U

each curve c in U from q

$$\boxed{\frac{\operatorname*{Avg}_{c}\left(|\dot{c}|_{h^\tau} + \frac{1}{2}n^\tau(\dot{c})\right)}{|\dot{c}(0)|_{h^\tau} + \frac{1}{2}n^\tau(\dot{c}(0))}}$$

is non-decreasing in τ

With all those conditions:

$$\left.\begin{aligned} \hat{M} &\cong (-\infty, 0] \times Q \\ \hat{\partial}(M) &= \{0\} \times Q \end{aligned}\right] \text{smoothly}$$

so we can ask whether

$h^{\tau} \to h^{0}$, non-degenerate

as $\tau \to 0$

$$h' = \frac{d}{d\tau} h^{\tau}$$

YES $\iff$ $\| h'^{\#} \|$ integrable

raise index

Assume:

The eigenspace of $h'^{\#}$ with largest eigenvalue has dimension ≥ 2.

Then $h^{\tau} \to h^{0}$ non-degenerate

$\iff$ for all planes Π^{τ} in $T^{\perp}$

$$\int_{\tau_0}^{0} \sqrt{|K(\Pi^{\tau})|}\, d\tau < \infty$$

Π^{τ}