Stationary Horizons of Type D

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*[4] DDR, Wojciech Kamiński, **JL**, AS 2018*

*[0] **JL,** TP 2002*

Local approach to black holes

Black holes in equilibrium:
non-expanding and shear-free null surfaces in 4d spacetime - horizons.

A. Ashtekar, C. Beetle, JL:

Mechanics of Rotating Isolated Horizons 2001

Geometry of Generic Isolated Horizons 2002

JL, Tomasz Pawłowski:
Geometric Characterizations of the Kerr Isolated Horizon 2002

Horizon geometry abstractly

H - a 3d manifold

g_{ab} - a degenerate metric tensor of signature $0++$

∇_a - a covariant derivative, metric, torsion free:

$$\nabla_a g_{bc} = 0 \qquad\qquad [\nabla_a, \nabla_b] f = 0$$

Lemma: $\ell^a g_{ab} = 0 \quad \Rightarrow \quad \mathcal{L}_\ell g_{ab} = 0$

rotation potential $\nabla_a \ell^b = \omega_a^{(\ell)} \ell^b$

rotation invariant $d\omega^{(\ell)} = d\omega^{(\ell')}$

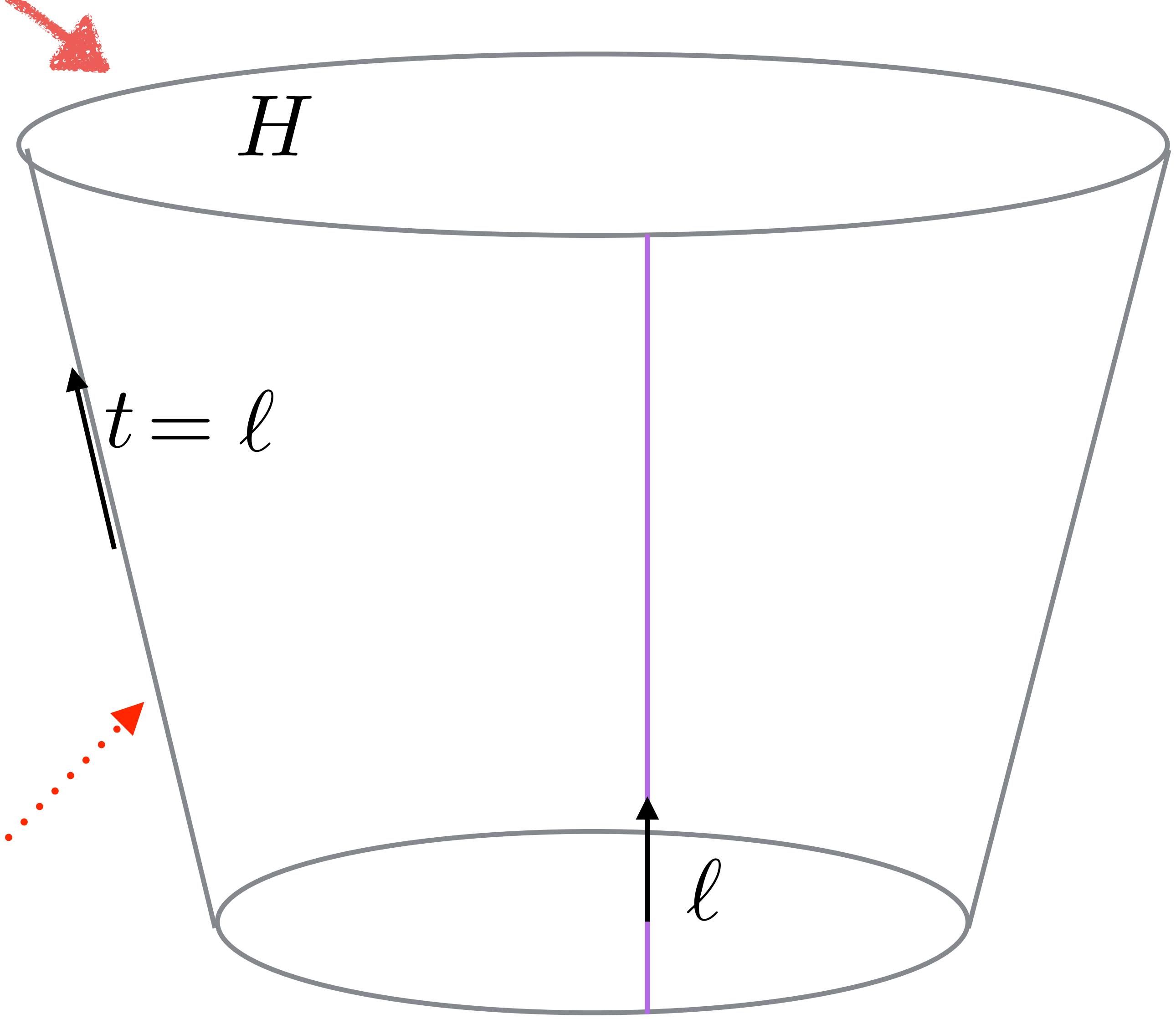
Every (H, g_{ab}, ∇_a) **is embeddable in a spacetime.**

What geometric conditions, if satisfied by $(g_{ab},\ \nabla_a)$ **make it embeddable in one of the Kerr spacetimes?**

Null surface stationary to the 2nd order

3d null surface in 4d spacetime $M\ ,\ g_{\mu\nu}$

Exists:



$\ell^\mu \ell_\mu = 0$

Such that

Killing horizon to 2nd order

$$\left.\begin{aligned} \mathcal{L}_t g_{\mu\nu} &= 0 \\ [\mathcal{L}_t, \nabla_\mu] &= 0 \\ \mathcal{L}_t R_{\mu\nu\alpha\beta} &= 0 \end{aligned}\right\}$$

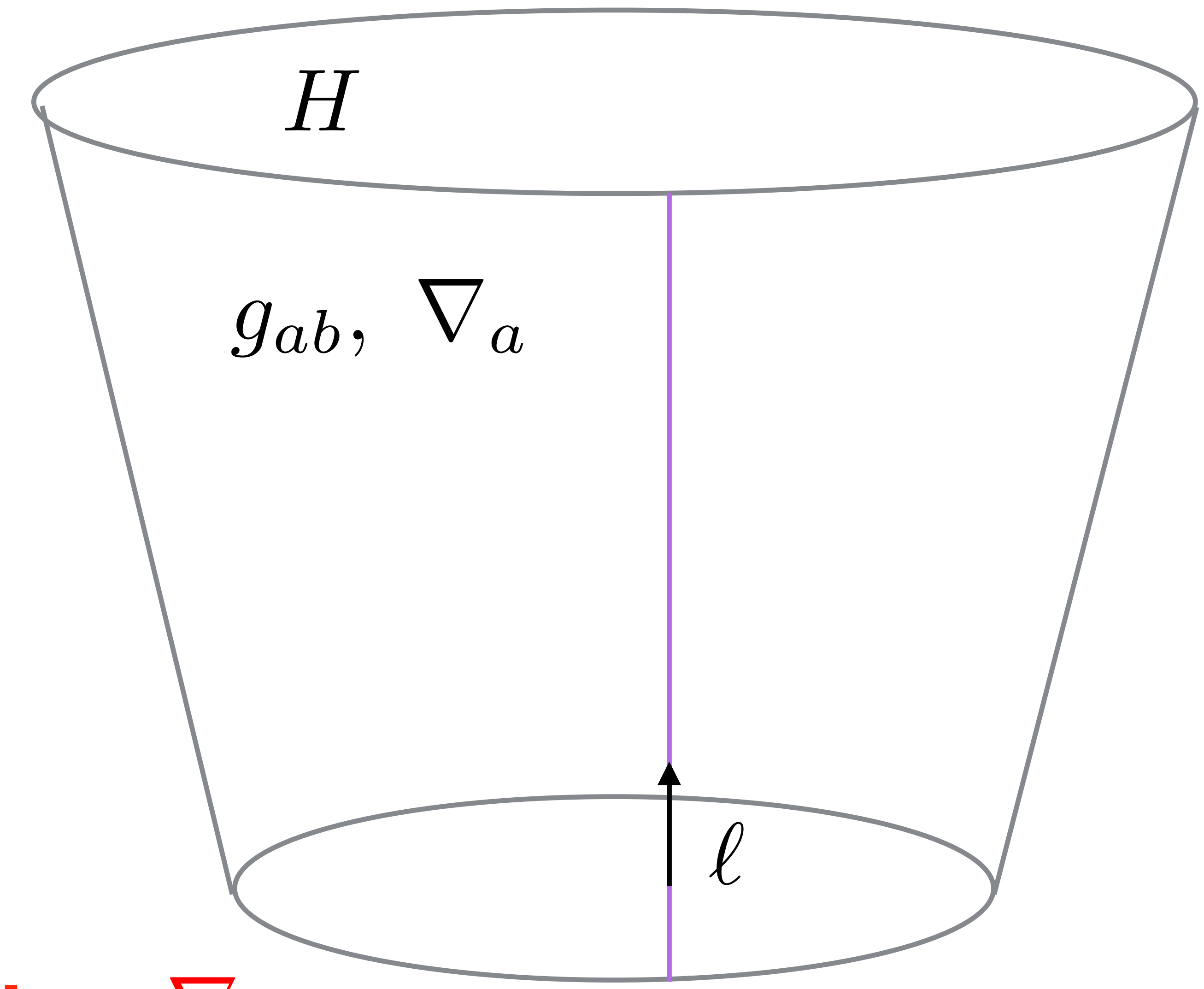
Assumption about $M\ ,\ g_{\mu\nu}$ **:** $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$

Resulting constraints on the horizon geometry

$\nabla_a \ell^b = \omega_a^{(\ell)} \ell^b$ rotation potential

$\omega_a^{(\ell)} \ell^a =: \kappa^{(\ell)}$ surface gravity

Assumption: $\kappa^{(\ell)} \neq 0$

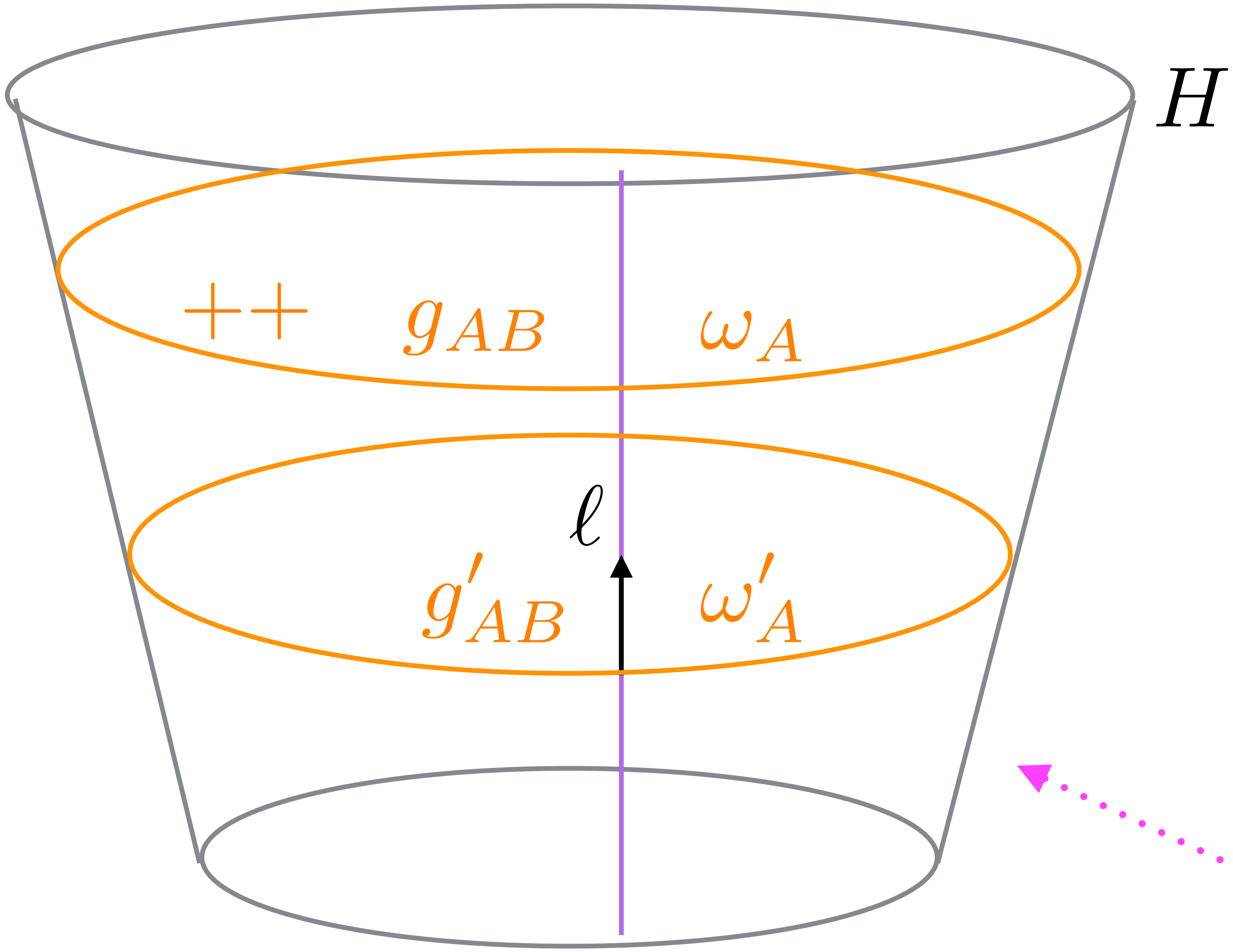


i) $\kappa^{(\ell)} = \mathrm{const}$

The 0th low of BH thermodynamics

ii) $g_{ab},\ \omega_a^{(\ell)}$ **determine all the** ∇_a

Free data on a 2d slice



$$g'_{AB} = g_{AB}$$
$$\omega'_A = \omega_A + f_{,A}$$

g_{AB}, ω_A **determines** $g_{ab}, \omega_a^{(\ell)}$ **determines** $g_{\mu\nu}, \nabla_\mu, R_{\mu\nu\alpha\beta}$

Technically important assumptions: global cross section and the null symmetry generator ℓ is transversal to the section at every point. In particular it nowhere vanishes at the section.

Horizon cross section geometry

2d-surface ———— S

Endowed with (g_{AB}, ω_A) modulo $\omega'_A = \omega_A + f_{,A}$

∇_A - the corresponding derivative $\nabla_A g_{BC} = 0$ $\nabla_{[A}\nabla_{B]} f = 0$

Scalar invariants:

Gaussian curvature: K

Rotation scalar: $d\omega =: \Omega \, d\mathrm{Area}$ combined:

$$\Psi := -\frac{1}{2}(K + i\Omega)$$

Possible Petrov types

The spacetime Weyl tensor at H is determined by the data

$$(S, g_{AB}, \omega_A)$$

Theorem:

The possible Petrov types of *H* are:

~~I~~, II, D, ~~III~~, ~~N~~, O

wherein:

$$\Psi + \frac{\Lambda}{6} = 0 \quad \Leftrightarrow \quad \textbf{O} \quad \Leftrightarrow \quad K = \frac{\Lambda}{3} \qquad d\omega = 0$$

$$\Psi + \frac{\Lambda}{6} \neq 0 \quad \Rightarrow$$ **generically II, unless…**

The Petrov type D equation

We use a null 2-frame

$$g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B \qquad d\mathrm{Area}_{BC} = i(\bar{m}_B m_C - \bar{m}_C m_B)$$

Theorem 1:

At *H* the spacetime Weyl tensor is of the Petrov type D iff the following two conditions are satisfied by the invariants of S:

$$\Psi + \frac{\Lambda}{6} \neq 0$$

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(\Psi + \frac{\Lambda}{6}\right)^{-\frac{1}{3}} = 0$$

Remark: a continues function $\left(\Psi + \frac{\Lambda}{6}\right)^{-\frac{1}{3}}$ exists locally on S

$$g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B \qquad d\mathrm{Area}_{BC} = i(\bar{m}_B m_C - \bar{m}_C m_B)$$

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(\Psi + \frac{\Lambda}{6}\right)^{-\frac{1}{3}} = 0 \qquad \Psi + \frac{\Lambda}{6} \neq 0$$

Symmetries:

$$\phi : S \to S \qquad g' = \phi^* g, \ \omega' = \phi^* \omega$$

$$g' = g, \ \omega' = \omega + df$$

$$m'_A = \bar{m}_A, \ \omega'_A = \omega_A$$

The non-continuity of $(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}$**:** use a covering

$(S, g_{AB}, \omega_A) \to (\tilde{S}, \tilde{g}_{AB}, \tilde{\omega}_A)$ s.t. $(\tilde{\Psi} + \frac{\Lambda}{6})^{-\frac{1}{3}}$ continues

Relation with the Near Horizon Geometry Equation

Theorem 2:

Suppose (g_{AB}, ω_A) **satisfy the NHG equation, namely**

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0$$

Then they also satisfy the Petrov type D equation:

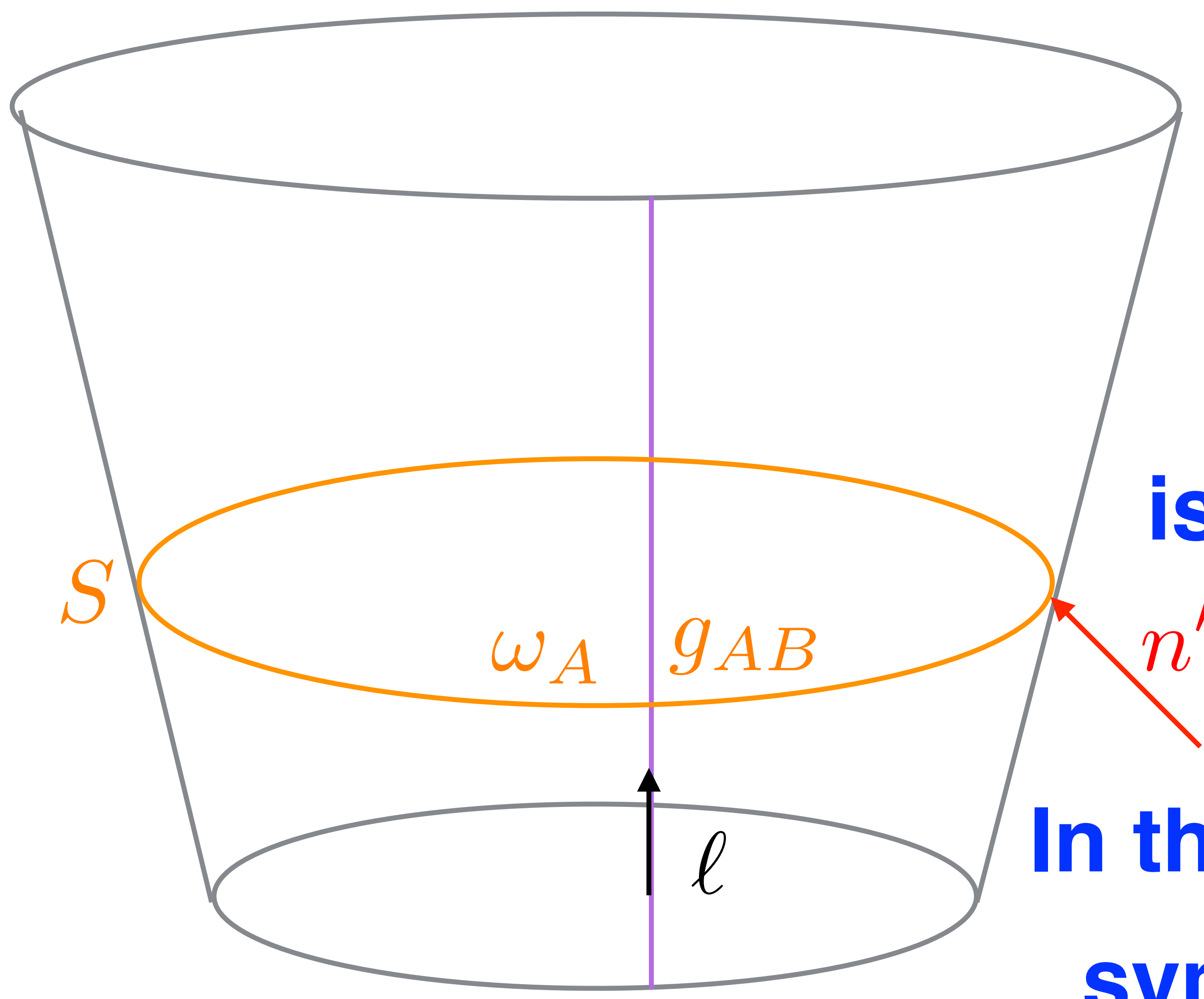
$$\bar{m}^A\bar{m}^B\nabla_A\nabla_B(\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

Non-twisting
of the second principal null direction of the Weyl tensor

Theorem:

Suppose (g_{AB}, ω_A) satisfy the NHG equation, namely

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0$$



Then the null vector n' orthogonal to the corresponding slice S is a double principal direction of the spacetime Weyl tensor at *H.*

In that case there exists another symmetry *t'* that is extremal

The holomorphic version

$$g_{AB}dx^A dx^B = \frac{2}{P^2}dz d\bar{z} \qquad\qquad m^A\partial_A = P\partial_z$$

$$z' = f(z)$$

$$X = X^z\partial_z + X^{\bar{z}}\partial_{\bar{z}} =: X^{(1,0)} + X^{(0,1)}$$

The Petrov type D equation reads:

$$\partial_{\bar{z}}(P^2\partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}) = 0$$

$$\Leftrightarrow$$

$$X := g^{AB}\partial_B(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}\partial_A$$

$$X^{(1,0)}$$

is a holomorphic vector

The Petrov type D equation on S of genus > 0

$$g_{AB}dx^A dx^B = \frac{2}{P^2} dz d\bar{z} \qquad m^A \partial_A = P\partial_z$$

The Petrov type D equation: $$\partial_{\bar{z}}(P^2 \partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}) = 0$$

$$\Rightarrow \quad \partial_{\bar{z}}\left(\Psi + \frac{\Lambda}{6}\right)^{-\frac{1}{3}} = \frac{F(z)}{P^2}$$

$$\Rightarrow \quad F(z)\partial_z$$

is a globally defined holomorphic vector field

$$\Rightarrow \quad F(z) = \text{const} \qquad \text{if genus} = 1$$

$$F(z) = 0 \qquad \text{if genus} > 1$$

The Petrov type D equation on S of genus > 0

Theorem 2.
Suppose S is a compact 2-surface of genus >0.
The only solutions to the Petrov type D equation with a cosmological constant Λ are (g, ω) such that

$$d\omega = 0 \qquad \textbf{and} \qquad K = \text{const} \neq \frac{\Lambda}{3}$$

Remark. There are no rotating solutions

The Petrov type D equation on topological spheres.

$S = S_2$, $g, d\omega$ **axi-symmetric**

Theorem 3:

The family of axisymmetric solutions of the Petrov type D equation with (or without) cosmological constant defined on a topological sphere can be parametrized by two numbers *(A,J)*:
the area and angular momentum, respectively. They take the following values in $\mathbb{R}^+ \times \mathbb{R}$:

$$\Lambda > 0$$

$$J \in \Big(-\infty, \infty \Big) \text{ for } A \in \Big(0, \frac{12\pi}{\Lambda} \Big) \text{ and } |J| \in \Big[0, \frac{A}{16\pi}\sqrt{\frac{\Lambda\ A}{12\pi} - 1} \Big) \text{ for } A \in \Big(\frac{12\pi}{\Lambda}, \infty \Big)$$

$$\Lambda \leq 0$$

$$J \in \Big(-\infty, \infty \Big) \text{ and } A \in \Big(0, \infty \Big)$$

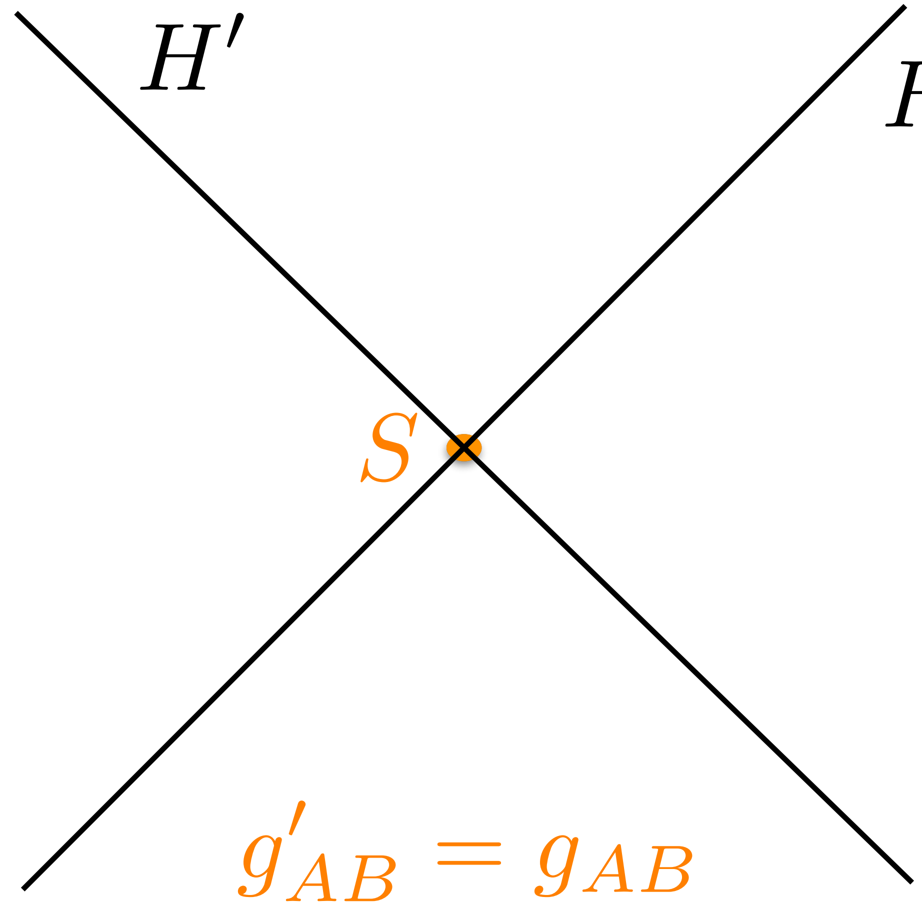
The embedding in spacetimes

All the horizons obtained via Theorem 2 can be embedded either in non-extremal Kerr-(anti) de Sitter spacetime or in so called near horizon geometry spacetime obtained by the near extremal horizon limit. The embedding preserves intrinsic geometry of each horizon:

$$g_{ab}, \nabla_a$$

The problem of non-axisymmetric solutions on topologically spherical S is open.

A bifurcated Petrov type D horizon: data



$$g'_{AB} = g_{AB}$$

$$\omega'_A = -\omega_A$$

$$\Psi' = \bar{\Psi}$$

A bifurcated Petrov type D horizon: equations

The Petrov type D equations

for H:

$$\bar{m}^A\bar{m}^B\nabla_A\nabla_B(\Psi+\frac{1}{6}\Lambda)^{-\frac{1}{3}}=0$$

and for *H':*

$$m^A m^B\nabla_A\nabla_B(\Psi+\frac{1}{6}\Lambda)^{-\frac{1}{3}}=0$$

hold simultaneously on S $\Rightarrow$ additional (axial) symmetry

M.J. Cole, I. Rácz, J.A. Valiente Kroon 2018

JL, A. Szereszewski 2018

In conformally flat coordinates

$$g_{AB}dx^A dx^B = \frac{2}{P^2} dz d\bar{z} \qquad m^A \partial_A = P\partial_z$$

$$\partial_{\bar{z}}(P^2 \partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}) = 0 \quad \Rightarrow \quad \partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = \frac{F(z)}{P^2}$$

$$\partial_z(P^2 \partial_z(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}) = 0 \quad \Rightarrow \quad \partial_z(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = \frac{\bar{G}(\bar{z})}{P^2}$$

$$\partial_z \partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = \partial_{\bar{z}} \partial_z(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} \quad \Rightarrow \quad \partial_z \left(\frac{F(z)}{P^2} \right) = \partial_{\bar{z}} \left(\frac{\bar{G}(\bar{z})}{P^2} \right)$$

$$\Rightarrow \quad \mathcal{L}_\Phi g_{AB} = 0 \qquad \Phi := F(z)\partial_z - \bar{G}(\bar{z})\partial_{\bar{z}}$$

$$= X^{(1,0)} - X^{(0,1)}$$

$$\Rightarrow \quad \mathcal{L}_\Phi \, d\omega = 0$$

The axial symmetry without the rigidity theorem

Theorem:

Suppose (g_{AB}, ω_A) **defined on** S **satisfy the Petrov type D equation**

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

and the conjugate one

$$m^A m^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

Then, there is a vector field Φ **at** S **such that**

$$\mathcal{L}_\Phi g_{AB} = 0 \qquad \textbf{and} \qquad \mathcal{L}_\Phi d\omega = 0$$

$$\Phi^A = \mathrm{Re}/\mathrm{Im} \left(d\mathrm{Area}^{AB} \partial_A (\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} \right)$$

Summary

- The type D equation:

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(-\frac{1}{2}K - \frac{1}{2}i\mathcal{O} + \frac{\Lambda}{6} \right)^{-\frac{1}{3}} = 0$$

- Non-twisting of the second double principal vector if:

$$\nabla_{(A}\omega_{B)} + \omega_A \omega_B + \frac{1}{2}(\Lambda - K) g_{AB} = 0$$

- All the axisymmetric solutions of the type D eq. on topological sphere parametrized by (A, J);
- All solutions on genus>0 derived (non-rotating);
- The extra (axial) symmetry in the case of bifurcated horizon;
- Open problems: existence of non-axisymmetric solutions on topological sphere

Thank you