Causal evolution of probability measures

Tomasz Miller

Joint project with **Michał Eckstein** (UG & CC, Cracow, Poland)

Warsaw University of Technology & Copernicus Center (Cracow)

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Motivations and goals

Causal precedence relation $\preceq$ (J^+) between **events**

$p \preceq q$ if $\exists$ a piecewise smooth fut-dir causal curve from p to q (or $p = q$).

Question: How would one extend $\preceq$ onto **probability measures** on a given spacetime?

Subquestion: Why probability measures?

- Besnard, Franco, Eckstein: causal relation between the **states** on a (possibly noncommutative) algebra $\mathcal{A}$.
- If $\mathcal{A} = C_0^\infty(\mathcal{M})$, then:
  - States on $\mathcal{A}$ $\leftrightsquigarrow$ Borel probability measures on $\mathcal{M}$ $\leftrightsquigarrow$
  - Pure states on $\mathcal{A}$ $\leftrightsquigarrow$ Dirac measures δ_p for $p \in \mathcal{M}$ $\leftrightsquigarrow$ **events**.

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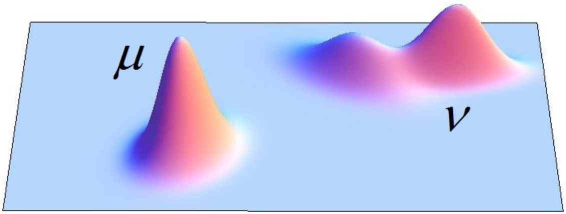

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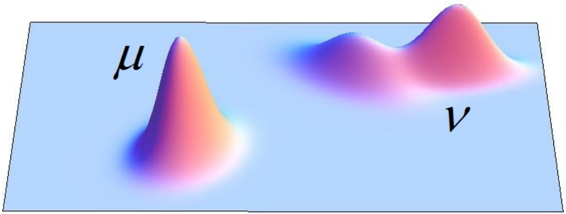

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Causality for probability measures

What does it mean that $\mu \preceq \nu$? [M. Eckstein, TM '17]

Let $\mathcal{M}$ be a spacetime. Then for any $\mu, \nu \in \mathscr{P}(\mathcal{M})$

$\mu \preceq \nu \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists\, \omega \in \mathscr{P}(\mathcal{M}^2)$ such that:

- $\forall_{B\text{ – Borel}} \quad \omega(B \times \mathcal{M}) = \mu(B), \;\; \omega(\mathcal{M} \times B) = \nu(B)$,
- $\omega(J^+) = 1$,

where $J^+ := \{(p,q) \in \mathcal{M}^2 \mid p \preceq q\}$.

- ω can be called a causal coupling or a causal transference plan.
- For $\mu = \delta_p$, $\nu = \delta_q$, the only coupling is $\omega = \delta_{(p,q)}$ and so $\delta_p \preceq \delta_q$ iff $p \preceq q$.
- $\preceq$ is reflexive and transitive. It is antisymmetric for $\mathcal{M}$ past/future distinguishing.

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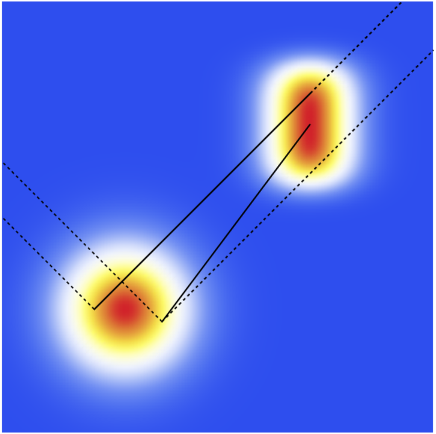
Each infinitesimal part of the probability measure should travel along a future-directed causal curve.



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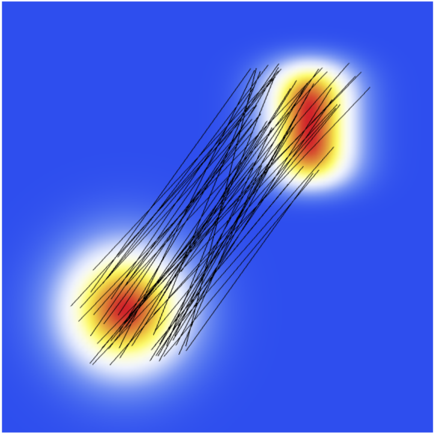
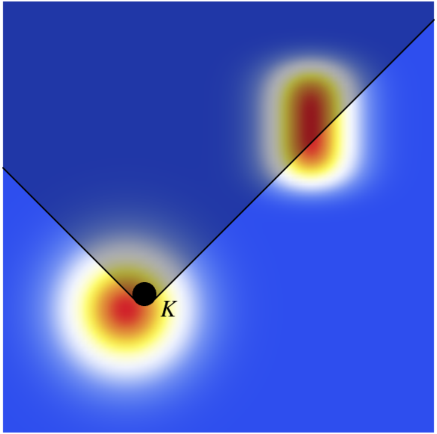


Causality for probability measures

For $\mathcal{M}$ causally simple:

$\mu \preceq \nu \quad \Longleftrightarrow \quad$ for any compact $\mathcal{K} \subseteq \mathrm{supp}\, \mu \quad \mu(\mathcal{K}) \leqslant \nu(J^+(\mathcal{K}))$



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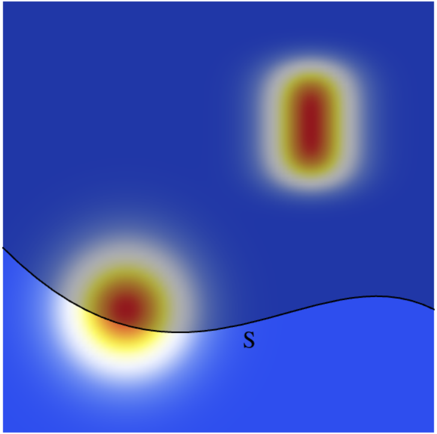
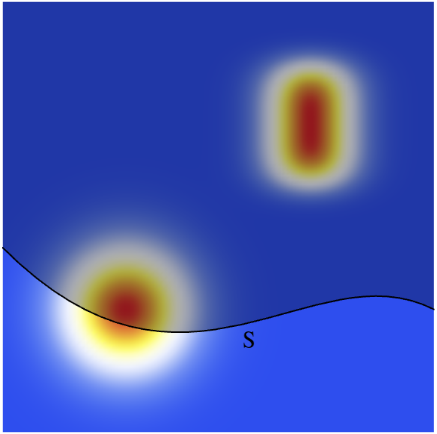


Causality for probability measures

For $\mathcal{M}$ causally simple:

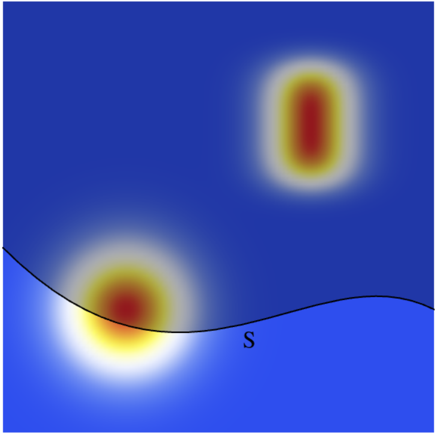
$\mu \preceq \nu \quad \Longleftrightarrow \quad$ for any time function $\mathcal{T}$ $\int_{\mathcal{M}} \mathcal{T} d\mu \leqslant \int_{\mathcal{M}} \mathcal{T} d\nu$



Causality for probability measures

For $\mathcal{M}$ globally hyperbolic:

$\mu \preceq \nu \quad \Longleftrightarrow \quad$ for any Cauchy hypersurface $\mathcal{S}$ $\quad \mu(J^+(\mathcal{S})) \leqslant \nu(J^+(\mathcal{S}))$



Causal time-evolution of measures ($\mathcal{M}$ – Minkowski)

Causal time-evolution of a pointlike particle

A curve $\gamma : I \to \mathcal{M}$ with $\gamma(t) = (t, x(t))$ is a worldline of a physical particle if

$$\forall s,t \in I \quad s \leqslant t \ \Rightarrow \ \gamma(s) \preceq \gamma(t).$$

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A map $\mu : I \to \mathscr{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\operatorname{supp} \mu_t \subseteq \{t\} \times \mathbb{R}^3$ for all $t \in I$ is a **causal evolution of a measure** if

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- Fix a Cauchy temporal function $\mathcal{T}$.

Causal time-evolution of a pointlike particle

A curve $\gamma : I \to \mathcal{M}$ such that $\mathcal{T}(\gamma(t)) = t$ is a worldline of a physical particle if

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A map $\mu : I \to \mathscr{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\operatorname{supp} \mu_t \subseteq \mathcal{T}^{-1}(t)$ for all $t \in I$ is a **causal evolution of a measure** if

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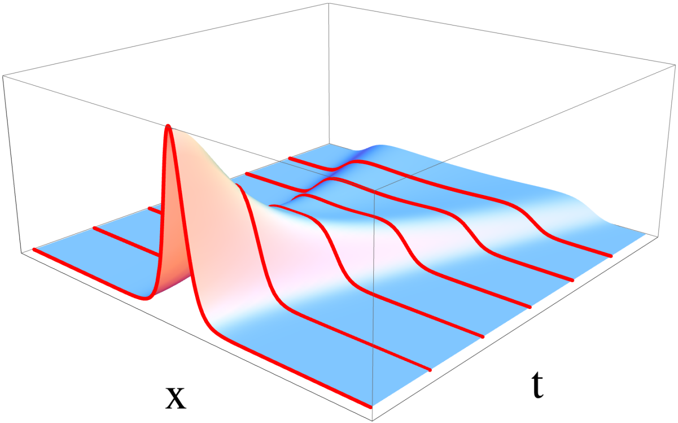

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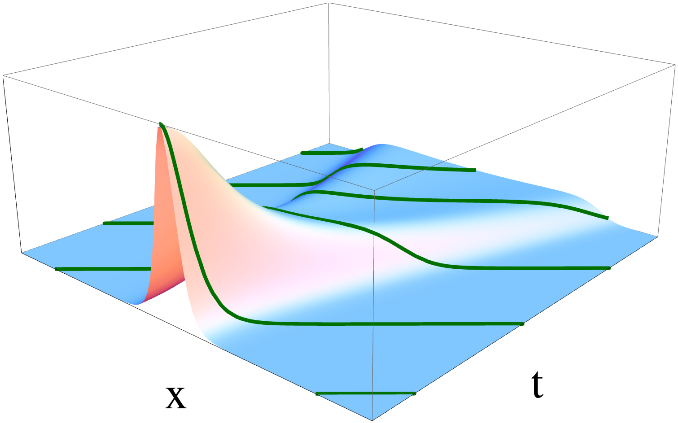

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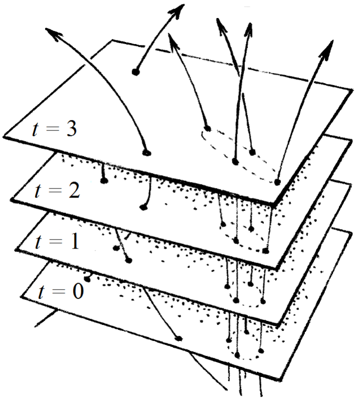
Theorem [TM '17]

Fix a Cauchy temporal function $\mathcal{T}$.
Consider a map $t \mapsto \mu_t \in \mathscr{P}(\mathcal{M})$ satisfying $\operatorname{supp} \mu_t \subseteq \mathcal{T}^{-1}(t)$ for all $t \in I$.
TFAE:

- The map $t \mapsto \mu_t$ is *causal*, i.e.
  $$\forall s, t \in I \quad s \leqslant t \;\Rightarrow\; \mu_s \preceq \mu_t.$$
- There exists a **probability measure on the space of worldlines**, from which one can recover μ_t for all $t \in I$.

The "space of worldlines" is suitably topologized so as to ensure **Polishness**.



Adapted from Penrose's "Road to Reality"

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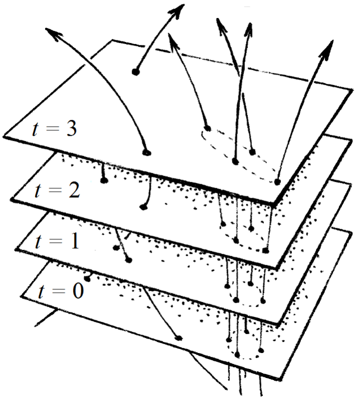

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Relationship with the continuity equation ($\mathcal{M}$ – Minkowski)

Theorem [M. Eckstein, TM '17]

Suppose $\rho(t,x)$ satisfies the **continuity equation** $\partial_t \rho + \nabla \cdot \rho \mathbf{v} = 0$ with a velocity field such that $\|\mathbf{v}(t,x)\| \leqslant 1$. Then μ_t defined via

$$d\mu_t = \delta_t \otimes \rho(t,x)\, d^3x$$

evolves **causally**.

More generally, suppose μ_t satisfies:

$$\forall\, \Phi \in C_c^\infty(I \times \mathbb{R}^n) \quad \int_I \int_{\mathcal{M}} (\partial_t + \mathbf{v} \cdot \nabla)\, \Phi\, d\mu_t dt = 0$$

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Conjecture

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$$\forall\, \Phi \in C_c^\infty(\mathcal{T}^{-1}(I)) \quad \int_I \int_{\mathcal{M}} X\Phi \, d\mu_t dt = 0 \qquad (\star)$$

with a certain **causal** vector field X. Then μ_t evolves causally.

Converse result (preliminary!)

Fix a Cauchy temporal function $\mathcal{T}$. Suppose μ_t evolves causally. Then there exists a **causal** vector field X such that $(\star)$ holds.

X is generally rather low-regular. Namely, $L^2(\mathcal{T}^{-1}(I), \int_I \mu_t dt)$-regular.

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- The causal relation J^+ can be **naturally** extended onto $\mathscr{P}(\mathcal{M})$ – the space of Borel probability measures on $\mathcal{M}$.
- One can use thus extended relations to describe the **causal evolution of probability measures** in glob. hyperbolic spacetimes.
  - Time-evolution of a **pointlike** particle $\leftrightsquigarrow$ **single** worldline.
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- The continuity equation $\partial_t \mu_t + \nabla \cdot \mu_t \mathbf{v} = 0$, when rewritten as $\int_I \int_{\mathcal{M}} X\Phi \, d\mu_t dt = 0$ for all test functions Φ, is nothing but a "nonlocal analogue" of the requirement that $\gamma'(t)$ is a causal vector.

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- The continuity equation $\partial_t \mu_t + \nabla \cdot \mu_t \mathbf{v} = 0$, when rewritten as $\int_I \int_{\mathcal{M}} X\Phi \, d\mu_t dt = 0$ for all test functions Φ, is nothing but a "nonlocal analogue" of the requirement that $\gamma'(t)$ is a causal vector.

Thank you for your attention!

M. Eckstein and T. Miller, *Causality for nonlocal phenomena*, Annales Henri Poincaré 2017, **18**(9), 3049–3096,

T. Miller, *Polish spaces of causal curves*, Journal of Geometry and Physics 2017 **116**, 295–315,

M. Eckstein and T. Miller, *Causal evolution of wave packets*, Physical Review A 2017 **95**, 032106,

T. Miller, *On the causality and K-causality between measures*, Universe 2017 3(1):27,

Bonus: Polish spaces of causal curves

- **Q**: How to topologize sets of (fut-dir) causal curves?
  A (naïve): Induce topology from $C(I, \mathcal{M})$ (the compact-open top.)
- Too large a space! Various parameterizations of an unparameterized curve treated as distinct elements!
- Two ways out:
  - Take a quotient modulo (continuous strictly increasing) reparameterizations $\Leftrightarrow$ focus on unparameterized curves, and use the C^0-**topology**.
  - Choose the "canonical" parameterization of each curve — e.g. the arc-length parameterization — and use the **compact-open topology**.

Polish spaces of causal curves

Spaces of causal curves parameterized "in accordance with $\mathcal{T}$"

$\mathcal{M}$ – stably causal spacetime, $\mathcal{T}$ – time function, I – interval.
$C^I_{\mathcal{T}} :=$ the space of all fut-dir causal curves $\gamma \in C(I, \mathcal{M})$ such that

$$\exists\, c_\gamma > 0 \ \ \forall s,t \in I \quad \mathcal{T}(\gamma(t)) - \mathcal{T}(\gamma(s)) = c_\gamma (t-s),$$

endowed with the compact-open topology induced from $C(I, \mathcal{M})$.

- $C^I_{\mathcal{T}}$ is separable and completely metrizable (i.e. Polish).
- $\mathscr{C} :=$ the space of all *compact* unparameterized causal curves with the C^0-topology. Theorem: $C^{[a,b]}_{\mathcal{T}} \cong \mathscr{C}$ and hence:
  - $\mathscr{C}$ is Polish!
  - $C^{[a,b]}_{\mathcal{T}_1} \cong C^{[c,d]}_{\mathcal{T}_2}$.
- $\mathcal{M}$ – glob. hyperbolic, $\mathcal{T}_1, \mathcal{T}_2$ – Cauchy temporal functions. Theorem: $C^{\mathbb{R}}_{\mathcal{T}_1} \cong C^{\mathbb{R}}_{\mathcal{T}_2}$.

Causal time-evolution of measures (full statement of the theorem)

Theorem [TM '17]

Fix a Cauchy temporal function $\mathcal{T}$.
Consider a map $t \mapsto \mu_t \in \mathscr{P}(\mathcal{M})$ satisfying $\operatorname{supp} \mu_t \subseteq \mathcal{T}^{-1}(t)$ for all $t \in I$.
TFAE:

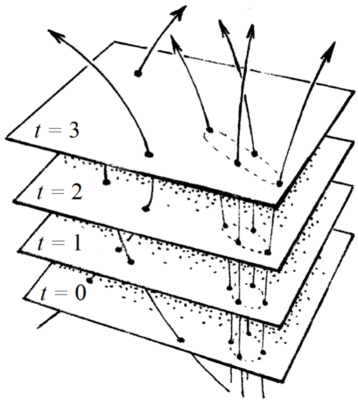
- The map $t \mapsto \mu_t$ is *causal*, i.e.

$$\forall s, t \in I \quad s \leqslant t \ \Rightarrow \ \mu_s \preceq \mu_t.$$

- $\exists\, \sigma \in \mathscr{P}(C^I_{\mathcal{T}})$ such that

$$(\mathrm{ev}_t)_{\#}\sigma = \mu_t,$$

where $\mathrm{ev}_t : C^I_{\mathcal{T}} \to \mathcal{M}$, $\gamma \mapsto \gamma(t)$.



Adapted from R. Penrose's "Road to Reality"