Hopf Real Hypersurfaces in the Indefinite Complex Projective Space

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Summary

Overview

This talk is based on the following joint work with
Makoto Kimura (Ibaraki University, Japan)

M. Kimura, —, *Hopf Real Hypersurfaces in the Indefinite Complex Projective Space*, https://arxiv.org/abs/1802.05556

Overview

The theory of real hypersurfaces in complex space forms is very well-developed.

J. Berdnt, T. Cecil, G. Kaimakamis, M. Kimura, S. Maeda, Y. Maeda, S. Montiel, K. Panagiotidou, Juan de Dios Pérez, P. Ryan, Y. J. Suh, R. Takagi...

Overview

R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*. Osaka J. Math. **10** (1973), 495–506

Theorem 1

Let M be a extrinsically homogeneous real hypersurface in $\mathbb{C}P^n$, $n \geq 2$. Then, M is a tube of radius r over one of the following:

A) *A totally geodesic* $\mathbb{C}P^k$, $0 \leq k \leq n-1$, $0 < r < \pi/2$;

B) *A complex quadric* $\mathbb{Q}^{n-1}$, $0 < r < \pi/4$;

C) $\mathbb{C}P^1 \times \mathbb{C}P^{(n-1)/2}$, $0 < r < \pi/4$, $n \geq 5$, n *odd;*

D) *Complex Grassmanian* $G_{2,5}(\mathbb{C})$, $0 < r < \pi/4$, $n = 9$;

E) *Hermitian Symmetric Space* $SO(10)/U(5)$, $0 < r < \pi/4$, $n = 15$.

If N is unit normal vector field to M in $\mathbb{C}P^n$, then $\xi = -JN$.
A: shape operator. All these examples satisfy $A\xi = \mu\xi$.

Overview

M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space, *Trans. Amer. Math. Soc.* **296** (1986) (1), 137-149.

Theorem 2

Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 2$, such that ξ is principal and M has constant principal curvatures. Then, M is an open subset of one of the real hypersurfaces in the Takagi's list.

J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. **395** (1989), 132?141.

Theorem 3

Let M be a real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, such that ξ is principal, and M has constant principal curvatures. Then, M is an open subset of one of the following:

A) *A tube of radius $r > 0$ over a totally geodesic $\mathbb{C}H^k$, $k = 0, \ldots, n-1$;*

B) *a tube of radius $r > 0$ over a totally geodesic $\mathbb{R}P^n$;*

C) *a horosphere.*

Overview

Hundreds of works about real hypersurfaces in non-flat complex space forms have appeared, also in

- the quaternionic space forms,
- the Grassmanian of 2-complex planes, and
- the complex quadric.

T. E. Cecil and P. J. Ryan, *Geometry of Hypersurfaces*, Springer Monographs in Mathematics, Springer, New York, NY (2015) DOI 10.1007/978-1-4939-3246-7

Overview

The study of real hypersurfaces in indefinite complex projective space seems to be initiated in

A. Bejancu, K. L. Duggal, *Real hypersurfaces of indefinite Kaehler manifolds*, Internat. J. Math. Math. Sci. **16** (1993), no. 3, 545–556.

They paid attention to real hypersurfaces in (flat) complex space forms, by considering the (ε)-Sasakian and (ε)-cosymplectic structures.

Overview

H. Anciaux, K. Panagiotidou, *Hopf Hypersurfaces in pseudo-Riemannian complex and para-complex space forms*, Diff. Geom. Appl. **42** (2015) 1-14 DOI: 10.1016/j.difgeo.2015.05.004

Anciaux and Panatiotidou studied the *almost contact structure* (g, ξ, η, ϕ) on a real hypersurface in $\mathbb{C}P^n_p$, and tubes over certain submanifolds.
A: shape operator.

Overview

H. Anciaux, K. Panagiotidou, *Hopf Hypersurfaces in pseudo-Riemannian complex and para-complex space forms*, Diff. Geom. Appl. **42** (2015) 1-14 DOI: 10.1016/j.difgeo.2015.05.004

Anciaux and Panatiotidou studied the *almost contact structure* (g, ξ, η, ϕ) on a real hypersurface in $\mathbb{C}P^n_p$, and tubes over certain submanifolds.
A: shape operator.

Problems

P1) Are there any real hypersurface s.t. $A\xi = \mu\xi$, $|\mu| \leq 2$?

P2) Classification of real hypersurfaces s.t. $A\phi = \phi A$.

When a real hypersurface had a timelike unit normal vector field, Anciaux and Panagiotidou always changed the metric g by $-g$.

Our aim

- Further develop Anciaux and Panagiotidou's ideas.
- Attack the problems they posed.

We will just focus on the indefinite complex projective space $\mathbb{C}P^n_p$ of index $1 \leq p \leq n-1$,

Overview

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  1. Families of non-degenerate real hypersurfaces whose shape operator is diagonalisable,
  2. A null example (with degenerate metric) and non-diagonalisable *shape operator*.

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- We allow the normal vector to have its own causal character, without changing the metric.
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  1. Families of non-degenerate real hypersurfaces whose shape operator is diagonalisable,
  2. A null example (with degenerate metric) and non-diagonalisable *shape operator*.
- A rigidity result.
- Real hypersurfaces which are η-umbilical.
- Real hypersurfaces whose ξ is Killing.
- Further problems.

Summary

Preliminaries

See [2] (Barros-Romero) for more details.
$\mathbb{C}_p^{n+1}$ the Euclidean complex space endowed with the following hermitian product and pseudo-Riemannian metric of index $2p$, $z = (z_1, \ldots, z_{n+1})$, $w = (w_1, \ldots, w_{n+1}) \in \mathbb{C}^{n+1}$,

$$g_{\mathbb{C}}(z, w) = -\sum_{j=1}^{p} z_j \bar{w}_j + \sum_{j=p+1}^{n+1} z_j \bar{w}_j, \quad g = \mathrm{Re}(g_{\mathbb{C}}), \tag{1}$$

where $\bar{w}$ is the complex conjugate of $w \in \mathbb{C}$.
J the natural complex structure.

$$\mathbb{S}^1 = \{a \in \mathbb{C} : a\bar{a} = 1\} = \{\mathrm{e}^{i\theta} : \theta \in \mathbb{R}\}.$$

$$\mathbb{S}_{2p}^{2n+1} = \{z \in \mathbb{C}_p^{n+1} : g(z, z) = 1\},$$

We define the action and its corresponding quotient

$$\mathbb{S}^1 \times \mathbb{S}^{2n+1}_{2p} \to \mathbb{S}^{2n+1}_{2p}, \ (a, (z_1, \ldots, z_{n+1})) \mapsto (az_1, \ldots, az_{n+1}),$$

$$\pi : \mathbb{S}^{2n+1}_{2p} \to \mathbb{C}P^n_p = \mathbb{S}^{2n+1}_{2p} / \sim .$$

Let g be the metric on $\mathbb{C}P^n_p$ such that π becomes a semi-Riemannian submersion. The manifold $\mathbb{C}P^n_p$ is called the *Indefinite Complex Projective Space*.

Let $\bar{\nabla}$ be its Levi-Civita connection. Then, $\mathbb{C}P^n_p$ admits a complex structure J induced by π, with Riemannian tensor

$$\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y$$
$$+g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ,$$

for any $X, Y, Z \in TM$.
$\mathbb{C}P^n_p$ has constant holomorphic sectional curvature 4.

Let M be a connected, non-degenerate, immersed real hypersurface in $\mathbb{C}P^n_p$.
N : a local unit normal vector field such that $\varepsilon = g(N,N) = \pm 1$.
$\xi = -JN$: The *structure* vector field on M. Clearly, $g(\xi,\xi) = \varepsilon$.
Given $X \in TM$, the vector JX might not be tangent to M. Then, we decompose it in its tangent and normal parts, namely

$$JX = \phi X + \varepsilon\, \eta(X) N,$$

where ϕX is the tangential part, and η is the 1-form on M. Given $X, Y \in TM$,

$$\eta(X) = g(X,\xi), \quad \phi\xi = 0, \quad \eta(\xi) = \varepsilon,$$
$$\phi^2 X = -X + \varepsilon\eta(X)\xi, \quad \eta(\phi X) = 0,$$
$$g(\phi X, \phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \quad g(\phi X, Y) + g(X, \phi Y) = 0.$$

Thus, the set (g, ϕ, η, ξ) is called an almost contact structure on M.

Next, if ∇ is the Levi-Civita connection of M, we have the Gauss and Weingarten formulae:

$$\bar{\nabla}_X Y = \nabla_X Y + \varepsilon g(AX, Y)N, \quad \bar{\nabla}_X N = -AX,$$

for any $X, Y \in TM$, where A is the shape operator associated with N. Note that

$$\nabla_X \xi = \phi AX.$$

The Codazzi equation is

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X + 2g(X, \phi Y)\xi,$$

for any $X, Y \in TM$. Let R be the curvature operator of M. Then, by using the Gauss equation, we obtain

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z + \varepsilon g(AY, Z)AX - \varepsilon g(AX, Z)AY, \end{aligned}$$

for any $X, Y, Z \in TM$.

Definition 4

Let M be a real hypersurface in $\mathbb{C}P_p^n$. We will say that M is *Hopf* when its structure vector field ξ is everywhere principal, i. e., it is an eigenvector of A.

Its associated principal curvature can be defined as $\mu = \varepsilon g(A\xi, \xi)$, and we will call it the *Hopf curvature*. Therefore, it holds $A\xi = \mu\xi$.

Theorem 5

[1] Let M be a non-degenerate Hopf real hypersurface in $\mathbb{C}P_p^n$ with $A\xi = \mu\xi$. Then, μ is (locally) constant.

(Anciaux-Panagiotidou)

Next lemma is essentially included in [1].

Lemma 6

Let M be a non-degenerate Hopf real hypersurface in $\mathbb{C}P_p^n$ with $A\xi = \mu\xi$. Assume that $X \in TM$ is a principal vector with associated principal curvature λ. Then,

$$(2\lambda - \mu)A\phi X = (\lambda\mu + 2\varepsilon)\phi X.$$

If $2\lambda - \mu \neq 0$, then $A\phi X = \dfrac{\lambda\mu + 2\varepsilon}{2\lambda - \mu}\phi X$, $X \in TM$.

Corollary 7

If $\mu = 2\lambda$, then $\varepsilon = -1$, $|\mu| = 2$ and $|\lambda| = 1$.

Summary

Since $\pi : \mathbb{S}^{2n+1}_{2p} \to \mathbb{C}P^n_p$ is a semi-Riemannian submersion and a *principal fiber bundle with structure Lie group* $\mathbb{S}^1$, we can call it the *Hopf map*.

Given a real hypersurface M^{2n-1} in $\mathbb{C}P^n_p$, then we construct its lift $\tilde{M}^{2n}$, i.e., the following commutative diagram:

$$\begin{array}{ccc} \tilde{M}^{2n} & \longrightarrow & \mathbb{S}^{2n+1}_{2p} \\ \downarrow & & \downarrow \\ M^{2n-1} & \longrightarrow & \mathbb{C}P^n_p \end{array}$$

It is important to point out that a real hypersurface in $\mathbb{C}P^n_p$ is a semi-Riemannian submanifold of arbitrary index, and therefore, its shape operator A might not be diagonalisable

Given $0 \leq q \leq p \leq m \leq n+2$, $m > q+1$, we define the following maps $\mathfrak{q}_1, \mathfrak{q}_2 : \mathbb{C}_p^{n+1} \to \mathbb{C}_p^{n+1}$. Given $z \in \mathbb{C}_p^{n+1}$, the case $q = 0$ and $m = n+2$ is not considered, and

- if $1 \leq q$ and $m \leq n+1$, $\mathfrak{q}_1(z) = (z_1, \dots, z_q, 0, \dots, 0, z_m, \dots, z_{n+1})$, $\mathfrak{q}_2(z) = (0, \dots, 0, z_{q+1}, \dots, z_{m-1}, 0, \dots, 0)$;
- if $q = 0$ and $m \leq n+1$, $\mathfrak{q}_1(z) = (0, \dots, 0, z_m, \dots, z_{n+1})$, $\mathfrak{q}_2(z) = (z_1, \dots, z_{m-1}, 0, \dots, 0)$;
- if $1 \leq q$ and $m = n+2$, $\mathfrak{q}_1(z) = (z_1, \dots, z_q, 0, \dots, 0)$, $\mathfrak{q}_2(z) = (0, \dots, 0, z_{q+1}, \dots, z_{n+1})$.

Example 8

Type A. Consider $t \in \mathbb{R}$, $t \neq 0, 1$, and $0 \leq q \leq p \leq m \leq n+2$, $m > q+1$. With this notation, we define

$$\begin{aligned}\tilde{\mathbf{M}}_q^m(t) &= \left\{ z = (z_1, \ldots, z_n) \in \mathbb{S}_{2p}^{2n+1} \ : \ g(\mathfrak{q}_1(z), \mathfrak{q}_1(z)) = t \right\} \\ &= \left\{ z = (z_1, \ldots, z_n) \in \mathbb{S}_{2p}^{2n+1} \ : \ g(\mathfrak{q}_2(z), \mathfrak{q}_2(z)) = 1 - t \right\}, \\ \mathbf{M}_q^m(t) &= \pi(\tilde{\mathbf{M}}_q^m(t)) \subset \mathbb{C}P_p^n.\end{aligned}$$

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$$A\xi = \mu\xi.$$

For a suitable $r > 0$,

(A_+) $\varepsilon = +1$, $0 < t = \cos^2(r) < 1$, $\mu = 2\cot(2r)$,
$\lambda_1 = -\tan(r)$, $\lambda_2 = \cot(r)$.

(A_-) $\varepsilon = -1$, $1 < t = \cosh^2(r)$, $\mu = 2\coth(2r)$,
$\lambda_1 = -\tanh(r)$, $\lambda_2 = \coth(r)$.

$$\dim V_{\lambda_1} = 2(m-q-2), \quad \dim V_{\lambda_2} = 2(n+q-m+1).$$

Type B

Example 9

Given $t > 0$, $t \neq 1$, $Q(z) = -\sum_{j=1}^{p} z_j^2 + \sum_{j=p+1}^{n+1} z_j^2$,

$$\tilde{\mathbf{M}}_t = \left\{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{S}_{2p}^{2n+1} \ : \ Q(z)\overline{Q(z)} = t \right\}, \ \mathbf{M}_t = \pi(\tilde{\mathbf{M}}_t).$$

$$\varepsilon = \mathrm{sign}(t(1-t)) = \pm 1, \quad A\xi = \mu\xi, \quad g(\xi, \xi) = \varepsilon.$$

(B_+) $\varepsilon = +1$, $0 < t = \sin^2(2r) < 1$, $\mu = 2\cot(2r)$, $\lambda_1 = \cot(r)$, $m_1 = n-1$, $\lambda_2 = \tan(r)$, $m_2 = n-1$, $\phi V_{\lambda_1} = V_{\lambda_2}$.

(B_0) $\varepsilon = -1$, $\mu = \sqrt{3}$, $\lambda = 1/\sqrt{3}$, $\dim V_\mu = n$, $\dim V_\lambda = n-1$, $\phi V_\mu = V_\lambda$, $\xi \in V_\mu$.

(B_-) $\varepsilon = -1$, $1 < t = \cosh^2(2r)$, $\mu = 2\tanh(2r)$, $\lambda_1 = \coth(r)$, $m_1 = n-1$, $\lambda_2 = \tanh(r)$, $m_2 = n-1$, $\phi V_{\lambda_1} = V_{\lambda_2}$.

Example 10

A degenerate example. Recall $Q(z) = -\sum_{j=1}^{p} z_j^2 + \sum_{j=p+1}^{n+1} z_j^2$.

$$\tilde{\mathbf{M}}_1 = \left\{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{S}_{2p}^{2n+1} \ : \ Q(z)\overline{Q(z)} = 1, \ z \neq Q(z)\bar{z} \right\}.$$

$\mathbf{M}_1 = \pi(\tilde{\mathbf{M}}_1)$ is a real hypersurface in $\mathbb{C}P_p^n$ such that:

1. The normal vector N is lightlike, so that $N \in T\mathbf{M}_1$.

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2. The induced metric g is degenerate, with $\{N, \xi\}$ spanning its radical.
3. If $AX = -\bar{\nabla}_X N$, for any $X \in TM$, then M is Hopf: $A\xi = 0$.
4. The shape operator is not diagonalisable: $A\xi = 0$, $\dim V_0 = n - 1$, $AN = \mathrm{Re}(Q(z) - 1)N - \mathrm{Im}(Q(z))\xi$, and there is another eigenvalue $\lambda_2 = 2$, $\dim V_2 = n - 1$. In addition, $\phi V_0 \subset V_2$.

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5. It is the tube of radius $s = \pi/4$ over a totally complex submanifold.

This example does not contradict Lemma 6, since ξ is lightlike.

Example 11

Type C, the Horosphere: Given $t > 0$,

$$\tilde{\mathbf{H}}(t) = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{S}^{2n+1}_{2p} : (z_1 - z_{n+1})(\bar{z}_1 - \bar{z}_{n+1}) = t \right\},$$

$$\mathbf{H}(t) = \pi(\tilde{\mathbf{H}}(t)).$$

There exists a global normal vector field along $\mathbf{H}(t)$, say N, which is a unit, time-like. The index of $\tilde{\mathbf{H}}(t)$ and $\mathbf{H}(t)$ are $2p-1$.

$$A\xi = 2\xi, \quad AX = X, \quad \forall X \in T\mathbf{H}(t),\ X \perp \xi.$$

Summary

A Ridigity Result

Theorem 12

Let $f_i : M_q^{2n-1} \to \mathbb{C}P_p^n$, $i = 1, 2$ *two isometric immersions of the same connected manifold in* $\mathbb{C}P_p^n$, *with Weingarten endomorphisms* A_1 *and* A_2. *If for each point* $p \in M$, $A_1(p) = A_2(p)$, *there exists an isometry* $\Phi : \mathbb{C}P_p^n \to \mathbb{C}P_p^n$ *such that* $f_2 = \Phi \circ f_1$.

A Ridigity Result

Proof.

$$\begin{array}{ccc} \tilde{M}^{2n} & \xrightarrow{\tilde{f}_i} & \mathbb{S}^{2n+1}_{2p} \\ \downarrow & & \downarrow \pi \\ M^{2n-1} & \xrightarrow{f_i} & \mathbb{C}P^n_p \end{array}$$

$$\tilde{A}, \quad A.$$

$\mathbb{S}^{2n+1}_{2p}$ is a space of constant curvature. By a similar way as in Riemannian Space Forms, there exist an isometry $\hat{\Phi}$ of $\mathbb{S}^{2n+1}_{2p}$ such that $\hat{\Phi} \circ \tilde{f}_1 = \tilde{f}_2$. $\hat{\Phi}$ can be chosen to be the restriction of an isometry of $\mathbb{C}^{n+1}_p$.
We can project and obtain our result. □

Summary

Further results

Definition 13

Let M be a real hypersurface in $\mathbb{C}P_p^n$, $n \geq 2$. We say that M is η-umbilical if its Weingarten endomorphism is of the form $AX = \lambda X + \rho\eta(X)\xi$ for any $X \in TM$, for some functions $\lambda, \rho \in C^\infty(M)$.

Theorem 14

Let M be a connected, non-degenerate, oriented real hypersurface in $\mathbb{C}P^n_p$, $n \geq 2$, such that it is η-umbilical. Then, M is locally congruent to one of the following real hypersurfaces:

1. *A real hypersurface of type A_+, with $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\cot(2r)$ and $\lambda = \cot(r)$, $r \in (0, \pi/2)$;*
2. *A real hypersurface of type A_+, with $m = n + q + 1$, $0 \leq q \leq 1$, $\mu = 2\cot(2r)$ and $\lambda = -\tan(r)$, $r \in (0, \pi/2)$;*
3. *A real hypersurface of type A_-, with $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\coth(2r)$, $r > 0$ and $\lambda = \coth(r)$;*
4. *A real hypersurface of type A_-, with $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\coth(2r)$, $r > 0$ and $\lambda = \tanh(r)$;*
5. *A horosphere.*

Corollary 15

Let M be a non-degenerate real hypersurface in $\mathbb{C}P^n_p$ such that its Weingarten endomorphism is diagonalisable. The following are equivalent:

1. ξ *is a Killing vector field;*
2. $A\phi = \phi A$;
3. M *is an open subset of one of the following:*
   - (a) *A real hypersurface of type* A_+, *with* $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\cot(2r)$ *and* $\lambda = \cot(r)$, $r \in (0, \pi/2)$;
   - (b) *A real hypersurface of type* A_+, *with* $m = n + q + 1$, $0 \leq q \leq 1$, $\mu = 2\cot(2r)$ *and* $\lambda = -\tan(r)$, $r \in (0, \pi/2)$;
   - (c) *A real hypersurface of type* A_-, *with* $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\coth(2r)$, $r > 0$ *and* $\lambda = \coth(r)$;
   - (d) *A real hypersurface of type* A_-, *with* $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\coth(2r)$, $r > 0$ *and* $\lambda = \tanh(r)$;
   - (e) *A horosphere.*

A Conjecture

We recall that J. Berndt in [4] and M. Kimura in [2] proved a very useful result, namely, that a real hypersurface in a complex space form is Hopf and has constant principal curvatures if, and only if, it is one of the examples in Montiel's list and Takagi's list, respectively.

Conjecture

Let M be a non-degenerate real hypersurface in $\mathbb{C}P_p^n$ whose shape operator is diagonalisable Then, M is Hopf and all its principal curvatures are constant if, and only, if, M is locally congruent to one of the examples A_+, A_-, B_0, B_+, B_-, or C.

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Thank you very much
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