Towards a new proof of the positive mass theorem

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The positive mass theorems:

- as GR is a **metric theory of gravity**
  - it is highly non-trivial to talk about, for instance, the mass—combining the gravitational and matter contributions—of bounded spatial regions
- **asymptotically flat spacetimes**: sensible notion of **total mass**
- **the proof of the positivity** of this global or ADM mass
  - the first attempt to prove by Geroch was quasi-local in its basic character
  - neither of the known generic proofs are so
- **the first complete proof** of the positive mass theorem by Schoen and Yau (1979 - 1981):
  - minimal surface theory and global existence of solutions to Jang's eqn.
- **Witten's proof** (1981):
  - inspired by positivity of energy in the context of supergravity, reduces the problem to proving solubility of the Dirac equation in asymptotically flat configurations

Motivations:

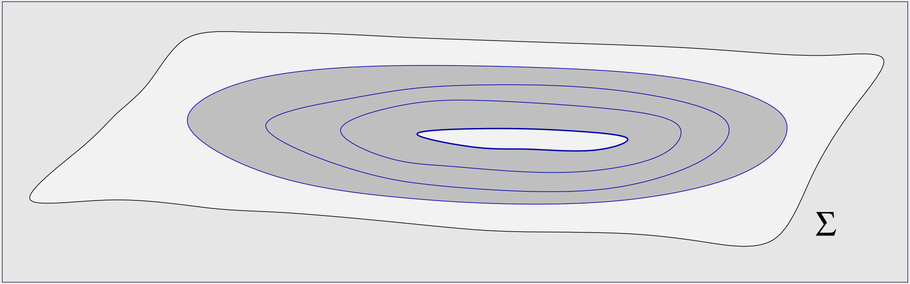
The exiting proofs:

- though they are generic the involved technicalities are considerable

The aims:

- outline **a relatively simple** alternative proof of the positive mass theorem
- try to restore its original quasi-local character
  - **demonstrating** that to a hypersurface with non-negative scalar curvature **flows can be constructed** such that the (quasi-local) **Geroch mass**—that can be evaluated on the leaves of the generated foliations—is **non-decreasing** with respect to these flows
- **the proposed procedure does not require asymptotic flatness**: applies to any subregion admitting non-negative scalar curvature (and suitable quasi-convex foliations)
- **the ultimate aim** is to show—in case of an asymptotically flat time-slices with non-negative scalar curvature—that the **desired flows exist globally**

Foliations by topological two-spheres:



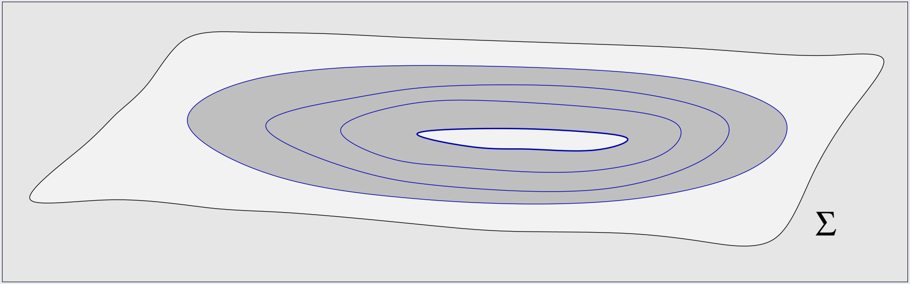
- consider a smooth 3-dimensional manifold Σ with a Riemannian metric h_{ij}
- assume

$$\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}}$$

i.e. Σ is smoothly foliated by a one-parameter family of two-surfaces $\mathscr{S}_\rho$: $\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

- $$\Longrightarrow \quad \boxed{\partial_i \rho \ \ \& \ \ h^{ij} \ \longrightarrow \ \widehat{n}_i \, , \widehat{n}^i = h^{ij}\widehat{n}_j \ \ \ldots \ \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i\widehat{n}_j}$$

Quasi-convex foliations:



- **the induced Riemannian metric** on the $\mathscr{S}_\rho$ level sets

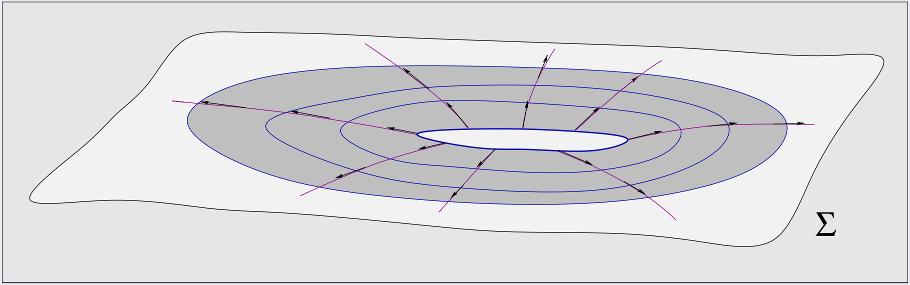
$$\boxed{\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j h_{kl}}$$

- **the extrinsic curvature** given by the symmetric tensor field

$$\boxed{\widehat{K}_{ij} = \widehat{\gamma}^l{}_i D_l \, \widehat{n}_j = \tfrac{1}{2} \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij} ,} \qquad D_i, \mathscr{L}_{\widehat{n}}$$

- a $\rho = const$ level surface is called to be **quasi-convex if its mean curvature**, $\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij} \widehat{K}_{ij}}$, is positive on $\mathscr{S}_\rho$

Flows:



- a smooth vector field ρ^i on Σ **is a flow**, w.r.t. $\mathscr{S}_\rho$
  - if the integral curves of ρ^i **intersect each leaves precisely once**, and
  - if ρ^i **is scaled such that** $\boxed{\rho^i \partial_i \rho = 1}$ holds throughout Σ
- any smooth flow can be decomposed in terms of its '**lapse**' and '**shift**' as

$$\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i} \qquad \boxed{\widehat{N} = \rho^i \widehat{n}_i\,, \quad \widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j}$$

  - the lapse **measures the normal separation of the surfaces** $\mathscr{S}_\rho$

The variation of the area:

- to any quasi-convex foliation $\exists$ a (quasi-local) **orientation of the leaves** $\mathscr{S}_\rho$
- a flow ρ^i is called **outward pointing** if the area is increasing w.r.t. it
- variation of the area $\boxed{\mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \widehat{\epsilon}}$ of the $\rho = const$ level surfaces, w.r.t. ρ^i

$$\boxed{\mathscr{L}_\rho \mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \mathscr{L}_\rho \,\widehat{\boldsymbol{\epsilon}} = \int_{\mathscr{S}_\rho} \left\{ \widehat{N}(\widehat{K}^l{}_l) + (\widehat{D}_i \widehat{N}^i) \right\} \widehat{\boldsymbol{\epsilon}} = \int_{\mathscr{S}_\rho} \widehat{N}(\widehat{K}^l{}_l)\ \widehat{\boldsymbol{\epsilon}},}$$

the relations $\mathscr{L}_{\widehat{n}}\,\widehat{\boldsymbol{\epsilon}} = (\widehat{K}^l{}_l)\,\widehat{\boldsymbol{\epsilon}}$ and $\mathscr{L}_{\widehat{N}}\,\widehat{\boldsymbol{\epsilon}} = \frac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_{\widehat{N}}\widehat{\gamma}_{ij}\,\widehat{\boldsymbol{\epsilon}} = (\widehat{D}_i\widehat{N}^i)\,\widehat{\boldsymbol{\epsilon}}$, along with the vanishing of the integral of the total divergence $\widehat{D}_i\widehat{N}^i$, were applied.

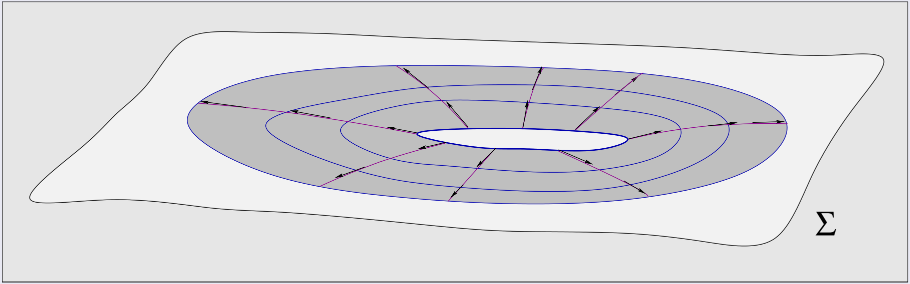
- $\widehat{N}$ does not vanish on Σ unless the Riemannian three-metric

$$\boxed{h^{ij} = \widehat{\gamma}^{ij} + \widehat{N}^{-2}(\rho^i - \widehat{N}^i)(\rho^j - \widehat{N}^j)}$$

gets to be singular

- for **quasi-convex foliations** $\boxed{\widehat{N}\widehat{K}^l{}_l > 0} \Longrightarrow$ the **area is increasing** w.r.t. ρ^i
- the orientations by $\widehat{n}^i$ and ρ^i coincide

Attempts to provide a quasi-local proof of the PMT:



- attempts all using the Geroch or the Hawking mass (are equal if $K^i{}_i = 0$)
  - Geroch (1973)
  - Wald & Jang (1977)
  - Jang (1978)
  - Kijowski (1986)
  - Jezieski & Kijowski (1987)
  - Huisken & Ilmanen (1997, 2001)
  - Frauendiener (2001)
  - Bray (2001), Bray & Lee (2009)

  ...

The Geroch mass:

- the (quasi-local) Geroch mass

$$m_{\mathcal{G}} = \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}} \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}$$

  where $\widehat{R}$ is the scalar curvature of the metric $\widehat{\gamma}_{ij}$ on the leaves
- for quasi-convex foliations the area $\mathscr{A}_\rho$ is monotonously increasing
- it suffices to investigate

$$W(\rho) = \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}$$

- **if $W(\rho)$ was non-decreasing**, and for some specific ρ_* value, $W(\rho_*)$ was zero or positive then $\boxed{m_{\mathcal{G}} \geq 0}$ would hold to the exterior of $\mathscr{S}_{\rho_*}$ in Σ

The variation of $W(\rho)$:

- the **key equation** we shall use **relates the scalar curvatures** of h_{ij} and $\widehat{\gamma}_{ij}$

$$ {}^{(3)}R = \widehat{R} - \Big\{ 2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N} \Big\} \qquad (*) $$

$$ \begin{aligned} \mathscr{L}_\rho W &= -\int_{\mathscr{S}_\rho} \mathscr{L}_\rho\Big[\,(\widehat{K}^l{}_l)^2\,\widehat{\boldsymbol\epsilon}\,\Big] = -\int_{\mathscr{S}_\rho} \Big\{\widehat{N}\,\mathscr{L}_{\widehat{n}}\Big[\,(\widehat{K}^l{}_l)^2\,\widehat{\boldsymbol\epsilon}\,\Big] + \mathscr{L}_{\widehat{N}}\Big[\,(\widehat{K}^l{}_l)^2\,\widehat{\boldsymbol\epsilon}\,\Big]\Big\} \\ &= -\int_{\mathscr{S}_\rho} (\widehat{N}\widehat{K}^l{}_l)\,\Big[2\,\mathscr{L}_{\widehat{n}}\,(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2\Big]\,\widehat{\boldsymbol\epsilon} - \int_{\mathscr{S}_\rho} \widehat{D}_i\big[(\widehat{K}^l{}_l)^2\,\widehat{N}^i\,\big]\widehat{\boldsymbol\epsilon} \\ &= -\int_{\mathscr{S}_\rho} (\widehat{N}\widehat{K}^l{}_l)\,\Big[\,(\,\widehat{R} - {}^{(3)}R\,) - \widehat{K}_{kl}\widehat{K}^{kl} - 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}\,\Big]\,\widehat{\boldsymbol\epsilon} \end{aligned} $$

- where on 1^{st} line $\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i$ and the Gauss-Bonnet theorem
- on 2^{nd} line the relations $\mathscr{L}_{\widehat{n}}\,\widehat{\boldsymbol\epsilon} = (\widehat{K}^l{}_l)\,\widehat{\boldsymbol\epsilon}$ and $\mathscr{L}_{\widehat{N}}\,\widehat{\boldsymbol\epsilon} = (\widehat{D}_i\widehat{N}^i)\,\widehat{\boldsymbol\epsilon}$
- on 3^{rd} line (*) and the vanishing of the integral of $\widehat{D}_i\big[(\widehat{K}^l{}_l)^2\,\widehat{N}^i\,\big]$ were used

The variation of $W(\rho)$:

- by the Leibniz rule

$$\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N} = \widehat{D}^l\big(\widehat{N}^{-1}\widehat{D}_l\widehat{N}\big) + \widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})$$

- and by using the trace-free part of $\widehat{K}_{ij}$

$$\mathring{\widehat{K}}_{ij} = \widehat{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\,(\widehat{K}^l{}_l), \qquad \widehat{K}_{kl}\widehat{K}^{kl} = \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + \tfrac{1}{2}\,(\widehat{K}^l{}_l)^2$$

- and using the vanishing of the integral of the total divergence $\widehat{D}^l(\widehat{N}^{-1}\widehat{D}_l\widehat{N})$

$$\begin{aligned}\mathscr{L}_\rho W = & -\tfrac{1}{2}\int_{\mathscr{S}_\rho}(\widehat{N}\widehat{K}^l{}_l)\left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2\right]\widehat{\boldsymbol{\epsilon}} \\ & + \int_{\mathscr{S}_\rho}(\widehat{N}\,\widehat{K}^l{}_l)\left[{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\right]\widehat{\boldsymbol{\epsilon}}\end{aligned}$$

A desired type flow:

- **once a foliation is fixed**, by specifying the function $\rho : \Sigma \to \mathbb{R}$, **not only the mean curvature**, $\widehat{K}^l{}_l$, **but the lapse** $\widehat{N}$, as well, **gets to be fixed**

$$\boxed{\widehat{n}_i = \widehat{N}\,(\partial_i\rho)}$$

- the **inverse mean curvature flow**

$$\boxed{\rho^i = \big(\widehat{K}^l{}_l\big)^{-1}\,\widehat{n}^i}$$

  if this flow existed globally the Geroch mass would be non-decreasing w.r.t it

- but **what is if only** the product $\boxed{\widehat{N}\widehat{K}^l{}_l}$ **is replaced** by its **mean value**

-
$$\boxed{\overline{\widehat{N}\widehat{K}^l{}_l} = \frac{\int_{\mathscr{S}_\rho}\widehat{N}\widehat{K}^l{}_l\,\widehat{\boldsymbol\epsilon}}{\int_{\mathscr{S}_\rho}\widehat{\boldsymbol\epsilon}}} \qquad \boxed{\overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$$

$$\boxed{[(64\,\pi^{3/2})/(\mathscr{A}_\rho)^{1/2}]\cdot\mathscr{L}_\rho\, m_{\mathcal{G}} = \mathscr{L}_\rho W + \tfrac{1}{2}\,(\mathscr{L}_\rho\log[\mathscr{A}_\rho])\,W \geq 0}$$

We can also adjust the shift

- $\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i}$: we have a freedom in choosing the shift $\widehat{N}^i$

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \tfrac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_i\widehat{N}^i}$$

- or equivalently, once $\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ is guaranteed to hold

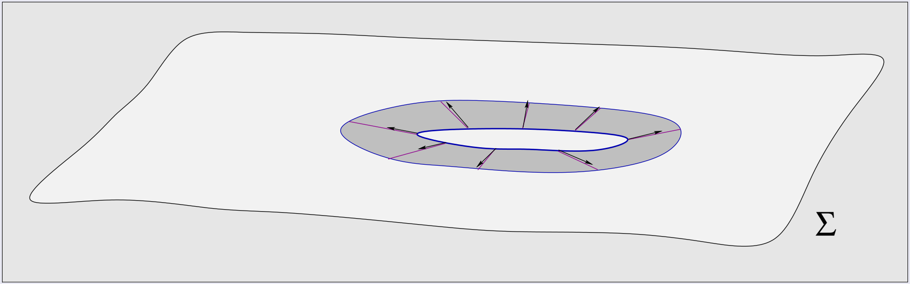
$$\boxed{\widehat{D}_A\widehat{N}^A = \mathscr{L}_\rho\sqrt{\det(\widehat{\gamma}_{ij})} - \mathscr{L}_\rho\log[\mathscr{A}_\rho] \qquad (**)}$$

- on topological two-spheres using then the Hodge decomposition of the shift $\boxed{\widehat{N}^A = \widehat{D}^A\,\chi + \widehat{\epsilon}^{AB}\,\widehat{D}_B\,\eta}$, χ and η are some smooth functions on $\mathscr{S}$, (**)

$$\boxed{\widehat{D}^A\widehat{D}_A\chi = \mathscr{L}_\rho\sqrt{\det(\widehat{\gamma}_{ij})} - \mathscr{L}_\rho\log[\mathscr{A}_\rho]}$$

- solubility in terms of spherical harmonics presumes that some standard polar coordinates (ϑ,φ) given on the unit sphere $\mathbb{S}^2$ are transfered to $\mathscr{S}$

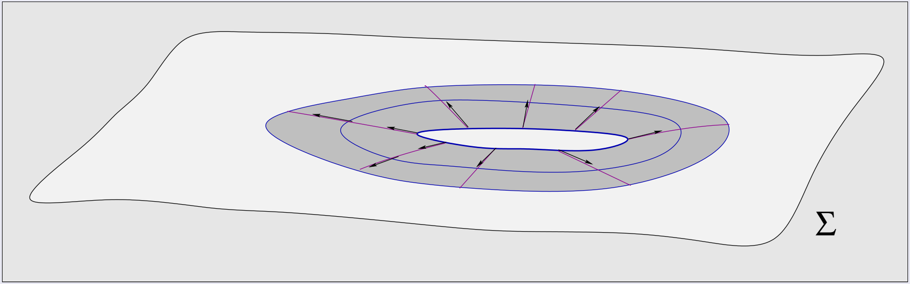
The construction of a flow:



the desired flow on the given Riemannian three-surface Σ, with the metric h_{ij}

1. start by choosing a topological two-sphere $\mathscr{S}$ in Σ, with induced metric $\widehat{\gamma}_{ij}$, such that it is quasi-convex, $\widehat{K}^l{}_l > 0$, and also $W \geq 0$ holds on $\mathscr{S}$
2. choose a small positive real number $A > 0$ and set $\widehat{N} = A \cdot (\widehat{K}^l{}_l)^{-1}$ on $\mathscr{S}$
3. construct an infinitesimally close two-surface $\mathscr{S}'$ simply by Lie dragging the points of $\mathscr{S}$ along the (auxiliary) flow $\rho^i = \widehat{N}\,\widehat{n}^i$ in Σ
4. by comparing the metric induced on $\mathscr{S}$ and $\mathscr{S}'$, respectively, both terms $\mathscr{L}_\rho \sqrt{\det(\widehat{\gamma}_{ij})}$ and $\mathscr{L}_\rho \log[\mathscr{A}_\rho]$ can be evaluated on $\mathscr{S}'$

The construction of a flow:



in the succeeding steps we have to update both the lapse and the shift such that the relation $\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l}}$ gets to be maintained in each of these steps

5. update first lapse on $\mathscr{S}'$ by setting $\boxed{\widehat{N} = \mathscr{L}_\rho \log[\mathscr{A}_\rho] \cdot (\widehat{K}^l{}_l)^{-1}}$, where $\boxed{\mathscr{L}_\rho \log[\mathscr{A}_\rho] > 0}$ is determined in the previous infinitesimal step
6. the key point here is that the shift can also be updated on $\mathscr{S}'$—such that $\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l}$ holds there—simply by solving (**) for $\widehat{N}^A$
7. the succeeding infinitesimal step: by Lie dragging the points of $\mathscr{S}'$ to $\mathscr{S}''$ ...

Limits and global existence:

- by performing analogous sequences of infinitesimal steps ultimately we get a one-parameter family of two-surfaces $\mathscr{S}_\rho$ foliating (at least) a one-sided neighborhood of $\mathscr{S}$ in Σ such that the product $\widehat{N}\widehat{K}^l{}_l$ is guaranteed to be positive and constant on each of the individual leaves
- the vanishing of $\mathscr{L}_\rho \log[\mathscr{A}_\rho]$ could get on the way of the applicability, i.e. minimal or maximal surfaces represent natural limits of applicability
  - the bifurcation surface of the Schwarzschild spacetime is a minimal surface on the $t_{Schw} = const$ time-slices, the Kerr-Schild $t_{KS} = const$ time-slices of the same spacetime can be foliated by metric spheres with area radius ranging from zero to infinity, and they do not contain any minimal surface
- it is also of obvious interest to know if the desired type of foliation would exist or could be constructed globally
- by inspecting the proposed construction it gets clear that all the steps are "safe" as far as the lapse $\widehat{N}$ is bounded and it is regular throughout Σ
- in clearing up the picture let us have a glance again of the key equation

$$ {}^{(3)}R = \widehat{R} - \left\{ 2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N} \right\} \qquad (*) $$

The parabolic equation governing the evolution of $\widehat{N}$:

- as noticed first by Bartnik (1993), while applying quasi-spherical foliations, (*) can be viewed as a parabolic equation for $\widehat{N}$
- remarkably, (*) **can always be put to be a parabolic equation** for the lapse provided that ${}^{(3)}R \geq 0$, $\widehat{\gamma}_{ij}$ and $\widehat{N}^i$ can be treated as prescribed fields
- with applying the notation $\boxed{\overset{\star}{K}_{ij} = \widehat{N}\widehat{K}_{ij}}$ and $\boxed{\overset{\star}{K} = \frac{1}{2}\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_i\widehat{N}^i}$ we can eliminate hidden presence of the lapse in (*) and get

$$\boxed{\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - \widehat{N}^l(\hat{D}_l\widehat{N})\,] = \widehat{N}^2(\widehat{D}^l\widehat{N}_l\widehat{N}) + \mathcal{A}\,\widehat{N} - \tfrac{1}{2}\,(\,\widehat{R} - {}^{(3)}R\,)\,\widehat{N}^3}$$

where $\boxed{\mathcal{A} = \partial_\rho\overset{\star}{K} + \frac{1}{2}\,[\overset{\star}{K}{}^2 + \overset{\star}{K}_{kl}\overset{\star}{K}{}^{kl}]}$, with $\boxed{\overset{\star}{K} = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho] > 0}$

- it is standard to obtain **existence of unique solutions to this uniformly parabolic PDE** in a sufficiently small one-sided neighborhood of $\mathscr{S}$ in Σ

The global existence of solutions to this parabolic equation:

- our main concern is **global existence (!)**
- it should not be a surprise that **an analogous parabolic equation** came up **in deriving the evolutionary form** of the Hamiltonian constraints in [Rácz I: *Constrains as evolutionary systems*, Class. Quant. Grav. **33** 015014 (2016)]
- (slightly generalizing Bartnik's results) **global existence of solutions** to the parabolic PDEs $\boxed{\overset{\star}{K}\,[\,(\partial_\rho \widehat{N}) - \widehat{N}^l(\hat{D}_l \widehat{N})\,] = \widehat{N}^2(\widehat{D}^l \widehat{N}_l \widehat{N}) + \mathcal{A}\,\widehat{N} - \tfrac{1}{2}\,(\,\hat{R} - {}^{(3)}\!R\,)\,\widehat{N}^3}$ could be derived
- assume now that ρ is the **area radius** such that $\boxed{\mathscr{A}_\rho = 4\pi\,\rho^2}$
- the condition guaranteeing that **for some positive and bounded initial data** for ${}_0\widehat{N}$ on $\mathscr{S}$ the solution $\widehat{N}$ **remains positive and bounded away from infinity for all** $\rho \geq \rho_0$ **ultimately** can be given by referring to

$$\boxed{\mathcal{K} = \sup_{\rho\in[\rho_0,\infty)} \left\{ \frac{1}{4\sqrt{\rho_0}} \int_{\rho_0}^{\rho} \rho'^{\,3/2} \cdot \left[\max_{\mathscr{S}_{\rho'}} \left({}^{(3)}\!R - \widehat{R} \right) \right] \mathrm{d}\rho' \right\}}$$

1. if $\mathcal{K} \leq 0$ then any smooth positive bounded initial data ${}_0\widehat{N}$ is fine
2. if $\mathcal{K} > 0$ then ${}_0\widehat{N}$ has to be chosen such that ${}_0\widehat{N} < 1/\sqrt{\mathcal{K}}$ [but choosing $A > 0$ small (!)]

Summary:

A relatively **simple method is proposed to generate a flow** on any three-dimensional Riemannian hypersurface, **with non-negative scalar curvature** in a four-dimensional ambient space.

1. it is **far more flexible** than the inverse mean curvature flow
2. this flow can be used **to construct** quasi-convex foliations
3. the (quasi-local) **Geroch mass**—associated with the foliating level surfaces—is **non-decreasing** w.r.t the proposed flow
4. **hints on the global existence and regularity** were provided
5. the construction **applies to wide range of geometrized theories of gravity**
    - no use of Einstein's equations or any other field equation on the metric of the ambient space had been applied anywhere in our construction
    - as only the Riemannian character of the metric on Σ was used the signature of the metric on the ambient space could be either Lor. or Euc.