Lorentzian length spaces

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Introduction

Why a *synthetic* approach to Lorentzian geometry?

- need for low regularity (of the metric): PDE point-of-view, physically relevant models (matched spacetimes, shock waves, impulsive gravitational waves, etc.)
- separate main concepts and derived notions of the causal structure
- minimal framework for causality and (timelike/causal) curvature bounds with continuous metrics
- timelike/causal curvature bounds without a Lorentzian metric
- possible applications to Quantum Gravity

Riemannian analogue: *Length spaces* . . . metric space (X, d) with $d(x, y) = \inf\{L_d(\lambda) : \lambda$ path connecting x and $y\} \rightsquigarrow$ synthetic curvature bounds via triangle comparison

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Lorentzian pre-length spaces

X set, $\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$, d metric on X, $\tau\colon X\times X\to[0,\infty]$ lower semicontinuous (with respect to d)

Definition

$(X,d,\ll,\leq,\tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z)\geq\tau(x,y)+\tau(y,z)\qquad (x\leq y\leq z)\,,$$

and $\tau(x,y)=0$ if $x\not\leq y$ and $\tau(x,y)>0\Leftrightarrow x\ll y$;
τ is called *time separation function*

examples

- smooth spacetimes (M,g) with usual time separation function $\tau(p,q):=\sup\{L_g(\gamma):\gamma$ f.d. causal from p to $q\}$
- finite directed graphs

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Causal curves (1/2)

Definition

$I \subseteq \mathbb{R}$ interval, $\gamma\colon I \to X$ non-constant is *future directed causal (timelike)* if γ locally Lipschitz continuous (wrt. d) and for $t_1, t_2 \in I$, $t_1 < t_2$: $\gamma(t_1) \leq \gamma(t_2)$ ($\gamma(t_1) \ll \gamma(t_2)$); analogously for *null* ($\gamma(t_1) \leq \gamma(t_2)$ and $\gamma(t_1) \not\ll \gamma(t_2)$) and *past directed* curves

- Lorentz cylinder $S^1_1 \times \mathbb{R}$: every non-constant locally Lipschitz curve is timelike and causal $\rightsquigarrow$ need causality conditions
- Minkowski spacetime $\mathbb{R}^3_1$: $t \mapsto (t, \cos(t), \sin(t))$ has null tangent but is timelike

Proposition

continuous, strongly causal spacetimes: different notions of causal curves agree

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Definition

$\gamma\colon [a,b] \to X$ f.d. causal, *τ-length* defined by

$$L_\tau(\gamma) := \inf\{\sum_{i=0}^{N-1} \tau(\gamma(t_i),\gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b\}$$

Proposition

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a smooth and strongly causal spacetime (M, g), then $L_\tau(\gamma) = L_g(\gamma)$

intrinsic notion of geodesics? $\rightsquigarrow$ *maximal causal curves*

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Further structure

- causality conditions (e.g. *strong causality*: topology generated by $I^+(x) \cap I^-(y) = \{x \ll z \ll y\}$ agrees with the metric topology, etc.)
- causal connectedness ($x < y$ or $x \ll y \Rightarrow \exists$ f.d. causal/timelike curve from x to y)
- limit curve theorems
- localizability (locally the geometry and causality of a (smooth) Lorentzian manifold is better behaved than globally)

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- synthetic notion of regularity $\Rightarrow$ maximal causal curves have causal character
- L_τ upper semicontinuous

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$(X, d, \ll, \leq, \tau)$ locally causally closed, causally path connected, localizable Lorentzian pre-length space; for $x, y \in X$ define

$$\mathcal{T}(x,y) := \sup\{L_\tau(\gamma) : \gamma \text{ f.d. causal from } x \text{ to } y\}\,,$$

if the set is not empty, otherwise $\mathcal{T}(x,y) := 0$

X is a *Lorentzian length space* if $\mathcal{T} = \tau$; if, in addition X is regularly localizing, then X is a *regular* Lorentzian length space

$(M, d^h, \ll, \leq, \tau)$ the Lorentzian pre-length space induced by a smooth and strongly causal spacetime (M, g) (since $L_\tau = L_g$) is a regular LLS

causal ladder for Lorentzian length spaces $\rightsquigarrow$ sufficient conditions (analogous to the smooth spacetime case) for τ continuous and finite (needed for triangle comparison)

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Timelike triangles

Definition

timelike geodesic triangle in Lorentzian pre-length space $(X, d, \ll, \leq, \tau)$ is triple $(x, y, z) \in X^3$ with $x \ll y \ll z$, $\tau(x, z) < \infty$ and s.t. sides are realized by f.d. causal curves

i.e., $\exists$ f.d. causal curves α, β, γ s.t. $L_\tau(\alpha) = \tau(x, y)$, $L_\tau(\beta) = \tau(y, z)$ and $L_\tau(\gamma) = \tau(x, z)$

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Timelike curvature bounds

Definition

Lorentzian pre-length space X has *timelike curvature bounded below (above)* by $K \in \mathbb{R}$ if all points in X have nhd. U s.t.:

1. $\tau|_{U \times U}$ finite and continuous
2. $x, y \in U$ with $x \ll y \Rightarrow \exists$ f.d. maximal causal curve in U from x to y
3. (x, y, z) small timelike geodesic triangle in U, $(\bar{x}, \bar{y}, \bar{z})$ comparison triangle of (x, y, z) in M_K, then for p, q points on the sides of (x, y, z) and $\bar{p}, \bar{q}$ corresponding points $(\bar{x}, \bar{y}, \bar{z})$:

$$\tau(p, q) \leq \bar{\tau}(\bar{p}, \bar{q}) \quad \text{(respectively } \tau(p, q) \geq \bar{\tau}(\bar{p}, \bar{q}))$$

Alexander, Bishop: smooth Lorentzian manifold with sectional curvature bounds

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Branching of maximal curves

Alexandrov spaces with curvature bounded below: geodesics do not branch

Definition

X Lorentzian pre-length space, $\gamma\colon [a,b] \to X$ maximal curve; $x := \gamma(t)$, $t \in (a,b)$ is *branching point* of γ if $\exists$ maximal curves $\alpha, \beta\colon [a,c] \to X$ with $c > b$ and $\alpha|_{[a,t]} = \beta|_{[a,t]} = \gamma|_{[a,t]}$, $\alpha([t,c]) \cap \beta([t,c]) = \{x\}$

Theorem

X strongly causal Lorentzian length space with timelike curvature bounded below by some $K \in \mathbb{R}$ s.t. X regular and locally compact or timelike locally uniquely geodesic, then maximal timelike curves do not have timelike branching points

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Alexandrov spaces with curvature bounded below: geodesics do not branch

Definition

X Lorentzian pre-length space, $\gamma\colon [a,b] \to X$ maximal curve; $x := \gamma(t)$, $t \in (a,b)$ is *branching point* of γ if $\exists$ maximal curves $\alpha, \beta\colon [a,c] \to X$ with $c > b$ and $\alpha|_{[a,t]} = \beta|_{[a,t]} = \gamma|_{[a,t]}$, $\alpha([t,c]) \cap \beta([t,c]) = \{x\}$

Theorem

X strongly causal Lorentzian length space with timelike curvature bounded below by some $K \in \mathbb{R}$ s.t. X regular and locally compact or timelike locally uniquely geodesic, then maximal timelike curves do not have timelike branching points

Causal curvature bounds

- *causal geodesic triangles*: $x, y, z \in X$ s.t. $x \ll y \leq z$ or $x \leq y \ll z$, $\rightsquigarrow$ one side possibly zero length (or collapsed)
- causal curvature bounds analogously to timelike curvature bounds except that one can only compare distances to the timelike sides
- length increasing push-up for smooth spacetimes via the Gauss Lemma; here new perspective

Proposition

X strongly causal Lorentzian pre-length space with causal curvature bounded above, $\gamma\colon [a,b] \to X$ f.d. causal curve with $\gamma(a) \ll \gamma(b)$ and $\exists$ sub-interval $[c,d]$ of $[a,b]$ s.t. $\gamma|_{[c,d]}$ null $\Rightarrow$ γ not maximal

Corollary

X strongly causal Lorentzian length space with causal curvature bounded above, then X is regular

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Curvature singularities

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Lorentzian pre-length space X has *timelike (respectively causal) curvature unbounded below/above* if $\forall p \in X$ $\exists$ nhd. U s.t. τ finite and continuous on U and maximal timelike/causal curves exist in U but triangle comparison fails for every $K \in \mathbb{R} \rightsquigarrow X$ has *curvature singularity*

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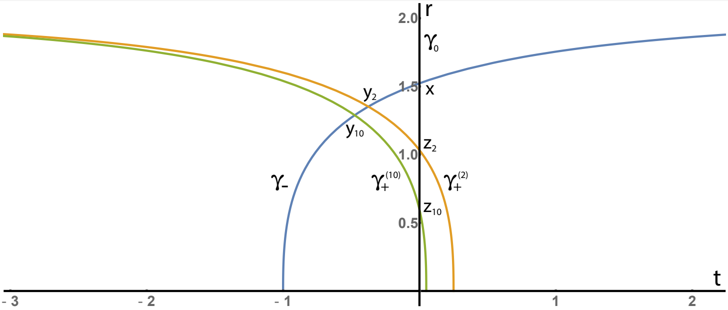

Figure : Schwarzschild has timelike curvature unbounded below

Inextendibility of spacetimes

joint work with M. Kunzinger, J.D.E. Grant, preprint arXiv:1804.10423

geodesics as locally maximizing causal curves

Theorem

X strongly causal Lorentzian length space s.t. all inextendible timelike geodesics have infinite τ-length, then X is inextendible as a regular Lorentzian length space

Corollary

(M, g) strongly causal, smooth and timelike geodesically complete spacetime, then (M, g) is inextendible as a regular Lorentzian length space

Theorem

(M, g) strongly causal, smooth and timelike geodesically complete spacetime, then every extension of (M, g) has causal curvature unbounded above, i.e., has a curvature singularity

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