Characterizations of spacelike submanifolds with constant scalar curvature in the de Sitter space

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IX International Meeting on Lorentzian Geometry

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Warsaw, Poland.

[1]Partially supported by Fundación Séneca project 19901/GERM/15, Spain.

Let $\mathbb{R}_p^{n+p+1}$ be the $(n+p+1)$-dimensional semi-Euclidean space of index p with metric tensor $\langle\, , \rangle$ given by

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - \sum_{j=n+2}^{n+p+1} v_j w_j,$$

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The de Sitter space of index p is the hyperquadric of $\mathbb{R}_p^{n+p+1}$ defined as the set of unit vectors of semi-Euclidean space

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- $\mathbb{S}_p^{n+p}$ is a complete semi-Euclidean manifold with constant sectional curvature one.

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For local semi-Riemannian orthonormal frame $\{e_1, \dots, e_{n+p}\}$ of $\mathbb{S}_p^{n+p}$ adapted to M^n we define the second fundamental form A of M^n and the square of the its norm by

$$A = \sum_{\alpha,i,j} h_{ij}^{\alpha} \omega_i \otimes \omega_j e_\alpha \quad \text{and} \quad |A|^2 = \sum_{\alpha,i,j} (h_{ij}^{\alpha})^2,$$

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the mean curvature vector h and the mean curvature function H of M^n by

$$h = \frac{1}{n} \sum_{\alpha} \left(\sum_i h_{ii}^{\alpha} \right) e_{\alpha} \quad \text{and} \quad H = |h| = \sqrt{\sum_{\alpha} \left(\sum_i h_{ii}^{\alpha} \right)^2},$$

for $n + 1 \leq \alpha \leq n + p$.

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Thus, after that choice

$$(1) \qquad H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} = \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0,\ \alpha \geq n+2,$$

where $h^{\alpha} = (h^{\alpha}_{ij})$ denotes the second fundamental form of M^n in direction e_{α} for every $n+1 \leq \alpha \leq n+p$.

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Associated by second fundamental form of M^n, we consider the traceless second fundamental form is given by

$$\Phi = A - H^{\alpha} I.$$

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From Gauss equation we get the following relation

$$(2) \qquad |\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 + n(n-1)(R-1) \geq 0,$$

with equality if and only if M^n is totally umbilical.

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Some remarks about pnmc submanifolds

- Submanifolds with nonzero parallel mean curvature vector field also have parallel normalized mean curvature vector field.
- The condition of having parallel normalized mean curvature vector field is much weaker than the condition of having parallel mean curvature vector field.
- Every hypersurface with nonzero mean curvature in a semi-Riemannian manifold always has parallel normalized mean curvature vector field.

The Main Result.

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Let M^n be a complete spacelike pnmc submanifold immersed in $\mathbb{S}_p^{n+p}$ with constant scalar curvature $0 < R \leq 1$. Then

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(i) *either* $\sup_M |\Phi| = 0$ *and M^n is a totally umbilical submanifold,*

(ii) *or*

$$\sup_M |\Phi|^2 \geq \alpha(n, p, R) > 0,$$

where $\alpha(n, p, R)$ is a positive constant depending only on n, p, R (see Remark 1).

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where $\alpha(n, p, R)$ *is a positive constant depending only on* n, p, R *(see Remark 1).*

Moreover, the equality $\sup_M |\Phi| = \alpha(n, p, R)$ *holds and this supremum is attained at some point of* M^n *if and only if* $p = 1$, $n \geq 3$ *and* M^n *is isometric to a hyperbolic cylinder*

$$\mathbb{H}^1 \times \mathbb{S}^{n-1}(\sqrt{1 + r^2})$$

of radius $r > 0$.

Geometrically, this result can be seen as Gap result for the traceless operator of M^n, close in the spirit other similar Gap results for the second fundamental form, as in the classical paper on minimal submanifolds by Simons [7] and Chern, do Carmo and Kobayashi [5].

A lower estimate for the Cheng-Yau operator

Proposition 1.

Let M^n be a spacelike pnmc submanifold in $\mathbb{S}_p^{n+p}$ with constant scalar curvature $R \leq 1$. Then

$$\frac{1}{2}L(|\Phi|^2) \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_R(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(1-R)},$$

where

$$Q_R(x) = \frac{(n-p-1)}{p}x^2 - (n-2)x\sqrt{x^2 + n(n-1)(1-R)} + n(n-1)R \tag{3}$$

and L is the Cheng-Yau operator defined by

$$L(\cdot) = \mathrm{tr}(P \circ \nabla^2(\cdot)), \tag{4}$$

with

$$P = nHI - h^{n+1}. \tag{5}$$

Lemma 2 (Camargo [4]).

Let M^n be a spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with constant scalar curvature $R \leq 1$. Then

$$|\nabla A|^2 = \sum_{\alpha,i,j,k} (h_{ijk}^{\alpha})^2 \geq n^2 |\nabla H|^2. \tag{6}$$

Moreover, if $R < 1$ and the equality holds on M^n, then H is constant on M^n.

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Lemma 3 (Santos [6]).

Let $A, B : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric linear maps such that $AB - BA = 0$ and $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$. Then

$$|\mathrm{tr}(A^2B)| \leq \frac{n-2}{\sqrt{n(n-1)}} N(A) \sqrt{N(B)},$$

where $N(B) = \mathrm{tr}(BB^t)$.

Moreover, the equality holds on the right hand side (resp. left hand side) if and only if $(n-1)$ of the eigenvalues of A and corresponding eigenvalues of B are equals.

Lemma 4.

Let M^n be a spacelike submanifold in the de Sitter space $\mathbb{S}_p^{n+p}$ with $H > 0$. Let μ_- and μ_+ be, respectively, the minimum and the maximum of the eigenvalues of the operator P at every point $p \in M^n$. If $R < 1$ (resp., $R \leq 1$ on M^n), then the operator L is elliptic (resp., semi-elliptic), with

$$\mu_- > 0 \quad (resp., \mu_- \geq 0)$$

and

$$\mu_+ < 2nH \quad (resp., \mu_+ \leq 2nH).$$

Proof of Proposition 1.

First of all, we recall the following Simons' type formula for Cheng-Yau operator (cf. [4]):

$$L(nH) = \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 - n^2|\nabla H|^2 + \sum_{\alpha,\beta} N\Big(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}\Big) + n|\Phi|^2 \tag{7}$$

$$+ \sum_{\alpha,\beta}\Big(\mathrm{tr}(h^{\alpha}h^{\beta})\Big)^2 - nH\sum_{\alpha}\mathrm{tr}\left(h^{n+1}(h^{\alpha})^2\right).$$

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Since R is constant and L is semi-elliptic, it follows from (2) that

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On the other hand, from $\Phi^\alpha = h^\alpha - H^\alpha I$,

$$-nH\sum_\alpha \mathrm{tr}\left[h^{n+1}(h^\alpha)^2\right] + \sum_{\alpha,\beta}\Big[\mathrm{tr}(h^\alpha h^\beta)\Big]^2 = -nH\sum_\alpha \mathrm{tr}\left[\Phi^{n+1}(\Phi^\alpha)^2\right] - nH^2|\Phi|^2 + \sum_{\alpha,\beta}\Big[\mathrm{tr}(\Phi^\alpha\Phi^\beta)\Big]^2 \tag{9}$$

and $N(h^\alpha h^\beta - h^\beta h^\alpha) = N(\Phi^\alpha\Phi^\beta - \Phi^\beta\Phi^\alpha) \geq 0$.

Since the normalized mean curvature vector of M^n is parallel, a straightforward computation allows us to check that Φ^{n+1} commutes with all the matrix Φ^α. In this setting, we can use Lemma 3, for $A = \Phi^\alpha$ and $B = \Phi^{n+1}$, in order to obtain

$$\sum_\alpha \left|\mathrm{tr}((\Phi^\alpha)^2\Phi^{n+1})\right| \leq \frac{n-2}{\sqrt{n(n-1)}} \sum_\alpha N(\Phi^\alpha)\sqrt{N(\Phi^{n+1})}. \tag{10}$$

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Moreover, $\sum_\alpha N(\Phi^\alpha) = |\Phi|^2$ and $N(\Phi^{n+1}) = \mathrm{tr}(\Phi^{n+1})^2 \leq |\Phi|^2$. Hence,

$$-nH\sum_\alpha |\mathrm{tr}(\Phi^{n+1}(\Phi^\alpha)^2)| \geq -\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3. \tag{11}$$

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Using Cauchy-Schwarz inequality,

$$\begin{aligned} p\sum_{\alpha,\beta}[\mathrm{tr}(\Phi^\alpha\Phi^\beta)]^2 \;&\ge\; p\sum_\alpha[\mathrm{tr}(\Phi^\alpha)^2]^2 \\ &=\; p\sum_\alpha[N(\Phi^\alpha)]^2 \ge \left(\sum_\alpha N(\Phi^\alpha)\right)^2 = |\Phi|^4. \end{aligned} \tag{12}$$

Thus, from (8) and previously inequalities, we get

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq H|\Phi|^2\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - n(H^2-1)\right). \tag{13}$$

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Besides, from (2) we have

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (1-R). \tag{14}$$

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Besides, from (2) we have

$$(14)\qquad H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (1-R).$$

Consequently, taking into account that $H > 0$, we can write

$$(15)\qquad H = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)(1-R)}.$$

Therefore, inserting (14) and (15) in (13) we obtain the lower estimate.

□

We say that *the Omori-Yau maximum principle* holds on M^n for the operator L if, for any function $u \in \mathcal{C}^2(M)$ with $u^* = \sup_M u < \infty$, there exists a sequence $\{p_k\}_{k\in\mathbb{N}} \subset M^n$ with the properties

$$u(p_k) > u^* - \frac{1}{k}, \quad |\nabla u(p_k)| < \frac{1}{k} \quad \text{and} \quad Lu(p_k) < \frac{1}{k}$$

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Our Omori-Yau maximum principle is obtained as an application of the following result, which is a particular case of:

Lemma 5 (Alías, Mastrolia and Rigoli [3]).

Let M^n be a complete, non-compact Riemannian manifold with sectional curvature bounded from below. Then the Omori-Yau maximum principle holds on M^n for any semi-elliptic operator

$$\mathcal{L} = \mathrm{tr}(P \circ \mathrm{Hess})$$

with $\sup_M \mathrm{tr}(P) < +\infty$.

Proposition 2.

Let M^n be a complete non-compact spacelike submanifold in $\mathbb{S}_p^{n+p}$ with constant scalar curvature. If

- $R \leq 1$
- $\sup_M |\Phi|^2 < +\infty$,

then the Omori-Yau maximum principle holds on M^n for the Cheng-Yau operator L.

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So, let us suppose that $0 < \sup_M |\Phi|^2 < +\infty$ and let us take $u = |\Phi|^2$.

Then, from Proposition 1 we get

$$L(u) \geq f(u), \tag{16}$$

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and $Q_R(x)$ is given by (3).

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and from (16) we get $f(u^*) \leq 0$.

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$$\frac{1}{k} > Lu(p_k) \geq f(u(p_k)). \tag{18}$$

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$$\frac{1}{k} > Lu(p_k) \geq f(u(p_k)). \tag{18}$$

Taking into (18) the limit when $k \to +\infty$, by continuity, we have

$$0 \geq f(u^*) = \frac{2}{\sqrt{n(n-1)}} u^* Q_R(\sqrt{u^*})\sqrt{u^* + n(n-1)(1-R)}.$$

Now, assume that M^n is complete and non-compact.

Since $u^* < +\infty$, Lemma 2 guarantees that there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset M^n$ satisfying

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Since $u^* > 0$ and $R \leq 1$, this implies

$$Q_R(\sqrt{u^*}) \leq 0. \tag{19}$$

Note that the hypothesis $R > 0$ guarantees us that

$$Q_R(0) = n(n-1)R > 0.$$

It is not difficult check that, this condition jointly with this inequality $Q_R(\sqrt{u^*}) \leq 0$ implies that

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Now, suppose that $R < 1$. Hence, Lemma 4 assures that the operator L is elliptic.

Therefore, if there exists a point $p_0 \in M^n$ such that $|\Phi(p_0)| = \sup_M |\Phi|$, we can apply the strong maximum principle, in order to obtain that the function $u = |\Phi|^2$ is constant and equal to x_0. Thus,

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$$0 = \frac{1}{2}L(|\Phi|^2) \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_R(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(1-R)} \geq 0. \tag{20}$$

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Thus, all inequalities obtained along the proof of Proposition 1 are, in fact, equalities. In particular, from inequality (11) we conclude that

$$\mathrm{tr}(\Phi^{n+1})^2 = |\Phi|^2.$$

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Thus, all inequalities obtained along the proof of Proposition 1 are, in fact, equalities. In particular, from inequality (11) we conclude that

$$\mathrm{tr}(\Phi^{n+1})^2 = |\Phi|^2.$$

So, from (2) we get

$$\mathrm{tr}(\Phi^{n+1})^2 = |\Phi|^2 = |A|^2 - nH^2. \tag{21}$$

On the other hand,

$$\mathrm{tr}(\Phi^{n+1})^2 = |A|^2 - \sum_{\alpha > n+1} \sum_{i,j} (h_{ij}^{\alpha})^2 - nH^2. \tag{22}$$

Thus, from (21) and (22) we conclude that $\sum_{\alpha>n+1}\sum_{i,j}(h_{ij}^{\alpha})^2 = 0$. But, from inequality (12) we also have that

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$$|\Phi|^4 = p \sum_{\alpha \geq n+1} [N(\Phi^{\alpha})]^2 = pN(\Phi^{n+1})^2 = p|\Phi|^4. \tag{23}$$

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In this setting, from (6) and (23) we get

$$\sum_{i,j,k} (h_{ijk}^{n+1})^2 = n^2 |\nabla H|^2 = 0,$$

that is, $h_{ijk}^{n+1} = 0$ for all i, j and M^n is an isoparametric hypersurface of $\mathbb{S}_1^{n+1}$.

Hence, since the equality occurs in (10), we have that also occurs the equality in Lemma 3.

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Therefore, we can apply classical congruence theorem due to Abe, Koike and Yamaguchi we conclude that M^n must be one of the two following standard product into $\mathbb{S}^{n+1}_1$:

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This conclude the proof.

□

Positive root of Q_R

Remark 1.

$$\alpha(n,p,R) = \frac{n(n-1)p}{2((n-p-1)^2 - (n-2)^2p^2)}\beta(n,p,R),$$

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$$\begin{aligned}\beta(n,p,R) &= (n-2)^2p(1-R)-2(n-p-1)R\\ &\quad -(n-2)\sqrt{p(1-R)((n-2)^2p(1-R)-4(n-p-1)R)+4p^2R^2}.\end{aligned}$$

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In particular, when $n=2$ the expression for $\alpha(n,p,R)$ reduces to

$$\alpha(2,p,R)=\frac{2pR}{p-1}.$$

Hence, for spacelike surfaces,

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Hence, for spacelike surfaces,

Corollary 6.

The only complete spacelike pnmc surfaces immersed in $\mathbb{S}_p^{2+p}$, $p \geq 2$, *with constant Gaussian curvature* $0 < K \leq 1$ *and such that* $\sup_M |\Phi|^2 < \frac{2p}{p-1}K$, *are the totally umbilical ones.*

We recall that a Riemannian manifold M^n is said to be parabolic (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above; that is, for a function $u \in \mathcal{C}^2(M)$

$$\Delta u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

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More generally, let M^n be a Riemannian manifold and consider a general class of second order differential operators on M^n given by

$$\mathcal{L}(u) = \mathrm{tr}(\mathcal{P} \circ \nabla^2 u) \tag{24}$$

for every $u \in \mathcal{C}^2(M)$, where $\mathcal{P} : TM \to TM$ is a symmetric operator on M^n.

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In this setting, M^n is said to be $\mathcal{L}$-parabolic (or parabolic with respect to the operator $\mathcal{L}$) if the constant functions are the only functions $u \in \mathcal{C}^2(M)$ which are bounded from above and satisfying $\mathcal{L}u \geq 0$.

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Theorem 7.

Let M^n, $n \geq 3$, be a complete L-parabolic spacelike pnmc submanifold in $\mathbb{S}_p^{n+p}$ with constant scalar curvature $0 < R \leq 1$. Suppose that M^n is not totally umbilical. Then

$$\sup_M |\Phi|^2 \geq \alpha(n, p, R) > 0, \tag{25}$$

with equality if and only if $p = 1$ and M^n is isometric to a hyperbolic cylinder

$$\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1 + r^2})$$

of radius $r > 0$.

Lemma 8.

Assume that $\mathcal{L}$ is semi-elliptic on a connected Riemannian manifold M^n. M^n is $\mathcal{L}$-parabolic if and only if every positive, bounded function u satisfying $\mathcal{L}(u) \geq 0$ is constant.

Lemma 8.

Assume that $\mathcal{L}$ is semi-elliptic on a connected Riemannian manifold M^n. M^n is $\mathcal{L}$-parabolic if and only if every positive, bounded function u satisfying $\mathcal{L}(u) \geq 0$ is constant.

Proposition 3.

Let $M = M_1 \times M_2$ be a Riemannian (connected) product manifold, where M_1 is parabolic (with respect to the Laplacian operator) and M_2 is compact. Let $\mathcal{P} : TM \to TM$ be a positive definite symmetric operator on M which splits as

$$\mathcal{P}(U, V) = (\lambda U, \mu V)$$

for every $U \in TM_1$ and $V \in TM_2$, with positive constants $\lambda, \mu \in \mathbb{R}$. Then M^n is $\mathcal{L}$-parabolic.

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Corollary 9.

The hyperbolic cylinders $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1 + r^2})$, as spacelike submanifolds of $\mathbb{S}_p^{n+p}$, are parabolic with respect to the Cheng-Yau operator L.

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Obrigado!!!