The classical singularity theorems of GR under optimal regularity conditions

(joint work with Melanie Graf, James D.E. Grant, Michael Kunzinger)

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Singularity Theorems in GR

- singularities occur in exact solutions; high degree of symmetries
- singularities as obstruction to extend causal geodesics [Penrose, 65]

Theorem (Pattern singularity theorem [Senovilla 98])

A spacetime is causal geodesically incomplete if we have

(i) *an energy/curvature condition,*
(ii) *a causality condition, and*
(iii) *an initial or boundary condition*

- (iii) initial condition $\rightsquigarrow$ causal geodesics start focussing
- (i) energy condition $\rightsquigarrow$ focusing goes on $\rightsquigarrow$ focal point
- (ii) causality condition $\rightsquigarrow$ no focal points
- way out: one causal geodesic has to be incomplete

The issue of regularity

Theorem (Pattern singularity theorem [Senovilla 98])

A C^2-spacetime [1] is causal geodesically incomplete if we have

(i) *an energy/curvature condition,*

(ii) *a causality condition, and*

(iii) *an initial or boundary condition*

- C^2 is too much to ask for: Realistic models (stars, matched spacetimes) involve jumps in matter variables $\rightsquigarrow g \in C^{1,1}$.
- Theorem allows (i)–(iii) plus completeness for $C^{1,1}$.
- But $C^{1,1}$-spacetimes are not 'singular' (curvature bd., geodesics ok).
- Below $C^{1,1}$: unbounded curvature, non-unique geodesics: singular.

Hence $C^{1,1}$ is the natural regularity for the singularity theorems.

[1] (M, g) with M smooth $g \in C^2$

Low ($=\mathcal{C}^{1,1}$) regularity: Problems & Solutions

Problems:

- Curvature tensor only $L^\infty \rightsquigarrow$ no Jacobi fields, conjugate/focal points
- No second variation of arclength
- exp_p not a local diffeomorphism.

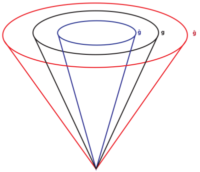
However:

- exp_p bi-Lipschitz homeomorphism and $\exists$ convex neighbourhoods, Gauss Lemma holds [Minguzzi 14], [Kunzinger, S, Stojković 14]
- Bulk of causality theory remains valid [Chruściel, Grant 12] [Minguzzi 15] [Kunzinger, S, Stojković, Vickers 14], [Sämann 16]
- The Hawking singularity theorem (big bang) holds in $C^{1,1}$ [Kunzinger, S, Stojković, Vickers 15]
- The Penrose singularity theorem (black hole) holds in $C^{1,1}$ [Kunzinger, S, Vickers 15]

Strategies in low regularity

(1) CG-regularization of the metric adapted to causal structure

[Chruściel, Grant 12]



Sandwich null cones of $g \in C^0$ between null cones of two approximating families of smooth metrics: $\check{\mathbf{g}}_\varepsilon \prec \mathbf{g} \prec \hat{\mathbf{g}}_\varepsilon$

(2) Use replacement for strong energy condition

Lemma (timelike case)

Let $(M, \mathbf{g})$ be a $\mathcal{C}^{1,1}$-spacetime satisfying the energy condition

$$\mathrm{Ric}\,[\mathbf{g}]\,(X, X) \geq 0 \text{ a.e. for all timelike local } \mathcal{C}^\infty\text{-vector fields } X.$$

Then for all $K \subset\subset M \quad \forall C > 0 \quad \forall \delta > 0 \quad \forall \kappa < 0 \quad \forall \varepsilon$ small

$$\mathrm{Ric}\,[\check{\mathbf{g}}_\varepsilon](X, X) > -\delta \quad \forall X \in TM|_K : \ \check{\mathbf{g}}_\varepsilon(X, X) \leq \kappa, \ \|X\|_h \leq C.$$

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The Hawking-Penrose Theorem

Theorem **[Hawking, Penrose 1970]**

A C^2-spacetime (M, g) is causally incomplete if

(i a) (SEC) $\mathbf{Ric}(X, X) \geq 0$ for every causal vector X

(i b) (Genericity) On every inext. causal geodesic γ, the tidal force operator is nontrivial at least at a point $\gamma(t_0)$

$$[R(t_0)] : [\dot\gamma(t_0)]^\perp \to [\dot\gamma(t_0)]^\perp , \quad v \mapsto \mathbf{R}(v, \dot\gamma(t_0))\dot\gamma(t_0) \not\equiv 0$$

(ii) (M, g) is chronological (no closed timelike curves)

(iii) M contains one of the following

- (a) a compact achronal set A without edge (cf. Hawking's thm.)
- (b) a trapped surface S (cf. Penrose's thm.)
- (c) a trapped point: the expansion θ becomes negative for any f.d. null geodesic starting at p

Comments on the classical proof

Proof rests on

The Hawking-Penrose Lemma **[Hawking, Penrose 1970]** **[Graf 2016]**

A C^2 $C^{1,1}$-spacetime (M, g) is causally incomplete if

(L1) M is chronological

(L2) Every complete causal geodesic contains a pair of conjugate ptsis not (globally) maximising

(L3) There is a trapped set (S achronal, $E^+(S) := J^+(S) \setminus I^+(S)$ cp.)

Good news: The H-P Lemma continues to hold in $C^{1,1}$ (causality)

Left to do: Show that

- appropriate version of the initial conditions $\Rightarrow$ (L3) (causality)
- appropriate version of genericity and SEC $\Rightarrow$ (L2) (analysis, here!)

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The $\mathcal{C}^{1,1}$-genericity condition

Definition ($\mathcal{C}^{1,1}$-genericity condition)

Genericity holds along a causal geodesic γ of a $\mathcal{C}^{1,1}$-metric g if near some $\gamma(t_0)$ there are continuous vector fields X, V with $X(\gamma(t)) = \dot\gamma(t)$, $V(\gamma(t)) \in \dot\gamma(t)^\perp$ such that

$$\langle \mathbf{R}(V, X)X, V\rangle > c.$$

- Equivalent to the usual condition for $g \in C^2$
- Survives approximation process (Friedrichs lemma): If $\gamma_\varepsilon \to \gamma$ in C^1

$$R[g_\varepsilon](t) > \mathrm{diag}(c, -C, \dots, -C) \text{ on } [t_0 - r, t_0 + r] \qquad (1)$$

  where $R[g_\varepsilon](t) := R[g_\varepsilon](\,.\,, \dot\gamma_\varepsilon(t))\dot\gamma_\varepsilon(t) : \dot\gamma_\varepsilon(t)^\perp \to \dot\gamma_\varepsilon(t)^\perp$
- to be fed into a matrix Riccati comparison argument later on...

Raychaudhuri argument (timelike case)

- γ tl. geodesic in approximating C^∞-spacetime, no conjugate pts.
- A (unique) Jacobi tensor with $A(-T) = 0$ and $A(t_0 = 0) = \mathrm{id}$
- $B := A' A^{-1}$, expansion $\theta = \mathrm{tr}(B)$ satisfies **Raychaudhuri eq.**:

$$\dot{\theta} = -\mathbf{Ric}(\dot{\gamma}, \dot{\gamma}) - \mathrm{tr}(\sigma^2) - (1/d)\,\theta^2$$

- 'old' (direct) argument: SEC $\Rightarrow$ $\dot{\theta} \le \delta - \frac{1}{d}\,\theta^2$; i.c. $\Rightarrow$ $\theta(0) < b < 0$
  $\Rightarrow$ upper bd. on first conj. pt in terms of b (scalar Riccati comp.)
- 'reverse' Raychaudhuri: no conj. pts. $\Rightarrow$ $|\theta|$ small initially

Boxing lemma

For $T > 0$ there is $\delta(T) > 0$ such that:
If γ is has no conjugate points on $[-T, T]$ then

$$\sup_{t \in [-\frac{T}{2}, \frac{T}{2}]} |\theta(t)| \le \frac{4d}{T}$$

provided that $\mathbf{Ric}(\dot{\gamma}, \dot{\gamma}) \ge -\delta$ on $[-T, T]$.

Matrix Riccati comparison argument

- $B := A' A^{-1}$ satisfies a **matrix Riccati eq.**: $\dot{B} + B^2 + R = 0$
- Comparison result [Eschenburg, Heintze 90]:

$$\dot{\tilde{B}} + \tilde{B}^2 + \tilde{R} = 0 \quad \text{and} \quad \begin{matrix} R \geq \tilde{R} \text{ on } I \\ B(0) \leq \tilde{B}(0) \end{matrix} \quad \Rightarrow \; B \leq \tilde{B} \text{ on } I \cap [0, \infty)$$

- Choosing $\tilde{R}$ and $\tilde{B}(t_0)$
  - (1) suggests $\tilde{R} := \mathrm{diag}(c, -C, \dots, -C)$, $I = [-r, r]$
  - reasonably $\tilde{B}(0) := f(T, \delta, r) \cdot \mathrm{id}$

$\Longrightarrow \; \tilde{B} = \frac{1}{d}\mathrm{diag}(H_{c,f}, \dots, H_{-C,f})$ (diagonal & explicit)
$\Longrightarrow$ eigenvalue $\beta_{\min}(t) \leq H_{c,f}(t) < H_{c,f}(\frac{r}{2}) < 0$ on $[\frac{r}{2}, r]$

- Feed into the shear term $\mathrm{tr}(\sigma^2)$ in the Raychaudhuri eq.: Integrating from $\frac{r}{2}$ to r contradicts boxing lemma for $T > T_0(r, c)$ and $\delta < \delta_0(r, c)$ $\Rightarrow$ conjugate points in $[-T, T]$.

  The bound T_0 depends only on c, r not on g_ε!

Going back to $g \in C^{1,1}$

Shown so far:

- $\check{g}_\varepsilon \in C^\infty$ close to $g \in C^{1,1}$ which satisfies genericity and SEC
- γ_ε causal $\check{g}_\varepsilon$-geodesics close to γ causal g-geodesic

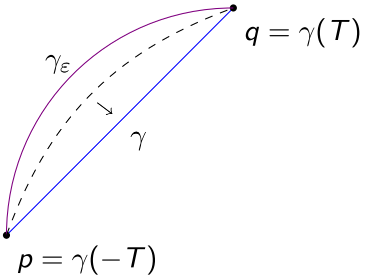
$\Rightarrow \gamma_\varepsilon$ have conj. pts. if too long (longer than bd. uniform in ε)

Want to show: γ is not g-maximizing

Theorem (timelike case)

Let $g \in C^{1,1}$ be a globally hyperbolic Lorentzian metric on M satisfying genericity and SEC.
Then any timelike geodesic γ is not globally maximising.

Proof



- Proof by contradiction, assume $\gamma : \mathbb{R} \to M$ is maximizing and satisfies genericity at $t = 0$
- Choose $T > T_0(c, r)$, set $p := \gamma(-T)$, $q := \gamma(T)$
- g glob. hyp. $\Rightarrow \check{g}_\varepsilon$ glob. hyp.
- $\exists\ \check{g}_\varepsilon$-maximizing geodesics $\gamma_\epsilon : I_\varepsilon \to M$ from p to q
- Extract a convergent subsequence
- Limit must equal γ (else two distinct g-maximizing curves)
- But then $I_\varepsilon \to [-T, T]$, contradicting γ_ϵ being $\check{g}_\varepsilon$-maximizing

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Further issues (mainly swept under the carpet)

The null case of the previous theorem

- We cannot use global hyperbolicity
- To produce long approximating null geodesics we need to rule out that they are closed null geodesics for g by hand

$\rightsquigarrow$ in the theorem we have to suppose the spacetime to be causal rather than chronological (as in the classical case)

The initial conditions:

(a) the hypersurface case simply rests on $C^{1,1}$-causality

(b) We extend the trapped (2D-)surface case to C^0-submanifolds of arbitrary codimensions generalising a condition by [Galloway, Senovilla 2010] using it in the support sense.

(c) The trapped point condition also needs to be formulated in the support sense using (b).

The Hawking–Penrose theorem in $C^{1,1}$

Theorem **[Hawking, Penrose 1970] [Graf, Grant, Kunzinger, S. 2017]**

A C^2 $C^{1,1}$-spacetime (M, g) is causally incomplete if

(i a) (SEC) $\mathbf{Ric}(X, X) \geq 0$ a.e. for every causal vector XLip. local vector field X

(i b) $C^{1,1}$-genericity holds

(ii) (M, g) is chronologicalcausal

(iii) M contains one of the following

- (a) a compact achronal set A without edge
- (b) a trapped C^0-surface S in the support sense
- (c) a trapped point in the support sense
- (d) a trapped C^0-submanifold of co-dimension $2 < m < n$ satisfying the Galloway-Senovilla condition in the support sense

Some References

P.T. Chruściel, J.D.E. Grant, *On Lorentzian causality with continuous metrics.* CQG 29 (2012).

G.J. Galloway, J.M.M. Senovilla, *Singularity theorems based on trapped submanifolds of arbitrary co-dimension.* CQG 27 (2010).

M. Graf, J.D.E. Grant, M. Kunzinger, R. Steinbauer, *The Hawking-Penrose singularity theorem for $\mathcal{C}^{1,1}$-Lorentzian metrics.* CMP 360 (2018).

J.D.E. Grant, J.-H. Treude, *Volume comparison for hypersurfaces in Lorentzian manifolds and singularity theorems.* Ann. Global Anal. Geom. 43 (2013).

S.W. Hawking, R. Penrose, *The singularities of gravitational collapse and cosmology.* Proc. Roy. Soc. London Ser. A 314 (1970).

M. Kunzinger, R. Steinbauer, M. Stojković, *The exponential map of a $\mathcal{C}^{1,1}$-metric.* Differential Geom. Appl. 34 (2014).

M. Kunzinger, R. Steinbauer, M. Stojković, J.A. Vickers, *Hawking's singularity theorem for $C^{1,1}$-metrics.* CQG 32 (2015).

M. Kunzinger, R. Steinbauer, J.A. Vickers. *The Penrose singularity theorem in regularity $C^{1,1}$.* CQG 32 (2015).

E. Minguzzi, *Convex neighborhoods for Lipschitz connections and sprays.* Monatsh. Math. 177 (2015).

C. Sämann, *Global hyperbolicity for spacetimes with continuous metrics.* Annales Henri Poincaré, 17 (2016).

Thanks for your attention