Mannheim Slant Helix in Lorentz-Minkowski Space

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17-24 June 2018

There are many studies about the associated curves such as Bertrand curve, Mannheim curve, involute and evolute, etc. If there is a relationship between the curves α and β such that principal normal lines of α coincide with the binormal lines of β, then this curves are called Mannheim pair.
Izumiya and Takeuchi [5] have introduced the concept of a slant helix in Euclidean 3-space. A slant helix in Euclidean space $\mathbb{E}^3$ is defined by the property that its principal normal vector field makes a constant angle with a fixed line u.

Helical structures are an important work frame in the differential geometry studies because they can be seen in many forms in nature, architecture, simulation of kinematic motion or design of highways, fractals, and mechanic tools. For example: biological macromolecules, macroscopic organs, protein secondary structure, magnetic trajectories, microscale using self organized nanoparticles or thin films. Some of research in this field are ([3],[4]).

In the recent article [11], the authors show that a family of slant helices are the exact solutions of the shape equations. Hence, the study of slant helices can help to understand better the biopolymer chains. In this talk we will metioned the slant Mannheim curves which satisfied both of characterizations of slant and Mannheim curves.

The Lorentz-Minkowski space $\mathbb{L}^3$ is the real vector space $\mathbb{R}^3$ endowed with the standart metric

$$\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) are the canonical coordinates of $\mathbb{R}^3$.

Definition

A unit speed curve α is called a slant helix if there exists a non-zero constant vector field U in $\mathbb{R}^3$ such that the function $\langle N(s), U\rangle = \cos\theta$ is constant [5].

Then they showed that a curve is a slant helix if and only if the function

$$\frac{\kappa^2}{(\tau^2+\kappa^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

is constant.
In other words geodesic curvature of spherical indicatrix curve N is constant.

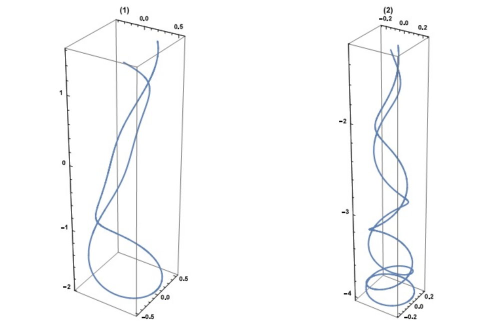

Figure: Some slant helices

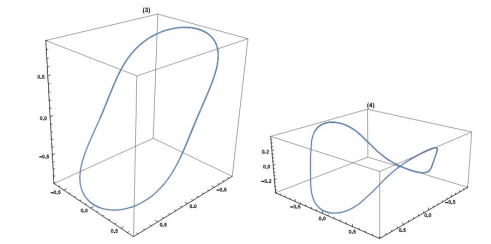

Figure: Some slant helices

Definition

A unit speed curve α is called a slant helix if there exists a non-zero constant vector field U in $\mathbb{L}^3$ such that the function $\langle N(s), U\rangle$ is constant [1].

Then the characterizations of the slant helices are given by the following theorems:

Theorem

Let α be a unit speed timelike curve in $\mathbb{L}^3$. Then α is a slant helix if and only if either one the next two functions

$$\frac{\kappa^2}{(\tau^2-\kappa^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \quad \text{or} \quad \frac{\kappa^2}{(\kappa^2-\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \tag{1}$$

is constant everywhere.

Theorem

Let α be a unit speed spacelike curve in $\mathbb{L}^3$.
(i) If the normal vector of α is spacelike, then α is a slant helix if and only if either one the next two functions

$$\frac{\kappa^2}{(\tau^2-\kappa^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \quad \text{or} \quad \frac{\kappa^2}{(\kappa^2-\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \tag{2}$$

is constant everywhere.

(ii) If the normal vector of α is timelike, then α is a slant helix if and only if the function

$$\frac{\kappa^2}{(\tau^2+\kappa^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \quad \text{or} \tag{3}$$

is constant.
(iii) Any spacelike curve with lightlike normal vector is a slant curve.

Theorem

Let α be a unit speed lightlike curve in $\mathbb{L}^3$. Then α is a slant helix if and only if the torsion is

$$\tau(s) = \frac{a}{(bs + c)^2} \tag{4}$$

where a, b and c are constants, $bs + c \neq 0$.

Definition

Let C and C^* be space curves in $\mathbb{R}^3$. If there exists a corresponding relationship between the space curves C and C^* such that, at the corresponding points of the curves, the principal normal lines of C coincides with the binormal lines of C^*, then C is called a Mannheim curve and C^* a Mannheim partner curve of α. The pair $\{C, C^*\}$ is said to be a Mannheim pair (see [8], [10] for more details).

Mannheim Curves in $\mathbb{L}^3$

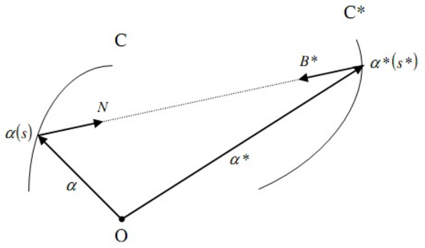

Figure: Mannheim Pair Curve

It is just known that a space curve in $\mathbb{R}^3$ is a Mannheim curve if and only if its curvature κ and torsion τ satisfy the relationship

$$\kappa = \lambda(\kappa^2 + \tau^2)$$

where λ is a nonzero constant.

Theorem

Let α be a non-null curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ in 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$. If α is a Mannheim curve, then the equation

$$\kappa(1 - \varepsilon_1 \lambda \kappa) - \varepsilon_3 \lambda \tau^2 = 0 \tag{5}$$

where $\varepsilon_1 = \langle T, T \rangle$ and $\varepsilon_3 = \langle B, B \rangle$ is hold [12].

Mannheim Slant Curves in $\mathbb{L}^3$

In this section we give characterization of the Mannheim curves which are slant helix in $\mathbb{L}^3$. Then we give curvature and torsion of these type of curves.

Theorem

Let α be a unit speed timelike Mannheim curve with non-zero curvature κ and torsion τ in $\mathbb{L}^3$. α is a slant helix if and only if

$$\frac{\tau}{\kappa} = \cosh(as + b).$$

Hence the curvatures are

$$\kappa = \frac{1}{\lambda}\frac{1}{\sinh^2(as+b)} \quad \text{and} \quad \tau = \frac{1}{\lambda}\frac{\cosh(as+b)}{\sinh^2(as+b)} \tag{6}$$

where *a* and *b* are non-zero constant hold.

Proof

Let α be a unit speed timelike curve in $\mathbb{L}^3$. Let α be a slant helix and its axis be u. Then the equation

$$\langle N(s), u\rangle = c$$

is satisfied. By differentiating the equation with respect to s, we have

$$\kappa\langle T(s), u\rangle + \tau\langle B(s), u\rangle = 0 \tag{7}$$

Again by differentiating the equation with respect to s, we have

$$\kappa'\langle T(s), u\rangle + \tau'\langle B(s), u\rangle = c(\tau^2 - \kappa^2) \tag{8}$$

We know that the timelike Mannheim curve satisfied the

$$\kappa = \lambda(\tau^2 - \kappa^2)$$

Hence we get the Eq.(8) as

$$\kappa' \langle T(s), u \rangle + \tau' \langle B(s), u \rangle = c\frac{\kappa}{\lambda} \tag{9}$$

If we consider the Eqs.(7) and (9) together,

$$\begin{aligned} \langle T(s), u \rangle &= -\frac{c\tau}{\lambda\kappa}\frac{1}{(\frac{\tau}{\kappa})'} \\ \langle B(s), u \rangle &= \frac{c}{\lambda}\frac{1}{(\frac{\tau}{\kappa})'} \end{aligned}$$

Denoted $\frac{\tau}{\kappa}$ by y then

$$\begin{aligned} \langle T(s), u \rangle &= \frac{c}{\lambda} \frac{y}{y'} \\ \langle B(s), u \rangle &= \frac{c}{\lambda} \frac{1}{y'} \end{aligned}$$

are obtained. Again differentiation of these equations give

$$\begin{aligned} \kappa &= -\frac{1}{\lambda} \frac{y'^2 - yy''}{y'^2} \\ \tau &= \frac{1}{\lambda} \frac{y''}{y'^2} \end{aligned}$$

Hence the differential equation

$$-yy'^2 + y''(y^2 - 1) = 0$$

is satisfied. Since $y = \frac{\tau}{\kappa} = \cosh(as + b)$ is the solution of the differential equation, the curvatures of the timelike Mannheim slant curve are

$$\kappa = \frac{1}{\lambda}\frac{1}{\sinh^2(as+b)} \quad \text{and} \quad \tau = \frac{1}{\lambda}\frac{\cosh(as+b)}{\sinh^2(as+b)} \tag{10}$$

Proposition

Let (α, β) be a Mannheim pair curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ respectively in $\mathbb{L}^3$. Let α be a timelike curve and β be a spacelike curve with timelike principal normal vector. α is slant helix if and only if

$$\lambda \tanh \varphi(s)\tau - \lambda\kappa = 1 \tag{11}$$

holds where λ is a constant and $\varphi(s)$ is a angle between the vector fields T and N^*.

Proof

Let α be a timelike curve and β be a spacelike curve with timelike principal normal vector. Then we can write

$$\begin{aligned} T^* &= \sinh\varphi(s)T(s) + \cosh\varphi(s)B(s) \\ N^* &= \cosh\varphi(s)T(s) + \sinh\varphi(s)B(s) \\ B^* &= N \end{aligned}$$

Differentiation of B^* with respect to arc-length parameter of the curve β gives

$$\frac{dB^*}{ds^*} = \tau^* N^* = \frac{dN}{ds}\frac{ds}{ds^*}$$

Proof

From

$$\tau^*(\cosh\varphi(s)T(s)+\sinh\varphi(s)B(s))=(\kappa T(s)+\tau B(s))\frac{ds}{ds^*}$$

We get,

$$\begin{aligned}\tau^*\cosh\varphi(s) &= \kappa\frac{ds}{ds^*}\\ \tau^*\sinh\varphi(s) &= \tau\frac{ds}{ds^*}\end{aligned}$$

and we have

$$\frac{\tau}{\kappa}=\tanh\varphi(s) \tag{12}$$

Proof

Also,

$$\begin{aligned} T^* &= \frac{d\beta}{ds}\frac{ds}{ds^*} \\ &= ((1+\lambda\kappa)T + \lambda\tau B)\frac{ds}{ds^*} \\ &= \sinh\varphi(s)T(s) + \cosh\varphi(s)B(s) \end{aligned}$$

and we have

$$\frac{1+\lambda\kappa}{\lambda\tau} = \tanh\varphi(s) \tag{13}$$

Proof

Hence,

$$-\lambda\kappa + \lambda\tau \tanh \varphi(s) = 1$$

is obtained.
From the Eq.(12) and Eq.(13), we have

$$\lambda\tau^2 - \lambda\kappa^2 = \kappa$$

Corollary

Let (α, β) be a Mannheim pair curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ respectively in $\mathbb{L}^3$. Let α be a timelike curve and β be a spacelike curve with timelike principal normal vector. α is slant helix if and only if

$$\lambda \cosh(as + b)\tau - \lambda\kappa = 1 \tag{14}$$

holds where λ is a constant and $\varphi(s)$ is a angle between the vector fields T and N^*.

Proposition

Let (α, β) be a Mannheim pair curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ respectively in $\mathbb{L}^3$. Let α be a timelike curve and β be a spacelike curve with timelike principal normal vector. α is a slant helix (β is a helix) if and only if

$$\varphi(s) = arc\tanh(\cosh(as + b))$$

where λ, a and b are non-zero constants and $\varphi(s)$ is the angle between the vector fields T and N^*.

Proposition

Let (α, β) be a timelike Mannheim pair curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ respectively in $\mathbb{L}^3$. Then

$$\lambda \coth \theta(s)\tau - \lambda\kappa = 1 \qquad (15)$$

where λ is a constant and $\theta(s)$ is angle between the vector fields T and T^*.

Remark

Let α be a curve in $\mathbb{L}^3$ with Frenet apparatus $\{T, N, B, \kappa, \tau\}$. α is a Bertrand curve if and only if

$$\lambda\kappa + \mu\tau = 1 \qquad (16)$$

is satisfied where λ and τ are constant.
But this relationship between the curvature functions of the curve is not satisfied for Mannheim curves.

Proposition

Let (α, β) be a timelike Mannheim pair curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ respectively in $\mathbb{L}^3$. Then

1. β is a helix (α is a slant helix) if and only if $\tau^* = -\frac{1}{\lambda \cosh(s)}$ and $\kappa^* = -\frac{1}{\lambda c \cosh(s)}$
2. $\theta = arc \tanh(-\lambda \tau^*)$
3. $\kappa^* = \frac{d\theta}{ds^*}$

Proof

1) Let α and β be timelike curves. Then

$$\begin{aligned} T^* &= \cosh\theta(s)T + \sinh\theta(s)B \\ N^* &= \sinh\theta(s)T + \cosh\theta(s)B \\ B^* &= N \end{aligned}$$

From the equation

$$\frac{dB^*}{ds^*} = -\tau^* N^* = \frac{dN}{ds}\frac{ds}{ds^*}$$

we get

$$\frac{\kappa}{\tau} = \tanh\theta$$

Proof

Also,

$$\begin{aligned} T &= \cosh\theta(s)T^* - \sinh\theta(s)N^* \\ B &= -\sinh\theta(s)T^* + \cosh\theta(s)N^* \\ N &= B^* \end{aligned}$$

Proof

$$\alpha(s) = \beta(s) - \lambda N(s) = \beta(s) - \lambda B^*(s)$$

$$\begin{aligned} T &= T^* \frac{ds^*}{ds} + \lambda \tau^* N^* \frac{ds^*}{ds} \\ &= \cosh\theta\, T^* - \sinh\theta\, N^* \end{aligned}$$

Proof

Then from the equations

$$\frac{ds^*}{ds} = \cosh\theta \quad \textit{and} \quad \lambda\tau^*\frac{ds^*}{ds} = -\sinh\theta$$

we have

$$\begin{aligned} \tau^* &= -\frac{1}{\lambda}\tanh\theta \\ &= -\frac{1}{\lambda}\frac{\kappa}{\tau} \\ &= -\frac{1}{\lambda}\frac{1}{\cosh(as+b)} \end{aligned}$$

and from $\frac{\tau^*}{\kappa^*} = c = const.$ we get $\kappa^* = -\frac{1}{\lambda c\cosh(s)}$

Proof

2) It is obvious from $\tau^* = -\frac{1}{\lambda}\tanh\theta$

Proof

3) Derivation of the equation

$$\langle T, T^* \rangle = -\cosh\theta$$

with respect to s^* gives

$$\begin{aligned} \langle T, \kappa^* N^* \rangle + \langle \kappa N \frac{ds}{ds^*}, T^* \rangle &= -\sinh\theta \frac{d\theta}{ds^*} \\ \kappa^*(-\sinh\theta) &= -\sinh\theta \frac{d\theta}{ds^*} \end{aligned}$$

Then we get

$$\kappa^* = \frac{d\theta}{ds^*}$$

Corollary

Let (α, β) be a timelike Mannheim pair curve with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ respectively in $\mathbb{L}^3$. Then the followings are satisfied

1. α is a helix if and only if β is a straight line
2. α is a helix if and only if θ is a constant.

Theorem

Let α be a unit speed spacelike Mannheim curve with spacelike normal vector in $\mathbb{L}^3$. α is a slant helix if and only if

$$\frac{\tau}{\kappa} = \cosh(as + b).$$

Hence the curvatures are

$$\kappa = -\frac{1}{\lambda}\frac{1}{\sinh^2(as + b)} \quad \text{and} \quad \tau = -\frac{1}{\lambda}\frac{\cosh(as + b)}{\sinh^2(as + b)} \tag{17}$$

hold.

Theorem

Let α be a unit speed spacelike Mannheim curve with timelike normal vector in $\mathbb{L}^3$. α is a slant helix if and only if

$$\frac{\tau}{\kappa} = \sinh(as + b).$$

Hence the curvatures are

$$\kappa = \frac{1}{\lambda}\frac{1}{\cosh^2(as+b)} \quad \text{and} \quad \tau = \frac{1}{\lambda}\frac{\sinh(as+b)}{\cosh^2(as+b)} \tag{18}$$

hold.

Theorem

Let α be a Cartan framed null curve in $\mathbb{L}^3$. α is a Mannheim slant helix if and only if

$$\kappa = \frac{c_2}{(s+2c_1)^2} \tag{19}$$

for non-zero constant c_1 and c_2. The torsion of the curve α is a constant. (see details [9])

Mannheim Slant Curves and Ruled Surfaces

In this section we will give the relationship between the Mannheim slant curve α and striction line of the ruled surface which is defined by

$$\varphi(s,u) = \alpha(s) + uN(s).$$

Proposition

Let α be a timelike Mannheim slant helix in $\mathbb{L}^3$. The striction line of the ruled surface

$$\varphi(s,u) = \alpha(s) + uN(s)$$

is the Mannheim partner of curve α where N is the normal vector field of the curve α.

Proof

Let α be a timelike Mannheim slant helix in $\mathbb{L}^3$. The striction line of the ruled surface

$$\varphi(s,u)=\alpha(s)+uN(s)$$

is given by

$$\begin{aligned}\beta &= \alpha-\frac{\langle T,N'\rangle}{\langle N',N'\rangle}N\\ &= \alpha-\frac{\langle T,\kappa T+\tau B\rangle}{\langle \kappa T+\tau B,\kappa T+\tau B\rangle}N\\ &= \alpha+\frac{\kappa}{\tau^2-\kappa^2}N\end{aligned}$$

Proof

Since the curve α satisfy the relation $\kappa = \lambda(\tau^2 - \kappa^2)$, the curve β is obtained as

$$\beta = \alpha + \lambda N$$

which means that the striction line of the ruled surface is Mannheim partner curve of α.

Theorem

Let $\varphi(s,u) = \alpha(s) + uN(s)$ be a ruled surface. Let α be a timelike Mannheim curve in $\mathbb{L}^3$ and let $\beta = \alpha + \lambda N$ be a curve on the surface φ. Then the followings are hold (a) β is a striction line of the surface
(b) β is a geodesic curve on the surface
(c) α is a slant helix iff β is a helix
(d) α is a clad helix iff β is a slant helix.

Proof

(a) Let $\varphi(s,u) = \alpha(s) + uN(s)$ be a ruled surface. Let α be a timelike Mannheim curve in $\mathbb{L}^3$. Then $\beta = \alpha + \lambda N$ is a striction line of the surface from the above proposition.
(b) Let $\{T, N, B\}$ be the Frenet frame and $\{N, C = \frac{N'}{\|N'\|}, W = N \times C\}$ be adopted frame of the curve α where W is the Darboux vector field of the Frenet frame. Then we have the Frenet frame of the curve β is as follows

$$\begin{aligned} \beta_T &= W \\ \beta_N &= -C \\ \beta_B &= N \end{aligned}$$

Proof

The normal vector field of the surface along the curve β is C. Also,

$$\beta'' = \lambda C$$

Hence the curve β is geodesic on the surface.
(c) If α is a slant helix, then

$$\begin{aligned} \langle N, u \rangle &= constant \\ \langle \beta_B, u \rangle &= constant \end{aligned}$$

Then β is a helix.

Proof

(d) If α is a clad helix, then

$$\begin{aligned} \langle C, u \rangle &= constant \\ \langle \beta_N, u \rangle &= constant \end{aligned}$$

Then β is a slant helix.

Proposition

Let α be a timelike Mannheim slant helix in $\mathbb{L}^3$. Drall of the ruled surface

$$\varphi(s,u) = \alpha(s) + uN(s)$$

is $P_x = -\frac{1}{\tau^*}$.

Proof

Let α be a timelike Mannheim slant helix in $\mathbb{L}^3$. Drall of the surface is

$$P_x = \frac{det(T, N, N')}{\langle N', N' \rangle} = \frac{\tau}{\tau^2 - \kappa^2}$$

Also, we know that

$$\begin{aligned} \tau^* &= -\frac{1}{\lambda}\frac{\kappa}{\tau} \\ &= -\frac{1}{\lambda}\frac{\lambda(\tau^2 - \kappa^2)}{\tau} \\ &= -\frac{\tau^2 - \kappa^2}{\tau} \end{aligned}$$

Then it completes the proof.

Proposition

Let α be a spacelike slant helix with spacelike or timelike normal vector in $\mathbb{L}^3$. The striction line of the ruled surface

$$\varphi(s, u) = \alpha(s) + uN(s)$$

is the Mannheim partner of curve α where N is the normal vector field of the curve α.

Kaynaklar

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Thank you for your attention