Trapped submanifolds in de Sitter space

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The results I am going to introduce in this talk have been obtained in collaboration with the following colleagues:

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- They will be part of **Veronica's PhD thesis**, to be defended in September 2018 (I hope so...)

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Let $\mathrm{II} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}^\perp(\Sigma)$ be the vector valued **second fundamental form** of the submanifold, that is the symmetric tensor

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- The extreme case $\mathbf{H} = 0$ corresponds to a **minimal** submanifold.

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- That is, the symmetric operators $A_\xi, A_\eta : \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}(\Sigma)$ given by

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- Therefore, in terms of $\{\xi, \eta\}$ we have

$$\mathbf{H} = -\theta_\eta \xi - \theta_\xi \eta$$

where

$$\theta_\xi = \frac{1}{n}\mathrm{trace}(A_\xi) \quad \text{and} \quad \theta_\eta = \frac{1}{n}\mathrm{trace}(A_\eta)$$

define the **null mean curvatures** (or **null expansion scalars**) of Σ.

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- This was the original formulation of trapped surfaces given by Penrose in terms of the signs or the vanishing of the null expansions.

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- In some sense, $\mathbb{S}_1^{n+2}$ can be seen, in Lorentzian geometry, as the equivalent of the Euclidean sphere.
- Consider on $\mathbb{S}_1^{n+2}$ the **time-orientation** induced by the globally defined timelike vector field $e_0^* \in \mathfrak{X}(\mathbb{S}_1^{n+2})$ given by

$$e_0^*(x) = e_0 - \langle e_0, x\rangle x = e_0 + x_0 x, \qquad e_0 = (1, 0, \ldots, 0),$$

  with

$$\langle e_0^*(x), e_0^*(x)\rangle = -1 - \langle e_0, x\rangle^2 \leq -1 < 0.$$

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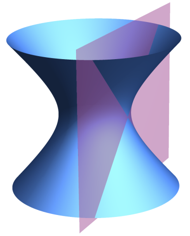
- Recall that a null hypersurface into a spacetime M is a smooth codimension one embedded submanifold such that the pull-back of the Lorentzian metric of M is degenerate.
- When the submanifold Σ is contained into a null hypersurface of M, there always exists a globally defined future-pointing normal null frame $\{\xi, \eta\}$ on Σ.

The light cone of de Sitter space

Light cone of de Sitter spacetime

Fix a point $\mathbf{a} \in \mathbb{S}_1^{n+2}$. The **light cone** in $\mathbb{S}_1^{n+2}$ with **vertex at a** is the subset

$$\Lambda_{\mathbf{a}} = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x \rangle = 1, x \neq \mathbf{a}\}.$$



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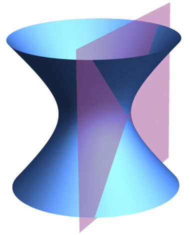

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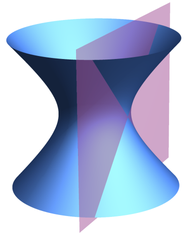

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- The **future** component of $\Lambda_{\mathbf{a}}$ is

$$\Lambda_{\mathbf{a}}^{+} = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x \rangle = 1, \langle x - \mathbf{a}, \mathbf{e}_0 \rangle = -x_0 + a_0 < 0\}.$$

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Future-pointing normal null frame

In these conditions

$$\xi = \psi - \mathbf{a} \quad \text{and} \quad \eta = -\frac{1 + \|\nabla u\|^2 + u^2}{2u^2}\xi + \frac{1}{u}e_0^\perp$$

gives two **future-pointing null normal** vector fields globally defined on Σ with $\langle \xi, \eta \rangle = -1$, where we are denoting

$$e_0 = e_0^\top(p) + e_0^\perp(p) + \langle \psi(p), e_0 \rangle \psi(p), \quad p \in \Sigma.$$

Null shape operators

The corresponding null second forms associated to the global null frame $\{\xi, \eta\}$ are given by

$$A_\xi = I \qquad \text{and} \qquad A_\eta = -\frac{1 + \|\nabla u\|^2 - u^2}{2u^2} I + \frac{1}{u}\nabla^2 u,$$

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- Therefore, Σ is **marginally trapped** if and only if $\theta_\eta = 0$, that is,

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- In that case, it is necessarily past marginally trapped since $\theta_\xi = 1 > 0$.

- On the other hand, it follows from the Gauss equation that the Ricci and the scalar curvatures of Σ are given by

$$\mathrm{Ric}(X,Y) = (n-1)(1+\langle \mathbf{H},\mathbf{H}\rangle)\langle X,Y\rangle + \frac{n-2}{nu}(\Delta u\langle X,Y\rangle - n\,\mathrm{Hess}\,u(X,Y)),$$

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- For each **positive** smooth function $f : \mathbb{S}^n \to (0, +\infty)$, consider the embedding $\psi_f : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ given by

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with $u = -\langle \psi, e_0 \rangle = \psi_0 > 0$, and $\psi = \psi_f \circ \Psi$ where $f = u \circ \Psi^{-1}$.

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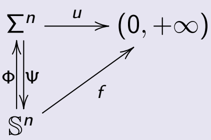

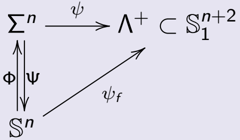

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- In particular, Ψ is a **local diffeomorphism**.
- Assume now that Σ is **complete** (that is, $\langle , \rangle$ is a complete Riemannian metric on Σ) and $u^* = \sup_\Sigma u < +\infty$.

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$$\limsup_{r \to +\infty} \frac{u}{r \log(r)} < +\infty$$

r the Riemannian distance from a fixed origin $o \in \Sigma$.

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- In particular, the conformal factor f is the conformal factor of a conformal diffeomorphism of the unit round sphere.
- Recall that, up to orthogonal transformations, every conformal diffeomorphism of $(\mathbb{S}^n,\langle,\rangle_0)$ is given by

$$F_{\mathbf{c}}(p)=\frac{p+(\mu\langle p,\mathbf{c}\rangle_0+\lambda)\mathbf{c}}{\lambda(1+\langle p,\mathbf{c}\rangle_0)}$$

for all $p\in\mathbb{S}^n$, where $\mathbf{c}\in\mathbb{B}^{n+1}$, $\mathbb{B}^{n+1}$ the open unit ball in $\mathbb{R}^{n+1}$, and

$$\lambda=(1-\|\mathbf{c}\|_0^2)^{-1/2} \quad \text{and} \quad \mu=(\lambda-1)\|\mathbf{c}\|_0^2.$$

- A direct computation shows that the conformal factor f of $F_{\mathbf{c}}$ is given by

$$f(p) = \frac{\sqrt{1 - \|\mathbf{c}\|_0^2}}{1 + \langle p, \mathbf{c} \rangle_0}$$

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- This completes the proof of Theorem 1.

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- Although all the embeddings $\psi_{\mathbf{b}}$ given in Example 2 are conformal to the round sphere and have the same constant sectional curvature 1, they are **not congruent** to each other.

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- Although all the embeddings $\psi_{\mathbf{b}}$ given in Example 2 are conformal to the round sphere and have the same constant sectional curvature 1, they are **not congruent** to each other.
- In other words, $\psi_{\mathbf{b}_1}$ is congruent to $\psi_{\mathbf{b}_2}$ for $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^{n+1}$

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{\psi_{\mathbf{b}_1}} & \Lambda^+ \subset \mathbb{S}^{n+2}_1 \\ \downarrow{\scriptstyle \psi_{\mathbf{b}_2}} & \swarrow{\scriptstyle A} & \\ \Lambda^+ \subset \mathbb{S}^{n+2}_1 & & \end{array}$$

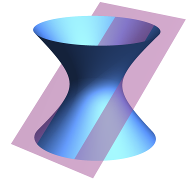
if and only if $\mathbf{b_1} = \mathbf{b_2}$.

The past infinity of the steady state space

Past infinity of steady state space

Fix a **null vector** $\mathbf{a} \in \mathbb{L}^{n+3}$, $a \neq 0$, and consider the null hypersurface in $\mathbb{S}_1^{n+2}$ given by

$$\mathcal{J}^- = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x \rangle = 0\}$$



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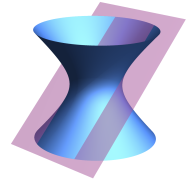

- Without loss of generality we may assume that $\mathbf{a}$ is **past-pointing**, $\langle \mathbf{a}, e_0 \rangle > 0$. The open region

$$\mathcal{H}^{n+2} = \{x \in \mathbb{S}_1^{n+2} : \langle x, \mathbf{a} \rangle > 0\}.$$

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The past infinity of the steady state space

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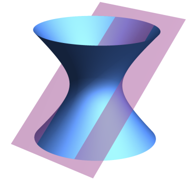

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  is the **steady state model** of the universe.
- The steady state space is a non-complete manifold, being only half of the de Sitter space and having as **boundary** the null hypersurface $\mathcal{J}^-$, which represents the **past infinity** of $\mathcal{H}^{n+2}$.

Marginally trapped submanifolds into $\mathcal{J}^-$

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Future-pointing normal null frame

In these conditions

$$\xi = -\mathbf{a} \quad \text{and} \quad \eta = -\frac{1 + \|\nabla u\|^2 + u^2}{2\langle \mathbf{a}, e_0 \rangle^2}\xi + \frac{1}{\langle \mathbf{a}, e_0 \rangle} e_0^\perp$$

gives two **future-pointing null normal** vector fields globally defined on Σ with $\langle \xi, \eta \rangle = -1$.

Null shape operators

The corresponding null second forms associated to the global null frame $\{\xi, \eta\}$ are given by

$$A_\xi = 0 \qquad \text{and} \qquad A_\eta = \frac{1}{\langle \mathbf{a}, e_0 \rangle}(\nabla^2 u + uI).$$

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$$\mathbf{H} = -\frac{1}{n\langle \mathbf{a}, e_0 \rangle}(\Delta u + nu)\xi$$

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Proposition 2

Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension-two spacelike submanifold which is contained in the **past infinite** of the steady state space. Then Σ is **always marginally trapped**, except at points where $\Delta u + nu = 0$ (if any), $u = -\langle \psi, e_0 \rangle$, where it is **minimal**.

Example 3

- For each smooth function $f : \mathbb{S}^n \to \mathbb{R}$, consider the embedding $\phi_f : \mathbb{S}^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ given by

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such that $\psi = \phi_f \circ \Psi$ where $f = u \circ \Psi^{-1}$ with $u = -\langle \psi, e_0 \rangle = \psi_0$.

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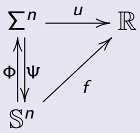

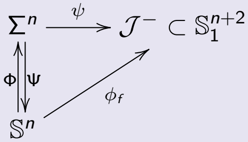

- In particular, the immersion ψ is an **embedding** and it is always **marginally trapped** except at points where $\Delta u + nu = 0$ (if any), where it is minimal.

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is a **local isometry**.
- Therefore, is we assume Σ to be **complete**, $\mathbb{S}^n$ being simply connected, we conclude that Ψ is in fact a **global isometry**.

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- Moreover:
  (I) Σ is minimal if and only if $c = 0$.
  (II) Σ is future marginally trapped if and only if $c < 0$.
  (III) Σ is past marginally trapped if and only if $c > 0$.

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- Therefore, the Laplacian of f satisfies $\Delta_0 f = -n(f - c)$ for a certain constant c.

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- The last assertions follow from (1) since $\mathbf{H} = c\mathbf{a}$, with $\mathbf{a}$ past-pointing.

Corollary 3

- Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension-two complete spacelike submanifold contained in $\mathcal{J}^-$.

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- Σ is **minimal** if and only if there exists an **isometry**

$$\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$$

such that $\psi = \phi_{\mathbf{b}} \circ \Psi$, where $\phi_{\mathbf{b}} : \mathbb{S}^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ is the embedding

$$\phi_{\mathbf{b}}(p) = (\langle p, \mathbf{b} \rangle_0, p, \langle p, \mathbf{b} \rangle_0).$$

for any $\mathbf{b} \in \mathbb{R}^{n+1}$.

A uniqueness result for the marginally trapped type equation on compact manifolds

- Motivated by the geometric meaning of the solutions to the partial differential equation $2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0$, we establish the following intrinsic uniqueness result for this equation.

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- The only positive solution to the partial differential equation

$$2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0 \qquad \text{(MT)}$$

on Σ is the constant function $u \equiv 1$.

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- Using this into (2) jointly with (MT) we obtain
$$\begin{aligned}\operatorname{div}(V) =& u^{-(n-1)}\left(\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n}\right) \\ & + u^{-(n-1)}\left(\operatorname{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2\right).\end{aligned}$$

- Integrating this and using the divergence theorem we obtain

$$\int_\Sigma u^{-(n-1)} \left(\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n} + \mathrm{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2 \right) = 0. \tag{3}$$

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- Since $K > (n-1)$, this last equation implies that u is constant and, by (MT) it must be $u \equiv 1$.

Remark: Theorem 2 is not true if $K = n - 1$

- When $K = n - 1$, if u is non-constant we conclude from the reasoning above that ∇u is a conformal vector field on Σ which is a direction of least Ricci curvature at points where $\nabla u(p) \neq 0$.

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- This is in fact what happens with the non-constant solutions given in Example 2, where

$$u(p) = f(p) = \frac{1}{\langle p, \mathbf{b} \rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}}$$

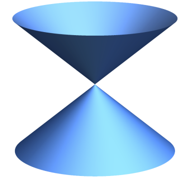
and $\Sigma = \mathbb{S}^n$ with the metric $\langle , \rangle = f^2 \langle , \rangle_0$.

Trapped submanifolds in the Lorentz-Minkowski space

Light cone of the Lorentz-Minkowski space

The **light cone** in $\mathbb{L}^{n+2}$ is the subset

$$\Lambda = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x \neq \mathbf{0}\}, \quad x = (x_1, \dots, x_{n+2}).$$



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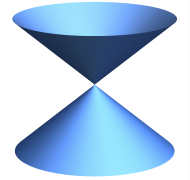

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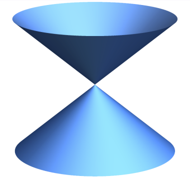

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- The **future** component of Λ is

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Future-pointing normal null frame

In these conditions

$$\xi = \psi \quad \text{and} \quad \eta = -\frac{1 + \|\nabla u\|^2}{2u^2}\xi + \frac{1}{u}e_1^\perp$$

gives two **future-pointing null normal** vector fields globally defined on Σ with $\langle \xi, \eta \rangle = -1$, where we are denoting

$$e_1 = e_1^\top(p) + e_1^\perp(p), \quad p \in \Sigma.$$

Null shape operators

The corresponding null second forms associated to the global null frame $\{\xi, \eta\}$ are given by

$$A_\xi = I \qquad \text{and} \qquad A_\eta = -\frac{1 + \|\nabla u\|^2}{2u^2} I + \frac{1}{u} \nabla^2 u,$$

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- In that case, it is necessarily past marginally trapped since $\theta_\xi = 1 > 0$.

- On the other hand, it follows from the Gauss equation that the Ricci and the scalar curvatures of Σ are given by

$$\mathrm{Ric}(X,Y) = (n-1)\langle \mathbf{H},\mathbf{H}\rangle\langle X,Y\rangle + \frac{n-2}{nu}(\Delta u\langle X,Y\rangle - n\mathrm{Hess}\,u(X,Y)),$$

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$$2u\Delta u - n(1 + \|\nabla u\|^2) \geq 0 \quad \text{on } \Sigma.$$

Examples

Example 4

- Let $\psi : \mathbb{R}^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be the map given by

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- In particular, $\nabla u(p) = \nabla^{\mathbb{R}^n} u(p) = p$ and $\Delta u(p) = \Delta_{\mathbb{R}^n} u(p) = n$, and u satisfies

$$2u\Delta u - n(1 + \|\nabla u\|^2) = n(\|p\|^2 + 1) - n(1 + \|p\|^2) = 0$$

which means ψ is a marginally trapped immersion of $\mathbb{R}^n$ into Λ^+.

Example 5

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- Therefore, Σ is a weakly trapped submanifold, and it is marginally trapped if, and only if $n = 2$.

Non-existence of weakly marginally trapped submanifolds into the light cone

Our first result establishes the non-existence of compact weakly trapped submanifolds into $\mathbb{L}^{n+2}$

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- The proof of Proposition 4 follows from that fact that

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- Therefore $\Delta u > 0$ (or $\Delta u < 0$) on Σ and from the divergence theorem we have

$$\int_\Sigma \Delta u \mathrm{d}\Sigma = 0$$

  what implies $\Delta u \equiv 0$ and gives us a contradiction.

As a consequence, and using a compactness result for codimension two spacelike submanifolds into the light cone of $\mathbb{L}^{n+2}$, we have the following.

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There is no codimension two complete weakly trapped immersed submanifold in $\Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$ satisfies

$$u \leq Cr \log r, \qquad r >> 1.$$

In particular, there is no codimension two complete weakly trapped immersed submanifold in $\Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function u is bounded from above.

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More generally, with the aid of the weak maximum principle we can extend this non-existence result to stochastically complete submanifolds as follows

Theorem 3

There is no codimension two stochastically complete weakly trapped immersed submanifold in $\Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function u is bounded from above.

Stochastic completeness and the weak maximum principle

- The **weak maximum principle** is said to hold on Σ if, for any $u \in \mathcal{C}^2(\Sigma)$ with $u^* < +\infty$ there is a sequence $\{p_k\}_{k\in\mathbb{N}}$ in Σ with

$$\text{(i)} \quad u(p_k) > u^* - \frac{1}{k}, \quad \text{and} \quad \text{(ii)} \quad \Delta u(p_k) < \frac{1}{k}.$$

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- This is equivalent (among other conditions) to the fact that for every $\lambda > 0$, the only non-negative bounded smooth solution u of $\Delta u \geq \lambda u$ on Σ is the constant $u = 0$.
- In particular, every **parabolic** manifold is stochastically complete. Hence, the weak max principle holds on every parabolic manifold.

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- Putting this into (4) we obtain

$$n \leq n(1 + \|\nabla u(p_k)\|^2) \leq 2u(p_k)\Delta u(p_k) < 2\frac{u(p_k)}{k},$$

and making $k \to +\infty$ we get

$$n \leq 0$$

which is not possible.

That's all !!

Thank you very much for your attention