Four-dimensional quasi-Einstein manifolds

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Miguel Brozos Vázquez

Departamento de Matemáticas

Escola Politécnica Superior

UNIVERSIDADE DA CORUÑA

Joint work with:

- Eduardo García Río,
- Peter Gilkey, and
- Xabier Valle Regueiro

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- M differentiable manifold of dimension 4
- g Pseudo-Riemannian metric.

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Curvature

1. ∇ denotes the Levi-Civita connection.
2. $R(x, y) = \nabla_{[x,y]} - [\nabla_x, \nabla_y]$ is the curvature operator.

For an orthonormal basis $\{e_1, \ldots, e_4\}$ with $\varepsilon_i = g(e_i, e_i)$:

Ricci tensor

$\rho(x, y) = \sum_i \varepsilon_i R(x, e_i, y, e_i) = g(Ric(x), y)$

Scalar curvature

$\tau = \sum_i \varepsilon_i \rho(e_i, e_i)$

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Weyl tensor

$$W(x, y, z, t) = R(x, y, z, t) + \frac{\tau}{6}\{g(x, z)g(y, t) - g(x, t)g(y, z))\}$$
$$+\frac{1}{2}\{\rho(x, t)g(y, z) - \rho(x, z)g(y, t) + \rho(y, z)g(x, t) - \rho(y, t)g(x, z)\}$$

Quasi-Einstein manifolds

Bakry-Émery-Ricci tensor on a manifold with density

Let (M, g) be a pseudo-Riemannian manifold and f a function on M. Then

$$\rho_f^\mu = \mathrm{Hes}_f + \rho - \mu df \otimes df, \text{ for } \mu \in \mathbb{R}.$$

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Einstein manifolds

For f constant, the QEE reduces to the Einstein equation:

$$\rho = \lambda g,$$

where $\lambda = \frac{\tau}{4}$ is constant.

Quasi-Einstein manifolds generalize other well-known families

Gradient Ricci almost solitons

For $\mu = 0$, the QEE reduces to the gradient Ricci almost soliton equation:

$$\mathrm{Hes}_f + \rho = \lambda g, \text{ for } \lambda \in \mathcal{C}^\infty(M)$$

- When λ is constant this is the *gradient Ricci soliton equation*, which identifies self-similar solutions of the Ricci flow: $\frac{\partial}{\partial t} g(t) = -2\rho(t)$.
- For $\lambda = \kappa\tau + \nu$, this identifies *$\kappa$-Einstein solitons*, which are self-similar solutions of the *Ricci-Bourguignon flow*: $\partial_t g(t) = -2(\rho(t) - \kappa\tau(t)\, g(t))$, $\kappa \in \mathbb{R}$.

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Conformally Einstein manifolds

The value $\mu = -\frac{1}{2}$ is exceptional :

$$(M, g) \text{ is generalized quasi-Einstein} \Leftrightarrow (M, e^{-f} g) \text{ is Einstein.}$$

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Static space-times

For $\mu = 1$, $h = e^{-f}$ and $\lambda = -\frac{\Delta h}{h}$, QEE becomes the defining equation of *static manifolds*:

$$\mathrm{Hes}_h - h\rho = \Delta h g.$$

Motivation of this talk

The QEE provides information directly on the Ricci tensor.

Decomposition of the curvature tensor

The space of curvature tensor decomposes under the action of the orthogonal group into orthogonal modules as follows:

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- $n = 4$: $\begin{cases} \mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \oplus \mathfrak{R}_3^+ \oplus \mathfrak{R}_3^- \\ \tau = Tr(\rho) \quad \rho_0 = \rho - \frac{\tau}{n} g \quad W^+ \quad W^- \end{cases}$

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A manifold is said to be **half conformally flat** if either $W^- = 0$ or $W^+ = 0$.

It seems reasonable to impose conditions on the Weyl tensor to obtain partial classification results for QE manifolds.

Motivation of this talk

There are natural conditions that one can impose related to the structure of the Weyl tensor:

- $W = 0$: (M, g) is locally conformally flat.
- $W^{\pm} = 0$: (M, g) is half conformally flat.
- $\operatorname{div}_4 W = 0$: the Weyl tensor is harmonic.

$$\operatorname{div}_4 W(X, Y, Z) = -\tfrac{1}{2} C(X, Y, Z) =$$
$$(\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z) - \tfrac{1}{6}(X(\tau)g(Y, Z) - Y(\tau)g(X, Z)).$$

- The Cotton tensor is preserved by a conformal change of the form $\tilde{g} = e^{-f} g$:
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Aim of the talk

1. To understand the local structure of **quasi-Einstein manifolds** in dimension four under "reasonable conditions" on the Weyl tensor.
2. To find examples with some of the conditions above but $W \neq 0$.

Basic equations and causal character of ∇f

Basic relations:

1. $\tau + \Delta f - \mu \|\nabla f\|^2 = n\lambda.$
2. $\nabla\tau + 2\mu(3\lambda - \tau)\nabla f + 2(\mu - 1)\,\mathrm{Ric}(\nabla f) = 6\nabla\lambda.$
3. $$R(X, Y, Z, \nabla f) = d\lambda(X)g(Y, Z) - d\lambda(Y)g(X, Z) + (\nabla_Y \rho)(X, Z) \\ -(\nabla_X \rho)(Y, Z) + \mu\,\{df(Y)\,\mathrm{Hes}_f(X, Z) - df(X)\,\mathrm{Hes}_f(Y, Z)\}.$$
4. Let $\eta = 2\mu + 1$. Then
$$W(X, Y, Z, \nabla f) = -C(X, Y, Z) + \frac{\tau\eta\{df(Y)g(X,Z) - df(X)g(Y,Z)\}}{6} \\ + \frac{\eta\{\rho(X,\nabla f)g(Y,Z) - \rho(Y,\nabla f)g(X,Z)\}}{6} + \frac{\eta\{\rho(Y,Z)df(X) - \rho(X,Z)df(Y)\}}{2}.$$

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In general, if (M,g) is QE, ∇f may have different causal characters.

We say that a gradient Ricci soliton (M,g,f) is

- **isotropic** if $\|\nabla f\| = 0$: the level sets of f are degenerate hypersurfaces.
- **non-isotropic** if $\|\nabla f\| \neq 0$: the level sets of f are non-degenerate hypersurfaces.

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Non-isotropic 4-dimensional manifolds

Theorem

Let (M, g) be a non-isotropic generalized QE manifold of dimension 4 with $\mu \neq -\frac{1}{2}$ and satisfying

- the Weyl tensor is harmonic and $W(\cdot, \nabla f, \cdot, \nabla f) = 0$, or
- $W^+ = 0$.

Then (M, g) decomposes locally as a warped product of the form $I \times_\phi N$, where N has constant sectional curvature. Hence (M, g) is locally conformally flat.

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Previous works in Riemannian signature:

- G. Catino; Generalized quasi-Einstein manifolds with harmonic Weyl tensor, *Math. Z.* **271** (2012).
- X. Chen, Y. Wang; On four-dimensional anti-self-dual gradient Ricci solitons, *J. Geom. Anal.* **25** 2, (2011).
- G. Catino; A note on four-dimensional (anti-)self-dual quasi-Einstein manifolds, *Differential Geom. Appl.*, **30** 6, (2012).

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$W(X, Y, Z, \nabla f) = -C(X, Y, Z) + \frac{\tau\eta\{df(Y)g(X,Z) - df(X)g(Y,Z)\}}{6}$
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3.
   - **∇f generates a totally geodesic distribution.**
   - **The level sets of f are totally umbilical hypersurfaces.**
     Use previous relations to show that:
     $\mathrm{Hes}_f(E_i, E_j) = (\lambda + \frac{1}{5}(\rho(V, V)\varepsilon - \tau))g(E_i, E_j)$.

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4. (M, g) is a **twisted** product $I \times_\omega N$.
5. Since $\rho(V, E_i) = 0$, the twisted product reduces to a **warped** product of the form
   $$(M, g) = (I \times N, \varepsilon dt^2 + \psi(t)^2 g_N)$$

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   $$(M, g) = (I \times N, \varepsilon dt^2 + \psi(t)^2 g_N)$$
6. Since the Weyl tensor is harmonic, (N, g_N) is Einstein and of dimension 3. Hence (M, g) is locally conformally flat.

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Isotropic 4-dimensional manifolds: the Lorentzian setting

Theorem

Let (M, g) be an isotropic generalized QE Lorentzian manifold of dimension 4 with $\mu \neq -\frac{1}{2}$. If

- the Weyl tensor is harmonic, and
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then

- $\lambda = 0$, and
- (M, g) is a *pp*-**wave**, i.e., (M, g) is locally isometric to $\mathbb{R}^2 \times \mathbb{R}^2$ with metric
$$g = 2\, dudv + H(u, x_1, x_2)du^2 + dx_1^2 + dx_2^2.$$

Sketch of the proof. Isotropic Lorentzian case.

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4. $\mathcal{D} = \mathrm{span}\{\nabla f\}$ is a null parallel distribution: Walker manifold.
5. $0 = R(E_i, U, \nabla f, E_i) = \frac{\lambda}{3} \Rightarrow \lambda = 0$.

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5. $0 = R(E_i, U, \nabla f, E_i) = \frac{\lambda}{3} \Rightarrow \lambda = 0.$
6. $R(\nabla f^{\perp}, \nabla f^{\perp}, \cdot, \cdot) = 0$ and the Ricci tensor is isotropic, so (M, g) **is a pp-wave**.

Isotropic Lorentzian *pp*-waves

Locally conformally flat quasi-Einstein *pp*-waves

A locally conformally flat *pp*-wave is locally isometric to $\mathbb{R}^2 \times \mathbb{R}^2$ with metric

$$g = 2\,dudv + H(u, x_1, x_2)du^2 + dx_1^2 + dx_2^2$$

where $H(u, x_1, x_2) = a(u)(x_1^2 + x_2^2) + b_1(u)x_1 + b_2(u)x_2 + c(u)$, and it is isotropic QE for f a function of u satisfying $f''(u) - \mu f'(u)^2 - 2a(u) = 0$.

—, E. García-Río and S. Gavino-Fernández; Locally conformally flat Lorentzian quasi-Einstein manifolds. *Monatsh. Math.* 173 (2014), 175–186.

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Non-locally conformally flat quasi-Einstein *pp*-waves

Let (M, g) be a *pp*-wave with $W \neq 0$, the following statements are equivalent:

- (M, g) is isotropic generalized quasi-Einstein,
- W is harmonic,
- $\Delta_x H = \phi(u)$.

If any of these conditions holds, then $W(\cdot, \cdot, \cdot, \nabla f) = 0$ and f is given by:

$$f''(u) + \mu f'(u)^2 - \frac{1}{2}\left(\frac{\partial^2 H}{\partial x_1^2}(u, x_1, x_2) + \frac{\partial^2 H}{\partial x_2^2}(u, x_1, x_2)\right) = 0$$

Index

Isotropic half conformally flat QE manifolds of signature $(2,2)$

Theorem

Let (M,g) be a **self-dual** isotropic generalized-quasi Einstein manifold of signature $(2,2)$, with $\mu \neq -\frac{1}{2}$. Then (M,g) is a Walker manifold with a 2-dimensional null parallel distribution.

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Walker metrics

The metric of a Walker manifold can be written in local coordinates as:

$$g_W(x_1,x_2,x_3,x_4) = \begin{pmatrix} a(x_1,x_2,x_3,x_4) & c(x_1,x_2,x_3,x_4) & 1 & 0 \\ c(x_1,x_2,x_3,x_4) & b(x_1,x_2,x_3,x_4) & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Sketch of the proof. Isotropic case.

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1. Choose a local appropriate frame of null vectors: $\{\nabla f, u, v, w\}$

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The self-dual condition expresses as:

$$\begin{aligned} W(\nabla f, v, z, t) &= W(u, w, z, t)\,, \\ W(u, v, z, t) &= 0\,, \\ W(\nabla f, w, z, t) &= 0\,. \end{aligned}$$

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2. Use these relations to show that $\lambda = \frac{\tau}{4}$ and the Ricci operator has the form:

$$\mathrm{Ric} = \begin{pmatrix} \lambda & 0 & a & c \\ 0 & \lambda & c & b \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

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There are several families of Walker manifolds that will play a role:

1. Deformed Riemannian extensions,
2. Modified Riemannian extensions.

Riemannian extensions

$$\begin{array}{c}(T^*\Sigma, g_D)\\ \big\downarrow \pi \\ (\Sigma, D)\end{array}$$

Reference:

Patterson and Walker; Riemann extensions, *Quart. J. Math., Oxford Ser. (2)* 3 1952.

Riemannian extensions

$$(T^*\Sigma, g_D) \xrightarrow{\pi} (\Sigma, D)$$

Parallel null distribution: $\ker \pi_*$

$$g_D(X^C, Y^C) = -\iota(D_X Y + D_Y X)$$

In local coordinates $(x_1, x_2, x_{1'}, x_{2'})$:

$$g_D = \begin{pmatrix} -2x_{1'}\Gamma_{11}^1 - 2x_{2'}\Gamma_{11}^2 & -2x_{1'}\Gamma_{12}^1 - 2x_{2'}\Gamma_{12}^2 & 1 & 0 \\ -2x_{1'}\Gamma_{12}^1 - 2x_{2'}\Gamma_{12}^2 & -2x_{1'}\Gamma_{22}^1 - 2x_{2'}\Gamma_{22}^2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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- $(T^*\Sigma, g_D)$ self-dual.
- $(T^*\Sigma, g_D)$ Einstein $\Leftrightarrow \rho^D_{sym} = 0$.
- $(T^*\Sigma, g_D)$ locally conformally flat $\Leftrightarrow$ (Σ, D) projectively flat.

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Deformed Riemannian extensions

Φ is a $(0,2)$-symmetric tensor field on Σ

$$(T^*\Sigma, g_{D,\Phi}) \xrightarrow{\pi} (\Sigma, D, \Phi)$$

$$g_{D,\Phi}(X^C, Y^C) = -\iota(D_X Y + D_Y X) + \pi^*\Phi$$

In local coordinates:

$$g_D = \begin{pmatrix} -2x_{1'}\Gamma_{11}^1 - 2x_{2'}\Gamma_{11}^2 + \Phi_{11} & -2x_{1'}\Gamma_{12}^1 - 2x_{2'}\Gamma_{12}^2 + \Phi_{12} & 1 & 0 \\ -2x_{1'}\Gamma_{12}^1 - 2x_{2'}\Gamma_{12}^2 + \Phi_{21} & -2x_{1'}\Gamma_{22}^1 - 2x_{2'}\Gamma_{22}^2 + \Phi_{22} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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(Σ, D, Φ)

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Isotropic half conformally flat QE manifolds of signature $(2,2)$

Quasi-Einstein manifolds with λ constant

Let (M,g) be an isotropic self-dual quasi-Einstein manifold of signature $(2,2)$ with $\mu \neq -\frac{1}{2}$ which is not Ricci flat. Then (M,g) is locally isometric to a deformed Riemannian extension $(T^*\Sigma, g_{D,\phi})$ of an affine surface (Σ, D) that satisfies the affine quasi-Einstein equation:

$$\mathrm{Hes}_{\hat{f}}^{D} + 2\rho_s^D - \mu d\hat{f} \otimes d\hat{f} = 0 \text{ for some } \hat{f} \in \mathcal{C}^\infty(\Sigma) \text{ and } \mu \in \mathbb{R}$$

and, moreover, $f = \pi^*\hat{f}$ and $\lambda = 0$.

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Remarks.

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Remarks.

- There exist examples of self-dual QE manifolds which are NOT locally conformally flat in dimension four.

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Remarks.

- There exist examples of self-dual QE manifolds which are NOT locally conformally flat in dimension four.
- The previous result suggest the new concept of affine quasi-Einstein manifold:
  (N, D) is *quasi-Einstein* if there exist a function $\hat{f}$ in N satisfying the *affine quasi-Einstein equation*
  $$\mathrm{Hes}_{\hat{f}}^D + 2\rho_s^D - \mu d\hat{f} \otimes d\hat{f} = 0.$$

Sketch of the proof. Isotropic case. λ constant

1. We use the previous pseudo-orthonormal frame $\{\nabla f, u, v, w\}$ where

$$\mathrm{Ric} = \begin{pmatrix} \lambda & 0 & a & c \\ 0 & \lambda & c & b \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

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2. We use that $\lambda = \mathit{const}.$ to see that $\lambda = 0$, $\tau = 0$ and $\mathrm{Ric}(\nabla f) = 0$.

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4. We see that $R(\cdot, \mathcal{D})\mathcal{D} = 0$, this shows that (M, g) is indeed a **deformed Riemannian extension**.

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4. We see that $R(\cdot, \mathcal{D})\mathcal{D} = 0$, this shows that (M, g) is indeed a **deformed Riemannian extension**.
5. We work in local coordinates and check that the condition for a deformed Riemannian extension to be quasi-Einstein is equivalent to the condition for the affine surface to be **affine quasi-Einstein**.

Reference:

Afifi; Riemann extensions of affine connected spaces, *Quart. J. Math., Oxford Ser. (2)* 5 1954.

Isotropic half conformally flat QE manifolds of signature $(2, 2)$

Generalized Quasi-Einstein manifolds (λ non-constant)

Let (M, g) be an isotropic self-dual generalized quasi-Einstein manifold of signature $(2, 2)$ with $\mu \neq \frac{1}{2}$ which is not Ricci flat. If λ is not constant then (M, g) is locally isometric to a modified Riemannian extension $(T^*\Sigma, g_{D,\Phi,T,Id})$ of an affine surface (Σ, D) with:

- $\Phi = \frac{2}{C} e^{\hat{f}} (\mathrm{Hes}^D_{\hat{f}} + 2\rho^D_s - \mu d\hat{f} \otimes d\hat{f})$,
- $T = Ce^{-\hat{f}} Id$,
- $\lambda = \frac{3}{2} Ce^{-f}$.

Modified Riemannian extensions

Φ is a $(0,2)$-symmetric tensor field on Σ

$$(T^*\Sigma, g_{D,\Phi}) \xrightarrow{\pi} (\Sigma, D, \Phi)$$

Parallel null distribution: $\ker \pi_*$

$g_{D,\Phi} \quad (X^{\mathcal{C}}, Y^{\mathcal{C}}) = -\iota(D_X Y + D_Y X) + \pi^*\Phi$

In local coordinates:

$$g_{D,\Phi} = \begin{pmatrix} g_{11} & g_{12} & 1 & 0 \\ g_{12} & g_{22} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$g_{ij} = -2\sum_k x_{k'} \Gamma_{ij}^k + \Phi_{ij}$$

Modified Riemannian extensions

Φ is a $(0,2)$-symmetric tensor field on Σ

T and S are $(1,1)$-tensor fields on Σ

$$(T^*\Sigma, g_{D,\Phi}) \xrightarrow{\pi} (\Sigma, D, \Phi)$$

Parallel null distribution: $\ker \pi_*$

$$g_{D,\Phi,T,S}(X^{\mathcal{C}}, Y^{\mathcal{C}}) = \iota T \circ \iota S - \iota(D_X Y + D_Y X) + \pi^*\Phi$$

In local coordinates:

$$g_{D,\Phi,T,S} = \begin{pmatrix} g_{11} & g_{12} & 1 & 0 \\ g_{12} & g_{22} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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Self-dual Walker manifolds (E. Calviño-Louzao, E. García-Río, R. Vázquez-Lorenzo)

A four-dimensional Walker metric is self-dual if and only if it is locally isometric to the cotangent bundle $(T^*\Sigma, g)$, where

$$g = \iota X(\iota\,\mathrm{Id} \circ \iota\,\mathrm{Id}) + \iota T \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi$$

Methods to construct examples

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1. Take any affine surface (Σ, D).

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Then $(T^*\Sigma, g_{D,\Phi,T,\mathrm{Id}}, f)$ is a self-dual generalized QE manifold with $\lambda = \frac{1}{4}\tau = \frac{3}{2} Ce^{-f}$.

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The affine quasi-Einstein equation

For an affine manifold (N, D) consider the QEE

$$\mathrm{Hes}^D_{\hat{f}} + 2\rho^D_s - \mu d\hat{f} \otimes d\hat{f} = 0.$$

Consider the change of variable $f = e^{-\frac{1}{2}\mu\hat{f}}$ to transform the equation into

$$\mathrm{Hes}_f = \mu f \rho_s.$$

Let $E(\mu)$ be the space of solutions for the affine QEE.

First results for the affine QEE.

If $f \in E(\mu)$ then

1. If X is Killing, then $Xf \in E(\mu)$.
2. $f \in C^\infty(N)$ and, if N is real analytic, then f is real analytic.
3. If $f(p) = 0$ and $df(p) = 0$, then $f = 0$ near p.
4. $\dim(E(\mu)) \leq \dim N + 1$.

References

Main references:

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Thank you!

Four-dimensional quasi-Einstein manifolds

Miguel Brozos Vázquez
Departamento de Matemáticas
Escola Politécnica Superior
UNIVERSIDADE DA CORUÑA

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Joint work with:

- Eduardo García Río,
- Peter Gilkey, and
- Xabier Valle Regueiro