Erasmo Caponio

Polytechnic University of Bari, Italy

On a Finslerian generalisation of standard static and stationary spacetimes

IX INTERNATIONAL MEETING ON LORENTZIAN GEOMETRY

17-23 June 2018 WARSAW

Based on :

E.C. - G. Stancarone

- "Standard static Finsler spacetimes",
  Int. J. Geom. Meth. Modern Phys. (2016).

- " On Finsler spacetimes with a timelike Killing vector field",
  Class. Quantum Grav. (2018).

Finsler manifold

M smooth manifold , $\dim M = n$

$F : TM \longrightarrow [0, \infty)$, $F \in C^0(TM) \cap C^k(TM \setminus 0)$

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Whenever $F(v) = F(-v)$, F is said reversible

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- $F(v) = 0 \iff v = 0$
- $F(\lambda v) = \lambda F(v)$, for all $\lambda > 0$ and $v \in TM$
- for all $v \in TM \setminus 0$ and $u_1, u_2 \in TM$, define

$$g_v(u_1, u_2) := \frac{\partial^2}{\partial s \partial t} \frac{1}{2} F^2(v + s u_1 + t u_2) \Big|_{(s,t)=(0,0)}$$

$$g_v : T_{\pi(v)} M \times T_{\pi(v)} M \longrightarrow \mathbb{R}$$

is positive definite for all $v \in TM \setminus 0$

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g_v is called the fundamental tensor of F

or the vertical (or fiberwise) Hessian of F^2

In particular Finsler geometry generalises Riemannian one :

If h is a Riemannian metric on M

$F(v) := \sqrt{h(v,v)}$ is Finsler.

This is the unique case where

$g \; (\equiv h)$ does not depend on $v \in TM \setminus 0$

but only on $x \; (= \pi(v)) \in M$

$\pi : TM \longrightarrow M$

$TM \setminus 0 , \quad \pi_{|TM \setminus 0}$

$\pi : TM \to M$

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$\pi^*(T^*M)$ over $TM \setminus 0$

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$\pi^*(T^*M)$ over $TM \setminus 0$

$$\rightsquigarrow$$

$$g \in \Gamma\left(\pi^*(T^*M) \otimes \pi^*(T^*M)\right)$$

symmetric and positive definite

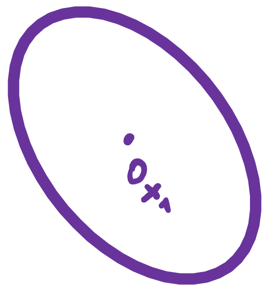
Indicatrix at $x \in M$:

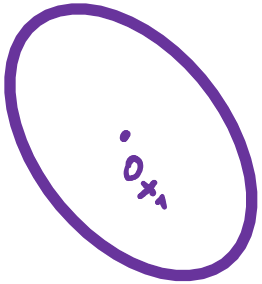
$$S_x = \{ v \in T_x M : F(v) = 1 \}$$

Due to the 0-homogeneity of g ($g_{\lambda v} = g_v$)
g can also viewed as a section of

$$\pi^*(T^*M) \otimes \pi^*(T^*M)$$

but now $\pi^*(T^*M)$ over the
indicatrix bundle $S = \bigcup_{x \in M} S_x$





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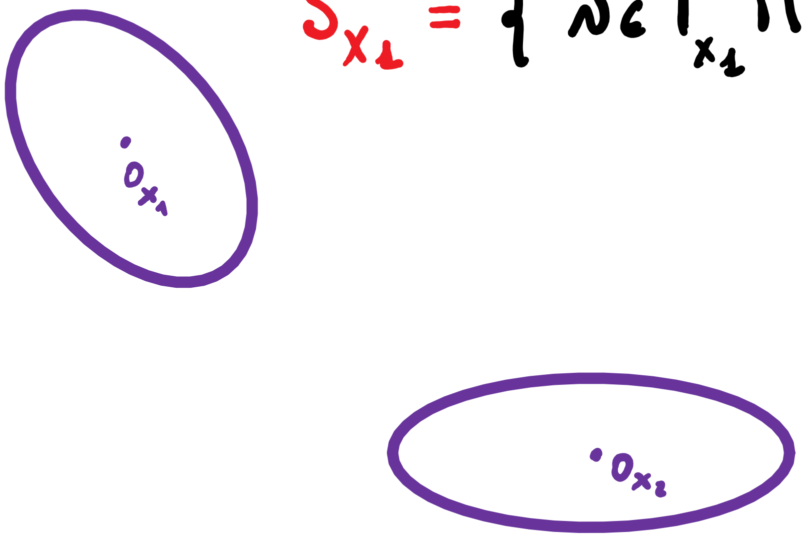

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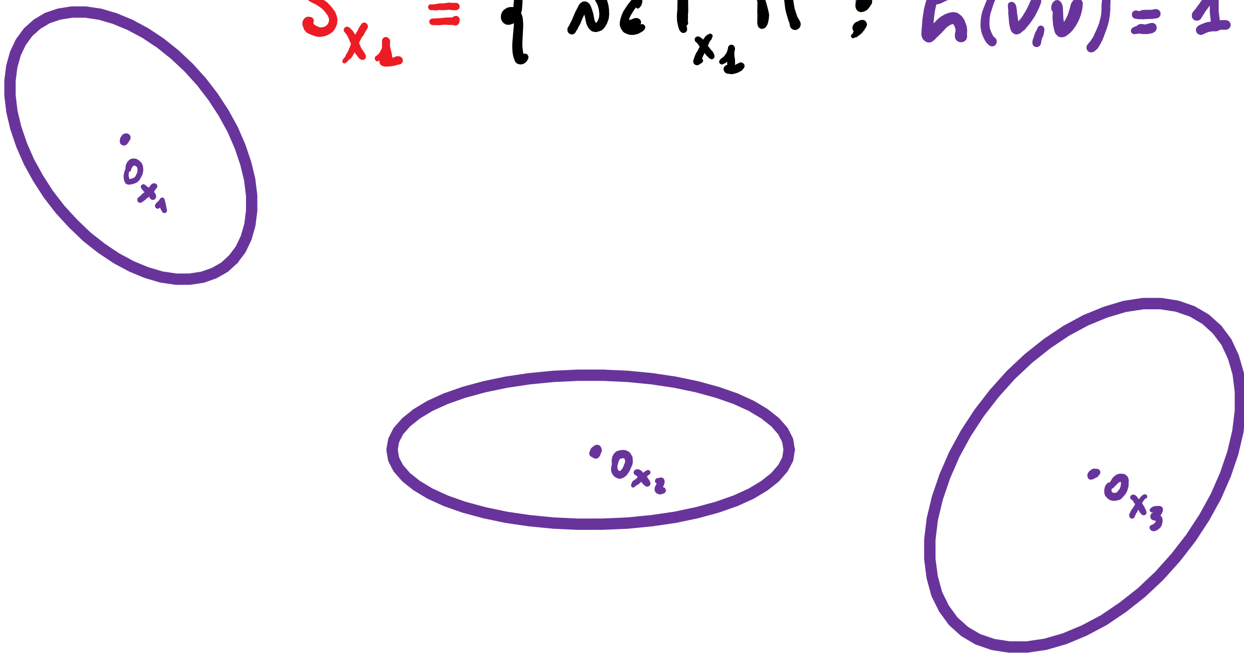

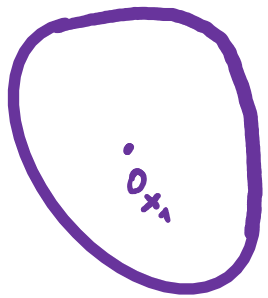

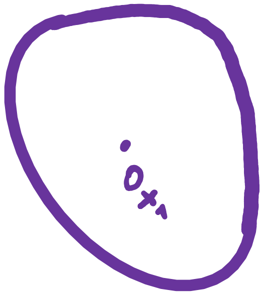

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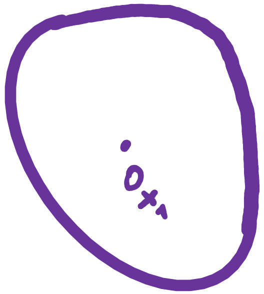

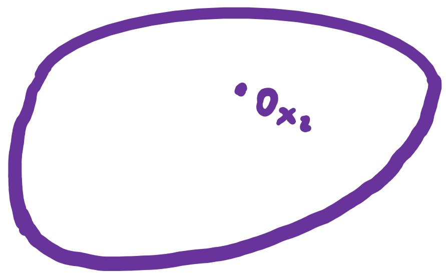

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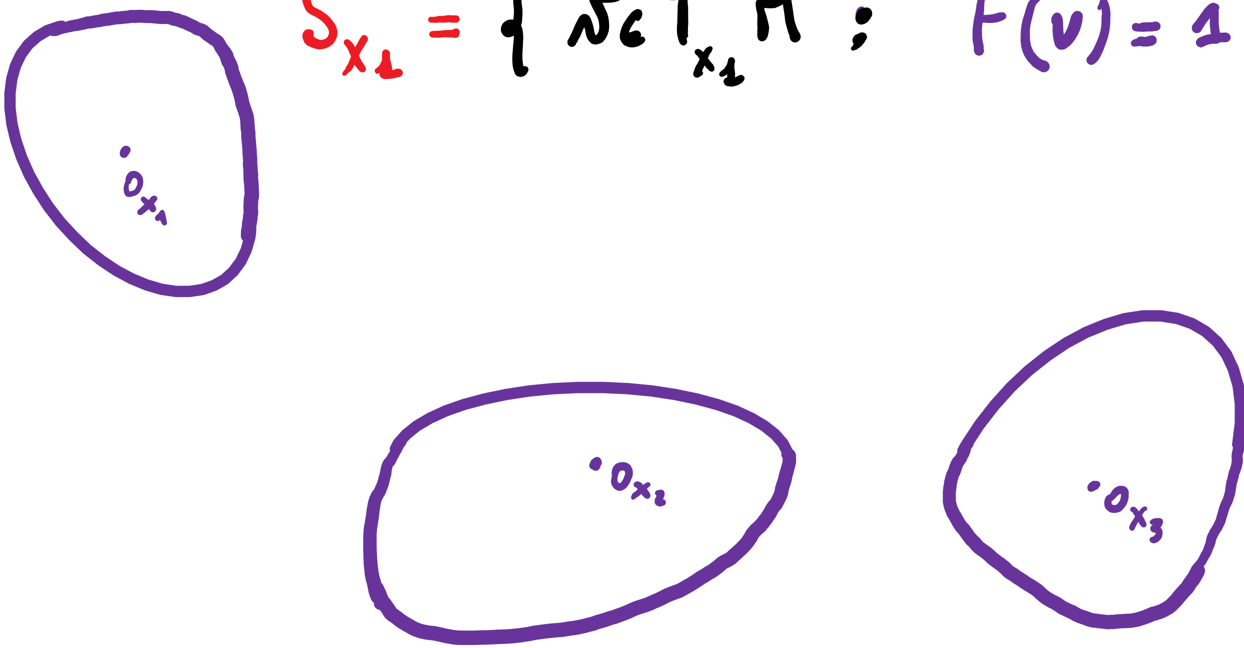
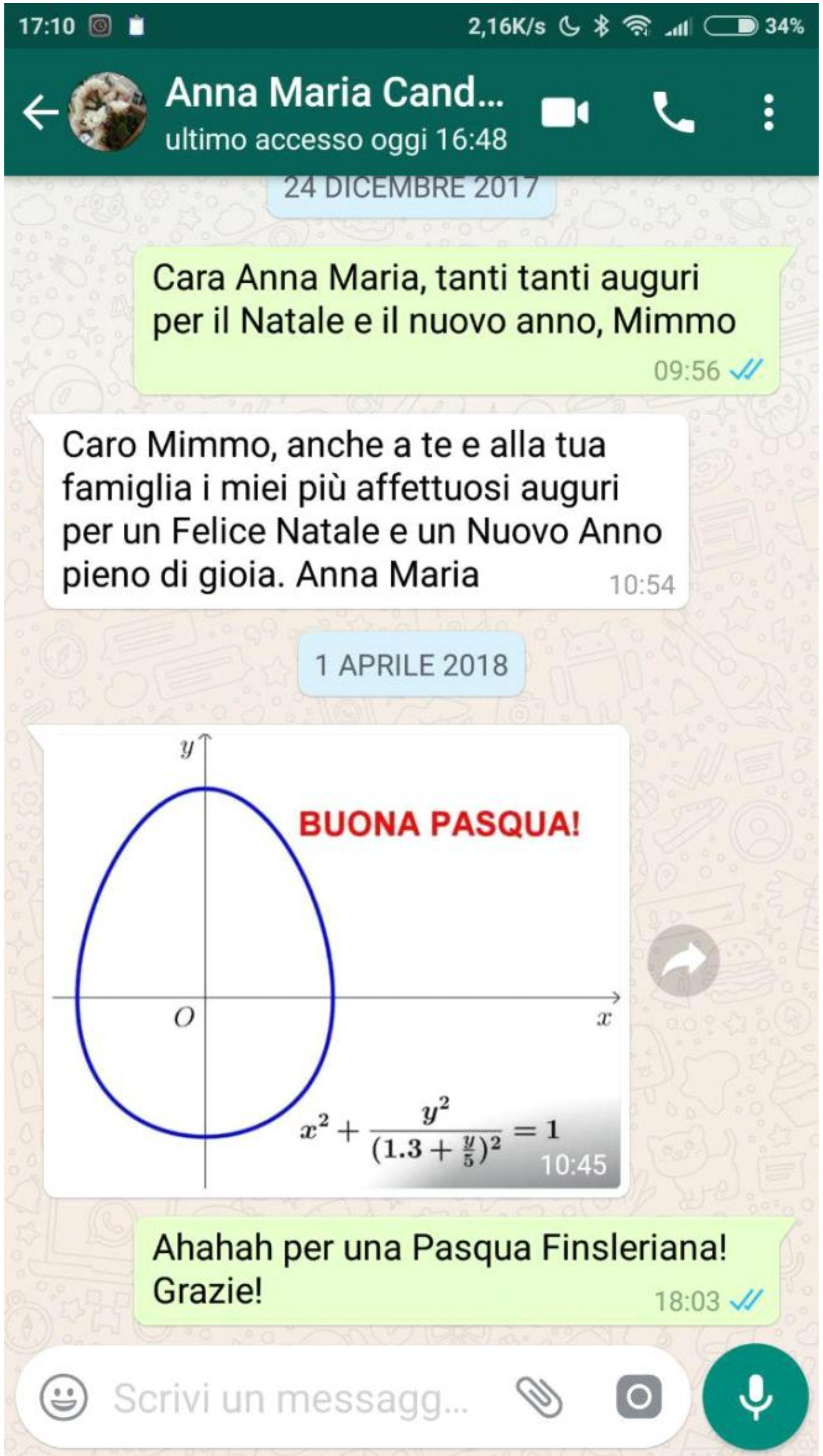
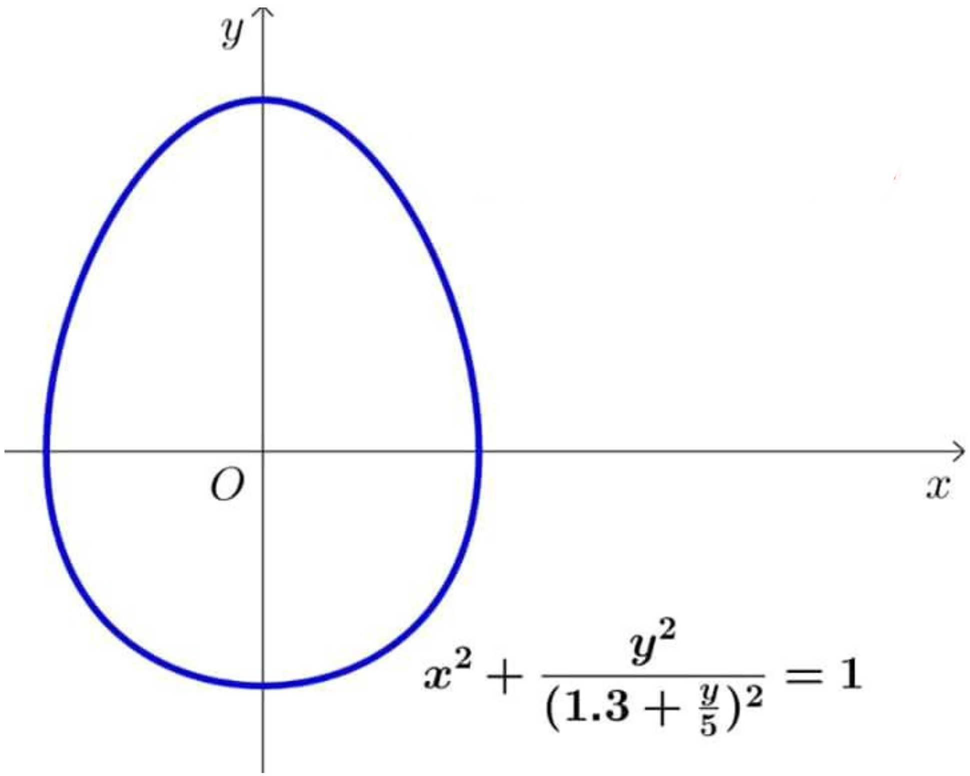



17:10
2,16K/s 34%
Anna Maria Cand...
ultimo accesso oggi 16:48
24 DICEMBRE 2017
Cara Anna Maria, tanti tanti auguri per il Natale e il nuovo anno, Mimmo
09:56
Caro Mimmo, anche a te e alla tua famiglia i miei più affettuosi auguri per un Felice Natale e un Nuovo Anno pieno di gioia. Anna Maria
10:54
1 APRILE 2018
y
BUONA PASQUA!
O
x
$x^2 + \frac{y^2}{(1.3 + \frac{y}{5})^2} = 1$
10:45
Ahahah per una Pasqua Finsleriana! Grazie!
18:03
Scrivi un messagg...


y
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x
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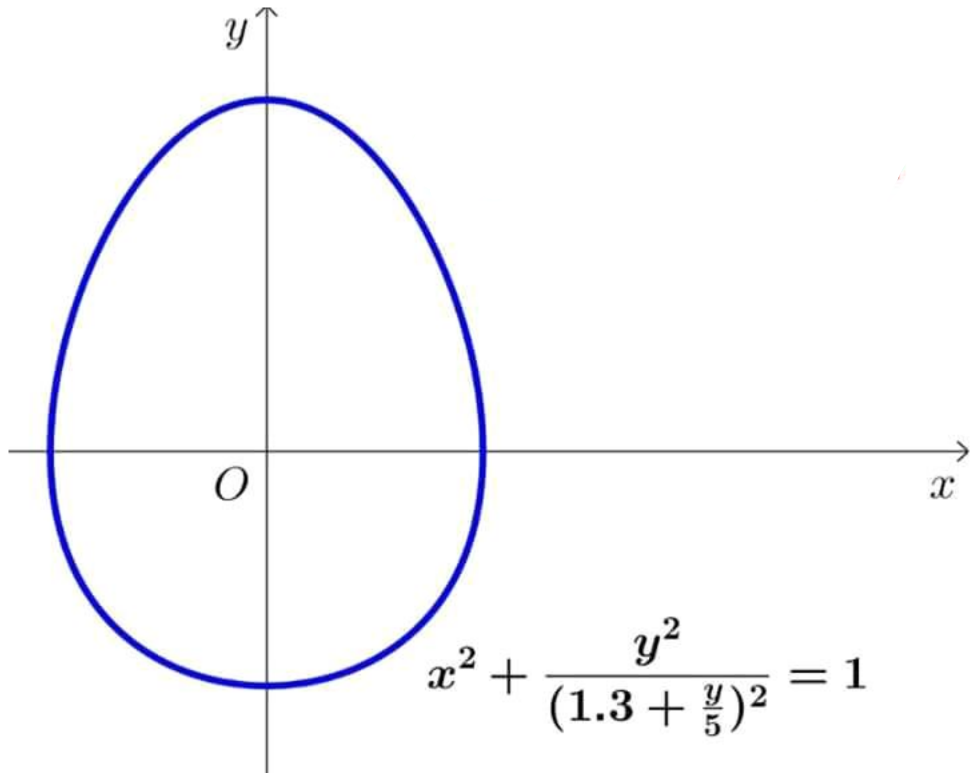


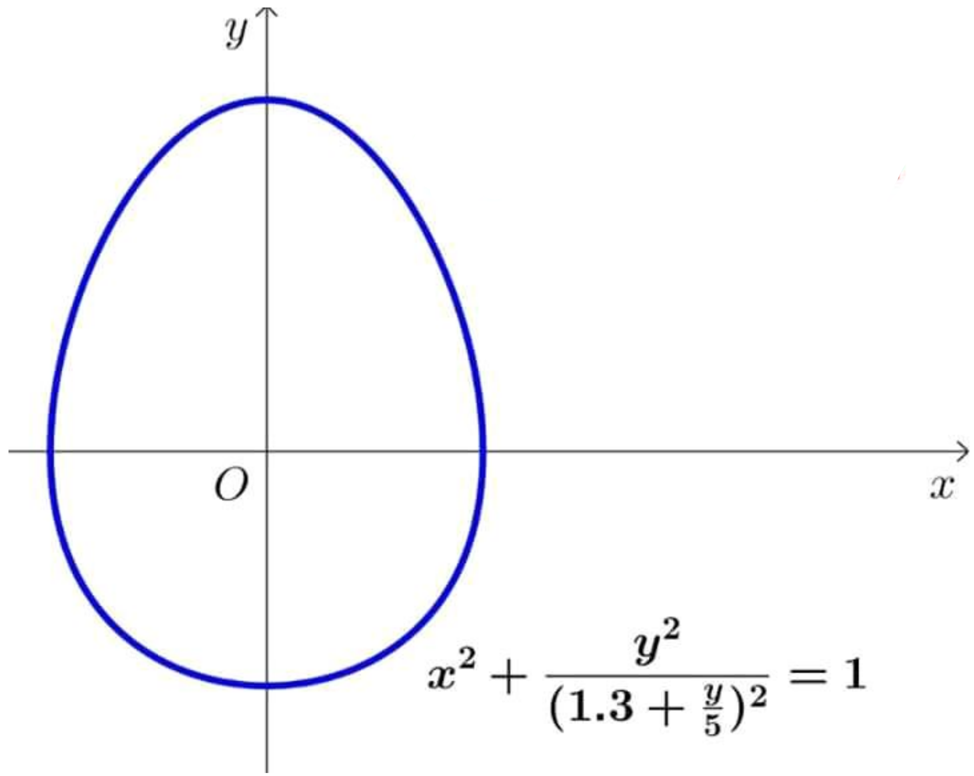
Here $v = (x,y) \in \mathbb{R}^2$ and

$$G(x,y) := x^2 + \frac{y^2}{(1.3+\frac{y}{5})^2}.$$

For any $(x,y) \in \mathbb{R}^2$:

$$F(x,y) = \lambda \quad F\left(\frac{1}{\lambda}x, \frac{1}{\lambda}y\right) = \lambda$$

iff $\left(\frac{1}{\lambda}x, \frac{1}{\lambda}y\right) \in G^{-1}(\{1\})$



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So we must solve

$$G\left(\frac{1}{\lambda}x, \frac{1}{\lambda}y\right) = 1,$$

w.r.t. λ in order to get

$$\lambda = F(x,y)$$

Input interpretation:

solve	$x^2 + \dfrac{y^2}{\left(1.3 + \frac{y}{5\,z}\right)^2} = z^2$
	$z \geq 0$

Solution over the reals: Exact form More digits

$x < 0$ and $y < 0$ and

$$
\begin{aligned}
z = &-0.5 \sqrt{(}0.00197239\,(-169\,x^2 - 96\,y^2) + 0.00591716 \\
&(169\,x^2 + 96\,y^2) + 0.00197239 \\
&(-4.82681\times10^6\,x^6 - 8.22557\times10^6\,x^4\,y^2 - 8.32291\times \\
&10^6\,x^2\,y^4 + 2702.\sqrt{(}4.82681\times10^6\,x^8\,y^4 + \\
&8.22557\times10^6\,x^6\,y^6 + 6.49771\times10^6\,x^4 \\
&y^8 + 884736\,x^2\,y^{10}) - 884736\,y^6)\,\hat{}\, \\
&(1/3) + (0.00197239\,(169\,x^2 + 96\,y^2)^2)/ \\
&((-4.82681\times10^6\,x^6 - 8.22557\times10^6\,x^4\,y^2 - 8.32291\times \\
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&y^4 + 8.22557\times10^6\,x^6\,y^6 + 6.49771\times \\
&10^6\,x^4\,y^8 + 884736\,x^2\,y^{10}) - \\
&884736\,y^6)\,\hat{}\,(1/3)) + 0.0236686\,y^2) + \\
&0.5\sqrt{(}0.00788955\,(169\,x^2 + 96\,y^2) - 0.00197239 \\
&(-4.82681\times10^6\,x^6 - 8.22557\times10^6\,x^4\,y^2 - \\
&8.32291\times10^6\,x^2\,y^4 + 2702. \\
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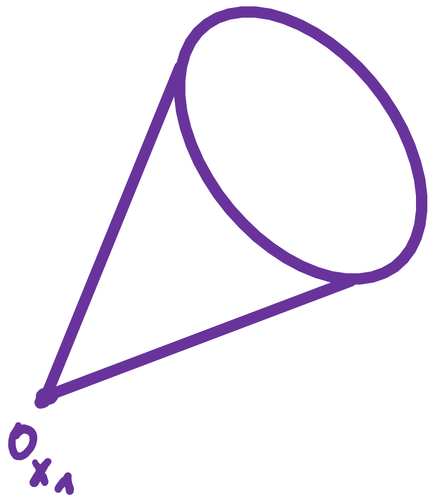
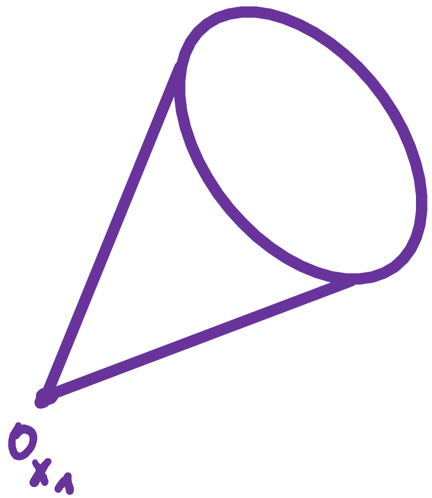
&2702.\sqrt{(}4.82681\times10^6\,x^8\,y^4 + 8.22557\times10^6\,x^6 \\
&y^6 + 6.49771\times10^6\,x^4\,y^8 + 884736 \\
&x^2\,y^{10}) - 884736\,y^6)\,\hat{}\,(1/3)) - \\
&(0.25\,(0.00728266\,y\,(-169\,x^2 - 96\,y^2) + \\
&2.46154\,x^2\,y - 0.0291306\,y^3))/ \\
&(\sqrt{(}(0.00197239\,(-169\,x^2 - 96\,y^2) +
\end{aligned}
$$

F(x,y)


O_{x_1}



$$\ell^+_{x_1} = \{ v \in T_{x_1}\tilde{M} : v \text{ f.p. and } h(v,v)=0 \}$$

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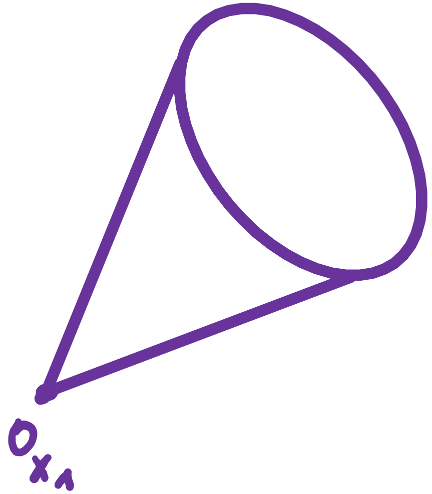

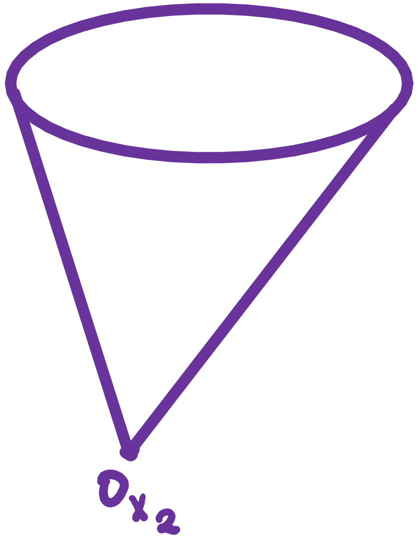

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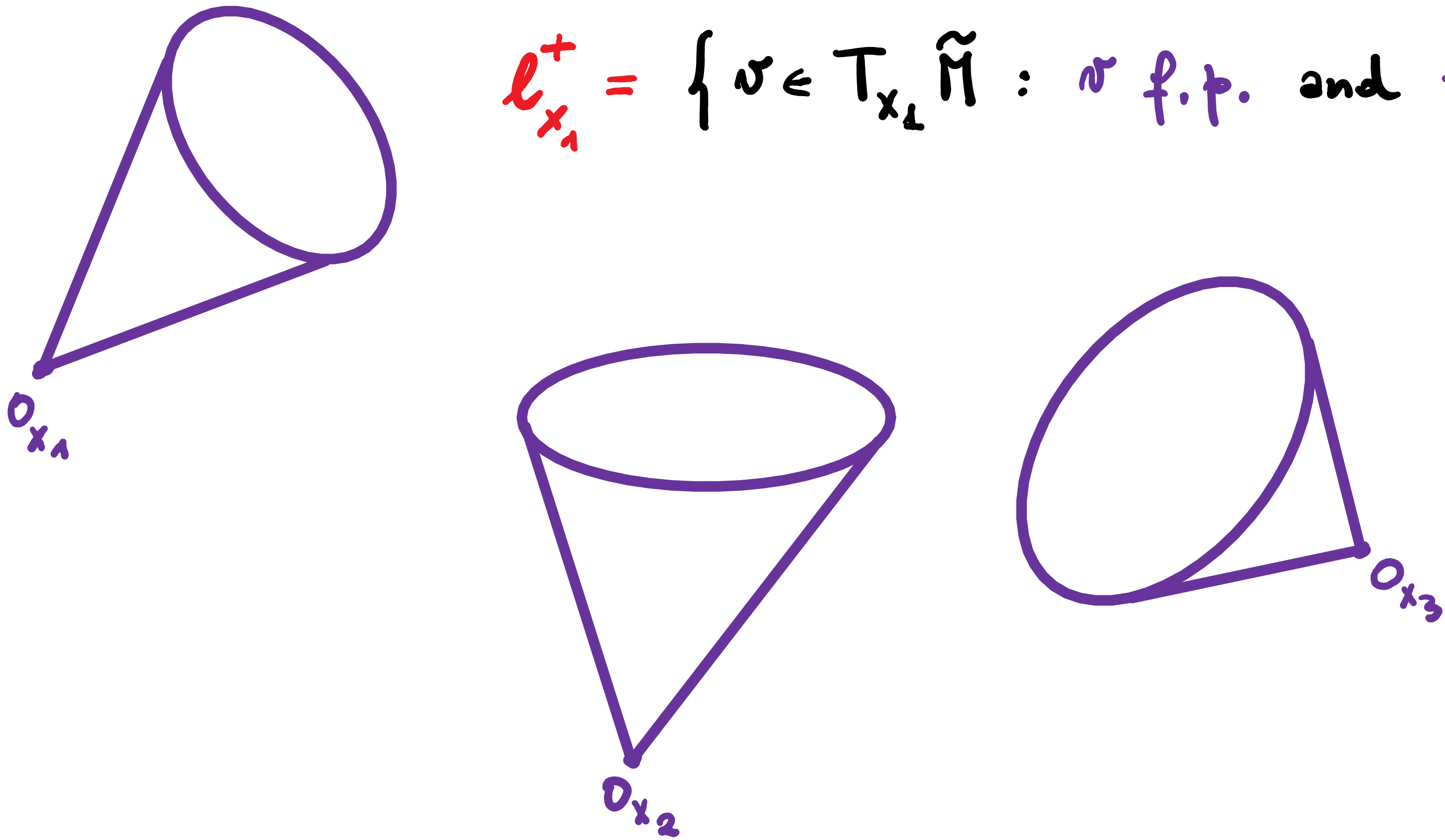
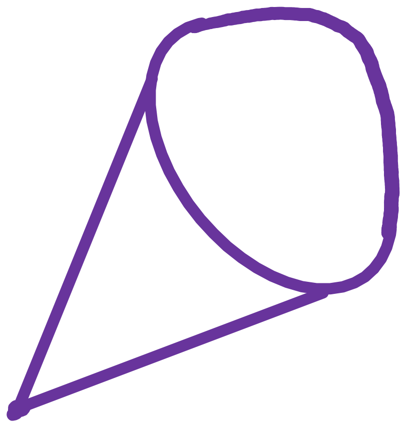
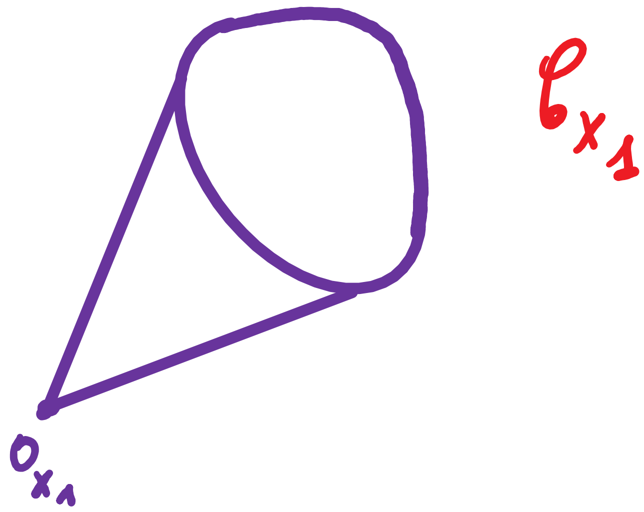
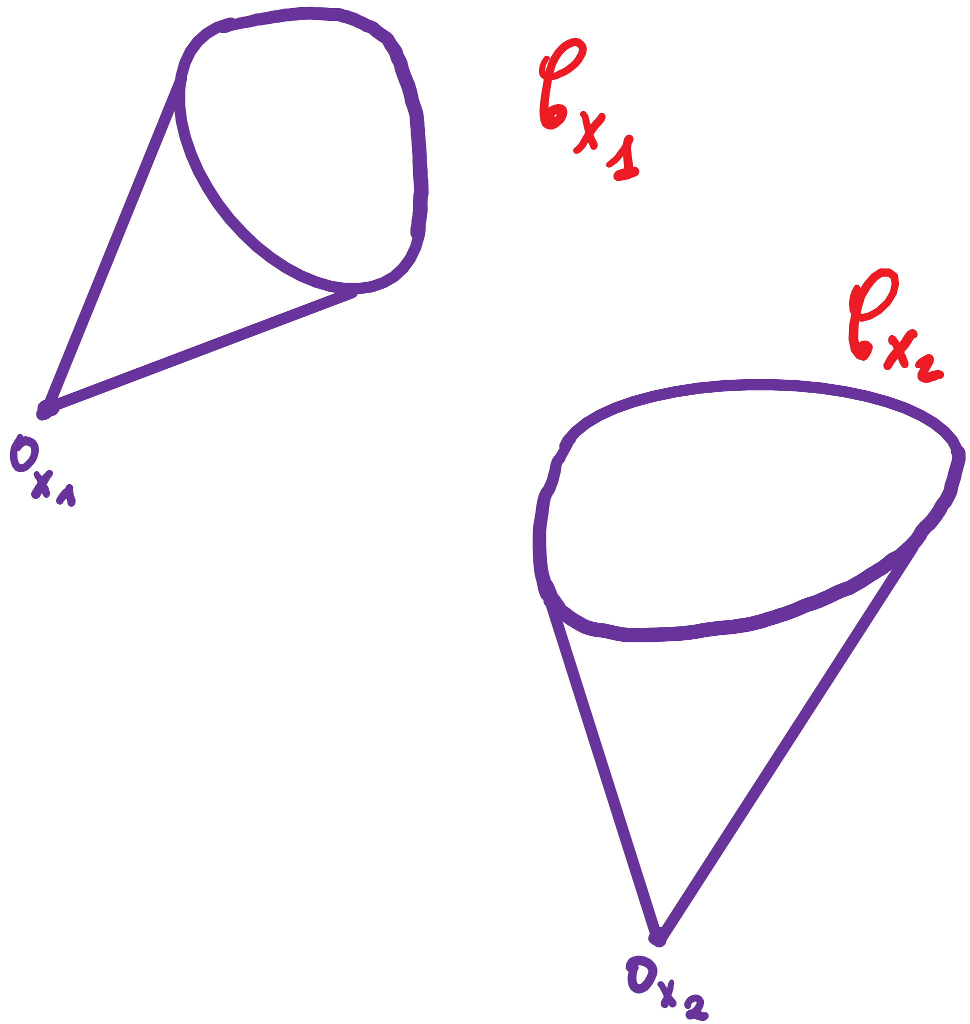
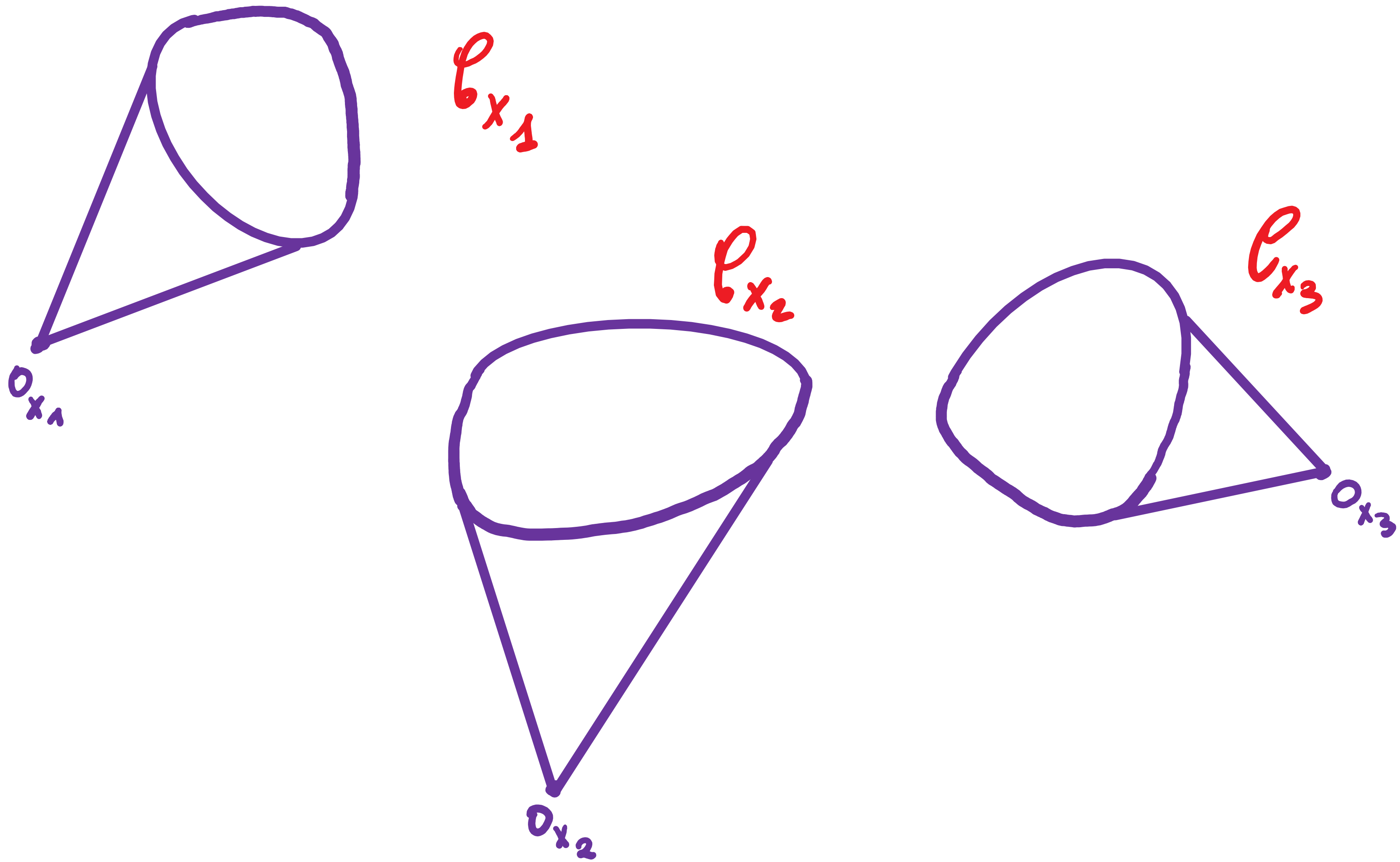




O_{x_1}


$\mathcal{C}_{x_1}$
O_{x_1}


ℓ_{x_1}
O_{x_1}
ℓ_{x_2}
O_{x_2}



ℓ_{x_1}
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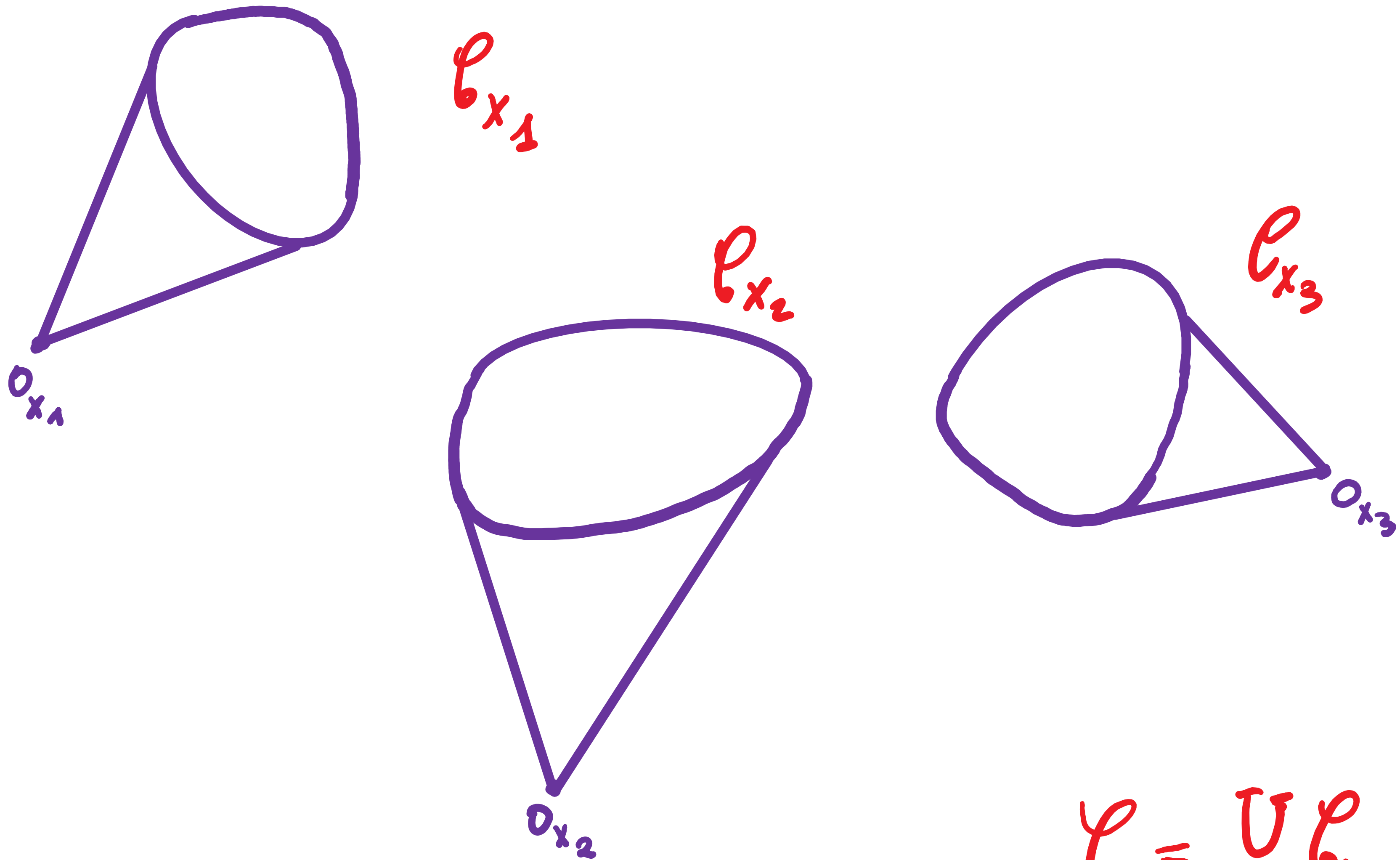

$$\mathcal{C} = \bigcup_{x} \mathcal{C}_x$$

CONE STRUCTURE

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- Fathi - Siconolfi, "On smooth time functions", MATH. PROC. CAMBRIDGE PHYL. SOC., 2012
- Minguzzi, "Affine sphere relativity", COMM. MATH. PHYS., 2017
- Minguzzi, "Causality theory for closed cones structures with applications", arXiv, 2017
- Bernard - Suhr, "Smoothing causal functions", J. PHYS. CONF. SERIES, 2018
- Javaloyes - Sanchez, "On the definition and examples of cones and Finsler spacetimes", arXiv, 2018

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A smooth, open, cone subset of $T\tilde{M} \setminus 0$

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FOR THE STATIC AND STATIONARY CLASS STUDIED $L^{-1}(\{0\}) \subset A$

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- L is C^3 on $A \subset T\tilde{M} \setminus 0$
- for all $y \in A$ and $u, v \in T_{\pi(y)}\tilde{M}$, let

$$\tilde{g}_y(u, v) := \frac{1}{2} \frac{\partial^2 L}{\partial s \partial t}(y + su + tv)\Big|_{(s,t)=(0,0)};$$

$\tilde{g}_y$ has index 1 for all $y \in A$

Def Let $v \in T\widetilde{M}$, we say that v is

timelike if $L(v) < 0$

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Def $v \in A$, v causal, is future-pointing if

$$\tilde{g}_v(v, Y) < 0$$

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- It was extended

Javaloyes - Sanchez (INT. J. GEOM. METH. MOD. PHYS. 2014) → conic domains

Lämmerzahl - Perlick - Hasse (PHYS. REV. D 2012)

Aazami - Javaloyes (CLASS. QUANTUM GRAV. 2016) → non smooth L

C- Stancarone (2016)

THE EQUIVALENCE BETWEEN CONE STRUCTURES & LORENT-FINSLER METRICS

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Th. (Minguzzi 2017 ; Javaloyes-Sanchez 2018)

Let $\mathcal{C} \subset T\tilde{M}$ be a smooth cone structure on $\tilde{M}$

(for each $p \in \tilde{M}$, $\mathcal{C}_p$ is a hypersurface in $T_p\tilde{M} \setminus \{0\}$ which is a strongly convex, salient cone & $p \longmapsto \mathcal{C}_p$ is smooth)

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which is a strongly convex, salient cone & $p \mapsto \mathcal{C}_p$ is smooth)

Then there exists a Lorentz-Finsler

function with $A = T\tilde{M} \setminus 0$ and

such that $\mathcal{C} = L^{-1}(\{0\}) \cap \{\text{f.p. vectors of } L\}$

(moreover, $D = L^{-1}((-\infty, 0)) \cap \{\text{f.p. vectors of } L\}$

is convex & $\partial D = \mathcal{C} \cup 0$)

Thm (Javaloyes-Sanchez 2018)

Any other Lorentz-Finsler metric L^0 such that

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$$\forall\ v \in \mathcal{C} : \mu(v) = \frac{\tilde{g}^\circ_v(v,w)}{\tilde{g}_v(v,w)},$$ for any $w \in T_{\pi(v)}\tilde{M}$

with $\tilde{g}_v(v,w) \neq 0$

STATIONARY SPLITTING FINSLER SPACETIMES

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$\boxed{\text{as generalisation of}}$

$(\mathbb{R} \times M, g)$, $\quad g((\tau,v),(\tau,v)) = -\Lambda \tau^2 + 2\omega(v)\tau + g_0(v,v)$

$$\left.\begin{array}{ll} \Lambda & \text{as above,} \\ \omega & \text{1-form} \\ g_0 & \text{Riemannian} \end{array}\right\} \text{on } M$$

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We call them

stationary splitting Finsler spacetimes

and, when B is a one-form on M,

standard stationary splitting Finsler spacetimes

The static case

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$$\tilde{g} : \quad \tilde{g}_{(\tau,v)}\big((\tau_1, v_1), (\tau_2, v_2)\big) = -\Lambda(v)\, \tau_1 \tau_2 + \tilde{g}_v(v_1, v_2)$$

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Yes, but causal cones can be non-convex

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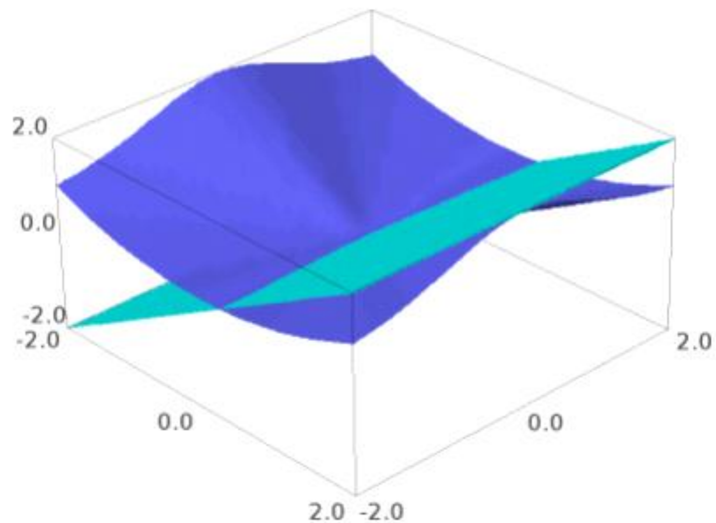
$$\tilde{g}_{(\tau,v_1,v_2)} = -e^{\frac{4v_2}{v_1^2+v_2^2}}dt^2 + dx^2 + dy^2$$

In cyan the plane of vectors which are $\tilde{g}_{(1,1,0)}$-orthogonal to $(1,1,0)$ which is lightlike, i.e. $\tilde{g}_{(1,1,0)}((1,1,0),(1,1,0)) = 0$



PHYSICAL REVIEW D **86,** 104042 (2012)

Observable effects in a class of spherically symmetric static Finsler spacetimes

Claus Lämmerzahl Volker Perlick Wolfgang Hasse

Definition A Finsler spacetime $(M, \mathcal{L})$ is *static* if M is diffeomorphic to a product, $M \simeq \mathbb{R} \times N$, and $\mathcal{L}$ is of the form

$$\mathcal{L}(x^1, x^2, x^3, \dot{t}, \dot{x}^1, \dot{x}^2, \dot{x}^3) = \frac{1}{2}(g_{tt}(x^1, x^2, x^3)\dot{t}^2 \quad + g_{ij}(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3)\dot{x}^i\dot{x}^j),$$

where t runs over $\mathbb{R}$ and (x^1, x^2, x^3) are coordinates on N; the temporal metric coefficient $g_{tt}(x^1, x^2, x^3)$ must be negative and the spatial metric $g_{ij}(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3)$ must be positive definite.

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$-\Lambda$ $F^2(\dot{x}^1, \dot{x}^2, \dot{x}^3)$

PHYSICAL REVIEW D **90,** 064049 (2014)

Exact solution of vacuum field equation in Finsler spacetime

Xin Li[1,2,3,*] and Zhe Chang[2,3,†]

B. Vacuum solution

Here, we propose an ansatz that the Finsler structure is of the form

$$F^2 = B(r)y^t y^t - A(r)y^r y^r - r^2 \bar{F}^2(\theta, \varphi, y^\theta, y^\varphi). \quad (27)$$

Then, the Finsler metric can be derived as

$$g_{\mu\nu} = \mathrm{diag}(B, -A, -r^2 \bar{g}_{ij}), \quad (28)$$

$$g^{\mu\nu} = \mathrm{diag}(B^{-1}, -A^{-1}, -r^{-2} \bar{g}^{ij}), \quad (29)$$

where $\bar{g}_{ij}$ and its reverse are the metrics derived from $\bar{F}$ and the indices i, j run over the angular coordinates θ, φ.

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$$\mathcal{L}_K \tilde{g} \in \Gamma\left(\pi_A^*(T^*\tilde{M}) \otimes \pi_A^*(T^*\tilde{M})\right)$$

$\pi_A : A \longrightarrow \tilde{M}$ canonical projection

$\pi_A = \pi|_A$

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Complete lift of K on $T\tilde{M}$:

$$K^c(f \circ \pi) := K(f) \quad \text{for all } f \in C^\infty(\tilde{M})$$

$$K^c(f^c) := (K(f))^c \quad \text{where } f^c \in C^\infty(T\tilde{M})$$

$$f^c(v) := v(f)$$

$$\forall v \in T\tilde{M}$$

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$\in C^1(A)$

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for all $W \in \mathcal{X}(\tilde{M})$, $\mathcal{L}_W$:

1) $\mathcal{L}_W(f) = W^c(f)$ for all $f \in C^\infty(A)$

2) for all $X \in \Gamma(\pi_A^*(T\tilde{M}))$, $\mathcal{L}_W X = i^{-1}([W^c, i(X)])$

where $i : \pi_A^*(T\tilde{M}) \longrightarrow \mathcal{X}(T\tilde{M})$ is the vertical homomorphism

Prop. let Ψ be the flow of K :

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As the flow of

K^c is $\tilde{\Psi} = \tilde{\Psi}(t,(x,y))$

$= (\Psi(t,x), d\Psi_t(x)[y])$,

this means that

L is invariant under the flow of K^c

Prop. let ψ be the flow of K :

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$$\tilde{g}_{d\psi_t(v)}\left(d\psi_t(v_1), d\psi_t(v_2)\right) = \tilde{g}_v(v_1, v_2), \ \forall v_1, v_2 \in T_{\pi(v)}\tilde{M}$$

and $\forall v \in A$ s.t. $d\psi_t(v) \in A$

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and $\forall v \in A$ s.t. $d\psi_t(v) \in A$

Actually, for each $p \in \tilde{M}$, there exists a neighborhood U of p and an interval $0 \in I_p \subset \mathbb{R}$ such that $\tilde{\Psi}(I_p \times (TU \cap A)) \subset A$

Def Let $L : T\tilde{M} \to \mathbb{R}$, $L \in C^1(T\tilde{M})$.

We say that

K is **static** if

$$D^K := \operatorname{Ker}\left(\partial_y L(K) \right) \text{ is integrable}$$

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Rmk We need that L is differentiable along K in order to have that D^K is a distribution

If L is twice differentiable along K

$$\operatorname{Ker}\left(\partial_y L(K) \right) = \operatorname{Ker}\left(\tilde{g}_K(K, \cdot) \right)$$

LOCAL STRUCTURE OF A STATIONARY FINSLER SPACETIME

Under some simple algebraic conditions on L we characterize stationary Finsler spacetimes that are, locally, standard stationary splitting

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Under some simple algebraic conditions on L we characterize stationary Finsler spacetimes that are, locally, standard stationary splitting

i.e. for all $p \in \tilde{M}$ $\exists$ $U \subset \tilde{M}$

and φ diffeomorphism $I \times S \xrightarrow{\varphi} U$.

s.t. $L \circ \varphi_* (\tau, v) = -\Lambda \tau^2 + 2\omega(v)\tau + F^2(v)$

Thm. Let $\mathcal{T}$ be a line subbundle of $T\tilde{M}$ and $\mathcal{A} = T\tilde{M} \setminus \mathcal{T}$; let $(\tilde{M}, L)$ be a stationary Finsler spacetime with timelike Killing K.

Thm. Let τ be a line subbundle of $T\tilde{M}$ and $\mathcal{A} = T\tilde{M} \setminus \tau$; let $(\tilde{M}, L)$ be a stationary Finsler spacetime with timelike Killing K. Then $(\tilde{M}, L)$ is

locally standard stationary splitting

Theor. Let τ be a line subbundle of $T\tilde{M}$ and $A = T\tilde{M} \setminus \tau$; let $(\tilde{M}, L)$ be a stationary Finsler spacetime with timelike Killing K. Then $(\tilde{M}, L)$ is

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$$\Updownarrow$$

1) $L \in C^1(T\tilde{M}) \cap C^2(T\tilde{M} \setminus \tau)$ and $K_z \in \tau_z$ at any point $z \in \tilde{M}$ where $L(z, \cdot)$ is not the quadratic form defined by a Lorentzian metric

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at any point $z \in \tilde{M}$ where $L(z, \cdot)$ is not the quadratic form defined by a Lorentzian metric

2) $L(K) = L(-K)$

3) $L(w \pm K_{\pi(w)}) = L(w) + L(K_{\pi(w)})$ for all $w \in \Delta^K$

Rmk 1 If L is $C^2(T\tilde{M})$ we recover the well known local structure of stationary Lorentzian spacetimes

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Rmk 2

The assumption $L \in C^1(T\breve{M})$ is needed to have a well defined distribution $D^K = \mathrm{Ker}\left(\partial_y L(K)\right)$

Actually L in a stationary splitting admits fiberwise derivative $\partial_y L(\tau,0)$ at $(\tau,0) \in \mathbb{R} \times TM$ iff B is odd. Moreover $\partial_y L(\tau,0)$ is linear iff B is linear too

<u>Rmk 3</u> If D^K is integrable then S can be taken such that $TS = D^K$ and then $\omega = 0$ in this case; thus we have that under the assumptions of previous theorem $(\tilde{M}, L, K)$ is locally standard static iff D^K is integrable

THE OPTICAL METRICS OF A STATIONARY SPLITTING FINSLER SPACETIME

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$$\Downarrow$$

$$F_B(v) = \frac{B(v)}{\Lambda} + \left(\frac{B(v)^2}{\Lambda^2} + \frac{F^2(v)}{\Lambda} \right)^{\frac{1}{2}}$$

$$\geq 0$$

$$F_B^-(v) = -\frac{B(v)}{\Lambda} + \left(\frac{B(v)^2}{\Lambda^2} + \frac{F^2(v)}{\Lambda} \right)^{\frac{1}{2}}$$

F_B is a (classical) Finsler metric on M

if a) $\partial^2_{yy} B(y)$ is positive semi-definite for all $y \in TM \setminus 0$

b) $B(x,y) \geq 0$ for all $y \in T_x M$ or $B(x \cdot)$ is linear on $T_x M$

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if a) $\partial^2_{yy} B(y)$ is positive semi-definite for all $y \in TM \setminus 0$

(resp. $\partial^2_{yy} B(y)$ is negative semi-definite for all $y \in TM \setminus 0$)

b) $B(x,y) \geq 0$ for all $y \in T_x M$ or $B(x,\cdot)$ is linear on $T_x M$

(resp. $B(x,y) \leq 0$ for all $y \in T_x M$ or $B(x,\cdot)$ is linear on $T_x M$)

F_B (resp. $F_{\bar{B}}$) is a (classical) Finsler metric on M

in particular this happens when

$$B = \pm(\omega + F_1)$$

when F_1 is a Finsler metric on M

and ω is a one-form on M

s.t. $\left|\omega\left(y/F_1(y)\right)\right| < 1$, for all $y \in TM$

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These assumptions are necessary and sufficient in order to have that

$\tilde{g}_{(\tau, y)}$ has index 1 for all $y \in TM \setminus 0$ and $\tau > 0$

(resp. " and $\tau < 0$)

F_B (resp. F_B^-) is a (classical) Finsler metric on M

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F_B and F_B^- come into play in the description of lightlike geodesics of $(\mathbb{R}\times M, L)$

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Th $\gamma:[a,b]\to\mathbb{R}\times M$, $\gamma(s) = (\theta(s), \sigma(s))$, $\dot\theta(s) > 0$

is a lightlike geodesic of $(\mathbb{R}\times M, L)$ iff

σ is a pregeodesic of (M, F_B) parametrized

with $B^2(\dot\sigma) + \Lambda F^2(\dot\sigma) \equiv -C_\gamma := \Lambda\dot\theta - B(\dot\sigma)$

and $\theta(s) = \theta(a) + \int_a^s F_B(\dot\sigma)\, ds$

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Th $\gamma:[a,b]\to\mathbb{R}\times M$, $\gamma(s)=(\theta(s),\sigma(s))$, $\dot\theta(s)>0$ is a lightlike geodesic of $(\mathbb{R}\times M, L)$ iff σ is a pregeodesic of (M, F_B) parametrized with $B^2(\dot\sigma)+\Lambda F^2(\dot\sigma)\equiv -C_\gamma := \Lambda\dot\theta - B(\dot\sigma)$ and $\theta(s)=\theta(a)+\int_a^s F_B(\dot\sigma)\,ds$

An analogous result holds for $\dot\theta(s)<0$ and preg. of F_B^-.

F_B (resp. F_B^-) is a (classical) Finsler metric on M

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Finsler metric on M

Thus we can consider the standard static Finsler spacetime $\mathbb{R} \times M$

$$L_B(\tau, y) = -\tau^2 + F_B^2(y)$$

F_B (resp. F_B^-) is a (classical)

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Thus we can consider the standard static

Finsler spacetime $\mathbb{R} \times M$

$$L_B(\tau, y) = -\tau^2 + F_B^2(y)$$

The future causal structure of $(\mathbb{R} \times M, L)$

is encoded in $(\mathbb{R} \times M, L_B)$

F_B (resp. F_B^-) is a (classical) Finsler metric on M

Thus we can consider the standard static Finsler spacetime $\mathbb{R} \times M$

$$L_B^-(\tau, y) = -\tau^2 + \left(F_B^-(y)\right)^2$$

The past causal structure of $(\mathbb{R} \times M, L)$ is encoded in $(\mathbb{R} \times M, L_B^-)$

This is true for standard stationary Lorentzian manifold too where

$$F_B = \frac{w}{\Lambda} + \left(\frac{w^2}{\Lambda} + g_0 \right)^{\frac{1}{2}}$$

$g_0 \to$ Riemannian metric

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and

$$F_B^- = -\frac{\omega}{\Lambda} + \left(\frac{\omega^2}{\Lambda} + g_0\right)^{\frac{1}{2}} \quad (= \text{reverse of } F_B)$$

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Thus $\mathbb{R} \times M$ with the Lorentzian metric

$$g = g_0 + 2w \otimes dt - \Lambda\, dt^2$$

and with the static Finsler metric

$$L_B(\tau, y) = -\tau^2 + F_B^2(y)$$

are trivially isocausal, i.e. they have the same future causal cones.