The mass of asymptotically hyperbolic manifolds

Piotr T. Chruściel

University of Vienna

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1. Total energy is useful in one-dimensional classical mechanics

Mass or energy?

What is it good for anyway?

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2. But less so for higher dimensional gravitating systems (many body Keppler problem: Xia's finite-time ejections to infinity)
3. energy and mass are *not always* the same

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Mass or energy?

What is it good for anyway? some good news in the asymptotically flat case

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3. $m \geq 0$ for AF metrics $\Longrightarrow$ suitably regular static black holes are Schwarzschild in all dimensions
4. *Hollands and Wald* (2016): variational identities involving total mass for AF metrics can be used to prove existence of instabilities in “black strings”

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1. Spacetime variational methods: "Noether charge" *à la Wald* (~ *1990*) ≡ geometric Hamiltonian methods *à la Kijowski-Tulczyjew (1979)*: "$H(\partial_t, \{t = 0\})$" is the energy
2. Hamiltonians for asymptotic symmetries: If $\mathbf{g}$ suitably approaches a background $\overline{\mathbf{g}}$ with a Killing vector field X, then the Hamiltonian is

$$H(X, \mathscr{S}) := \frac{1}{2} \int_{\partial \mathscr{S}} \left(\mathbb{U}^{\nu\lambda} - \mathbb{U}^{\nu\lambda} \big|_{\mathbf{g}=\overline{\mathbf{g}}} \right) dS_{\nu\lambda} \,, \qquad (1)$$

$$\mathbb{U}^{\nu\lambda} = \mathbb{U}^{\nu\lambda}{}_{\beta} X^{\beta} - \frac{1}{8\pi} \sqrt{|\det \mathbf{g}|}\, \mathbf{g}^{\alpha[\nu} \delta^{\lambda]}_{\beta} \overline{\nabla}_{\alpha} X^{\beta} \,, \qquad (2)$$

$$\mathbb{U}^{\nu\lambda}{}_{\beta} = \frac{2|\det \overline{\mathbf{g}}|}{16\pi \sqrt{|\det \mathbf{g}|}} \mathbf{g}_{\beta\gamma} \overline{\nabla}_{\kappa} \left(e^2 \mathbf{g}^{\gamma[\lambda} \mathbf{g}^{\nu]\kappa} \right) \,, \qquad (3)$$

where $\overline{\nabla}$ is the covariant derivative of $\overline{\mathbf{g}}_{\mu\nu}$ and

$$e^2 \equiv \frac{\det \mathbf{g}}{\det \overline{\mathbf{g}}} \,. \qquad (4)$$

Asymptotically locally hyperbolic (ALH) metrics

Asymptotically hyperbolic if $(N^{n-1}, \mathring{h})$ is the unit round sphere

$$g = \ell^2 x^{-2}\Big(\mathrm{d}x^2 + (1 - \tfrac{k}{4}x^2)^2 \mathring{h} + x^n \mu\Big) + o(x^{n-2})\mathrm{d}x^i \mathrm{d}x^j \,,$$

$$\mathring{h} = \mathring{h}_{AB}(x^C)\mathrm{d}x^A \mathrm{d}x^B \,, \quad \mu = \mu_{AB}(x^C)\mathrm{d}x^A \mathrm{d}x^B \,,$$

$\ell > 0$ is a constant related to Λ, $\mathring{h}$ is a Riemannian metric on N^{n-1} with scalar curvature

$$R[\mathring{h}] = (n-1)(n-2)k \,, \quad k \in \{0, \pm 1\} \,. \qquad (5)$$

The mass aspect function is

$$\theta := \mathrm{tr}_{\mathring{h}} \mu$$

uniquely defined unless the conformal infinity is a round sphere
The total mass is

$$m_0 = c_n \int_{N^{n-1}} \theta \,, \qquad m_i = c_n \int_{S^{n-1}} \theta x^i$$

(defines a “Minkowskian” vector on a sphere).

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Hyperbolic mass (also known as *holographic energy*, cf. “holographic stress-energy tensor”).

- We only have satisfactory understanding of mass and related invariants in the asymptotically Euclidean setting. (Spectacular progress by Schoen and Yau 2017.)
- Asymptotically hyperbolic setting: Positivity? Spin structure or other topological restrictions? Sharp and insightful inequalities in higher dim? e.g., on spin manifolds with spherical infinity, in

$$E^2 \geq |\vec{j}|^2 , \tag{6}$$

where $\vec{j}$ is the total angular momentum

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$$E^2 - |\vec{p}|^2 \geq -\Lambda/3 \left(|\vec{c}|^2 + |\vec{j}|^2 + 2|\vec{c} \times \vec{j}|\right), \tag{6}$$

where $\vec{j}$ is the total angular momentum and $\vec{c}$ the centre of mass.

What backgrounds **g**?

- For simplicity, assume vacuum Einstein equations throughout:

$$R(\mathbf{g})_{\mu\nu} = c_n \Lambda g_{\mu\nu} \tag{7}$$

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What are the spacelike manifolds $\mathscr{S}$ we are interested in?

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- This talk: mostly $\Lambda < 0$
- What kind of spacelike hypersurfaces are compatible with (7) when $\Lambda \neq 0$

Constraint equations, cosmological constant Λ

Does the curvature scalar know about Λ? ($\rho = j^k = 0$ in vacuum)

- The scalar constraint equation:

$$R(g) = 16\pi\rho + |K|^2 - (\mathrm{tr}K)^2 + 2\Lambda \tag{8}$$

where ρ is the energy density of matter fields, $R(g)$ is the scalar curvature of the space metric

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- ⚠ You can fool around with Λ by playing with the trace of K

$$K \to K + ag \quad \Longrightarrow \quad \tilde{\Lambda} \to \tilde{\Lambda} - \frac{(n-1)}{2n}(2a\,\mathrm{tr}K + a^2)$$

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$$K \to K + ag \quad \Longrightarrow \quad \tilde{\Lambda} \to \tilde{\Lambda} - \frac{(n-1)}{2n}(2a\,\mathrm{tr}K + a^2)$$

- This is compatible with the vector constraint equation:

$$D_i(K^{ik} - \mathrm{tr}K\, g^{ik}) = 8\pi\, j^k$$

Constraint equations, cosmological constant Λ

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- *Corollary*: The Trautman-Bondi mass m_{TB} is ~~*the same as*~~ *related to* the hyperbolic mass (⚠ pure trace K + constraint equations + $\Lambda = 0 \Longrightarrow$ no gravitational radiation ⚠)

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PTC, Tod 2007

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• *Corollary*: positivity theorems for asymptotically hyperbolic initial data ($\Lambda < 0$) translate to angular momentum bounds with $\Lambda = 0$

$$m_{TB} \geq \frac{|\mathrm{tr} K|}{3} |\vec{J}| , \quad m_{TB} \geq \frac{|\mathrm{tr} K|}{3} |\vec{c}| ,$$

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- *Corollary*: positivity theorems for asymptotically hyperbolic initial data ($\Lambda < 0$) translate to angular momentum bounds with $\Lambda = 0$ *on CMC hypersurfaces $\mathscr{S}$ when there is no-radiation at the conformal boundary of $\mathscr{S}$*

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Asymptotically Anti-de Sitter metrics

- Asymptotically anti-de Sitter metrics:

$$\mathbf{g} \to_{r\to\infty} \overline{\mathbf{g}} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2 \,, \qquad V = r^2 + 1 \,.$$

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PTC, Barzegar, Nguyen (2018), space-dimension n

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- Elementary positive energy theorem: in a suitable gauge, for $h := g - \overline{g}$ small, ($E := H(\partial_t, \{t = 0\})$)

$$E$$

$$\geq \int_M \left[R - \overline{R} + \frac{n-2}{16n} |\overline{D}h|^2_{\mathbf{g}} \right] V \,.$$

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$$\begin{aligned} E &\geq \int_M \Big[R - \overline{R} + \frac{n-2-\epsilon}{8n} |\overline{D}\mathrm{tr}\, h|^2_{\overline{\mathbf{g}}} + \frac{1-\epsilon}{4} |\overline{D}\hat{h}|^2_{\overline{\mathbf{g}}} \\ &\quad - \frac{1+\epsilon}{1} |\hat{h}|^2_{\overline{\mathbf{g}}} \Big] V \sqrt{\det \overline{\mathbf{g}}} \\ &\geq \int_M \Big[R - \overline{R} + \frac{n-2}{16n} |\overline{D}h|^2_{\overline{\mathbf{g}}} \Big] V \,. \end{aligned}$$

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- but *no stability*: arbitrarily small generic perturbations of initial data for the spherically symmetric Einstein-scalar field equations produce arbitrarily small black holes (?)

$$\mathbf{g} \to_{r\to\infty} \overline{\mathbf{g}} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2 \,, \qquad V = r^2 + 1 \,.$$

- For any Killing vector X of $\overline{\mathbf{g}}$ we have

$$H_b(X, \mathscr{S}) \;=\; \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{t=0,r=R} X^\nu Z^\xi W^{\alpha\beta}{}_{\nu\xi} dS_{\alpha\beta} \,,$$

where $W^{\alpha\beta}{}_{\nu\xi}$ is the Weyl tensor of $\mathbf{g}$ and $Z = r\partial_r$ is the dilation vector field

Asymptotically Anti-de Sitter metrics

Geometric formulae for total energy (Ashtekar Romano 1992; Herzlich 2015; PTC, Barzegar, Höerzinger 2017), space-dimension n

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- Riemannian version, asymptotically hyperbolic Riemannian metrics g, $\mathrm{R}^i{}_j$ is the Ricci tensor of g:

$$H_b(X, \mathscr{S}) = -\frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{r=R} X^0 V Z^j (\mathrm{R}^i{}_j - \frac{\mathrm{R}}{n} \delta^i_j) dS_i \,.$$

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- If X is a Killing vector of both $\mathbf{g}$ and $\overline{\mathbf{g}}$ we have

$$\begin{aligned} H_b(X, \mathscr{S}) \quad = \quad & \lim_{R\to\infty} \Bigg\{ \frac{n-1}{16(n-2)\pi} \int_{r=R} X^{[\alpha;\beta]} dS_{\alpha\beta} \\ & - \frac{\Lambda}{4(n-2)(n-1)n\pi} \int_{r=R} X^\alpha Z^\beta dS_{\alpha\beta} \Bigg\} \,, \end{aligned}$$

where $\Lambda < 0$ is the cosmological constant.

Other asymptotic backgrounds: Kottler-Birmingham metrics

Static vacuum solutions of Einstein equations with a negative cosmological constant

$$\mathbf{g}_m = -V_m^2 dt^2 + V_m^{-2} dr^2 + r^2 h_\kappa \,, \qquad V_m = r^2 + \kappa - \frac{2m}{r^{n-2}} \,.$$

where h_κ is a t- and r-independent Einstein metric on a $(n-1)$-dim compact manifold, with scalar curvature $R(h) = (n-1)(n-2)\kappa$.

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- The mass of $\mathbf{g}_m$ relative to $\bar{\mathbf{g}} := \mathbf{g}_0$ is proportional to m

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$$\mathbf{g}_m = -V_m^2 dt^2 + V_m^{-2} dr^2 + r^2 h_\kappa \,, \qquad V_m = r^2 + \kappa - \frac{2m}{r^{n-2}} \,.$$

where h_κ is a t- and r-independent Einstein metric on a $(n-1)$-dim compact manifold, with scalar curvature $R(h) = (n-1)(n-2)\kappa$.

- The mass of $\mathbf{g}_m$ relative to $\bar{\mathbf{g}} := \mathbf{g}_0$ is proportional to m
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Other asymptotic backgrounds: Kottler-Birmingham metrics Lee & Neves, $n = 3$, 2015

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Horowitz-Myers Instantons

$$\mathbf{g}_m = \cancel{-V_m^2 dt^2}\, V_m^2 d\theta^2 + V_m^{-2} dr^2 + r^2(\cancel{d\theta^2} - dt^2 + h'_0)\,, \quad V_m = r^2 \cancel{+k} - \frac{2m}{r^{n-2}}\,.$$

where h'_0 is a t-, θ-, and r-independent Ricci flat metric on a $(n-3)$-dim compact manifold.

- Naked singularity for $m < 0$.

Horowitz-Myers Instantons

$$\mathbf{g}_m = V_m^2 d\theta^2 + V_m^{-2} dr^2 + r^2 (-dt^2 + h_0') \,, \quad V_m = r^2 - \frac{2m}{r^{n-2}} \,.$$

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- Conjecture: these are local minima of energy.

$h = g - \overline{g}$, $\hat{h} =$ trace-free part of h:

$$
\begin{aligned}
m \;=\; \int_M \Big[\;& (R - \overline{R})V + \Big(\frac{n+2}{8n}|\overline{D}\phi|^2_{\mathbf{g}} + \frac{1}{4}|\overline{D}\hat{h}|^2_{\mathbf{g}} \\
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& - \frac{1}{2}\left(|\check{\psi}|^2_{\mathbf{g}} - \check{\psi}^i\overline{D}_i\phi\right)\Big)V + \left(h^k{}_i\check{\psi}^i + \frac{1}{2}\phi\check{\psi}^k\right)\overline{D}_k V \\
& + \left(O\left(|h|^3_{\mathbf{g}}\right) + O\left(|h|_{\mathbf{g}}\,|\overline{D}h|^2_{\mathbf{g}}\right)\right)V \\
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~~*gauge terms*~~

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gauge terms error terms

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gauge terms error terms ???

Horowitz-Myers Instantons

the Woolgar-Horowitz-Myers conjecture for nearby metrics??????

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Incidentally: *Uniqueness theorems* for the Horowitz-Myers instanton by Galloway and Woolgar, and by M. Anderson

Reminder: Asymptotically locally hyperbolic (ALH) metrics

Asymptotically hyperbolic if $(N^{n-1}, \mathring{h})$ is the unit round sphere

$$g = \ell^2 x^{-2}\Big(\mathrm{d}x^2 + (1 - \tfrac{k}{4}x^2)^2 \mathring{h} + x^n \mu\Big) + o(x^{n-2})\mathrm{d}x^i \mathrm{d}x^j ,$$

$$\mathring{h} = \mathring{h}_{AB}(x^C)\mathrm{d}x^A \mathrm{d}x^B , \quad \mu = \mu_{AB}(x^C)\mathrm{d}x^A \mathrm{d}x^B ,$$

$\ell > 0$ is a constant related to Λ, $\mathring{h}$ is a Riemannian metric on N^{n-1} with scalar curvature

$$R[\mathring{h}] = (n-1)(n-2)k , \quad k \in \{0, \pm 1\} . \qquad (11)$$

The mass aspect function is

$$\theta := \mathrm{tr}_{\mathring{h}} \mu$$

uniquely defined unless the conformal infinity is a round sphere
The total mass is

$$m_0 = c_n \int_{N^{n-1}} \theta , \qquad m_i = c_n \int_{S^{n-1}} \theta x^i$$

(defines a “Minkowskian” vector on a sphere).

A positive mass theorem without spin hypotheses
PTC, Galloway, Nguyen, Paetz, 2018

Theorem

Let (M^n, g), $4 \le n \le 7$, be a C^{n+5}–conformally compactifiable asymptotically locally hyperbolic (ALH) Riemannian manifold diffeomorphic to $[r_0, \infty) \times N^{n-1}$ with a compact boundary $N_0 := \{r_0\} \times N^{n-1}$ and with well defined total mass. Suppose that:

1. *The mean curvature of N_0 satisfies $H < n - 1$, where H is the divergence $D_i n^i$ of the unit normal n^i pointing into M.*
2. *The scalar curvature $R = R[g]$ of M satisfies $R \ge -n(n-1)$.*
3. *Either $(N, \mathring{h})$ is a flat torus, or $(N, \mathring{h})$ is a nontrivial quotient of a round sphere.*

Then the mass of (M^n, g) is nonnegative, $m \ge 0$.

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1. *g_ϵ has a pure monopole-dipole mass aspect function Θ_ϵ if $(N^{n-1}, \mathring{h})$ is conformal to the standard sphere, and has constant mass aspect function otherwise;*
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Mass aspect deformation theorem

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