Euclidian-Hyperboloidal foliations and CMC-harmonic coordinates

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- Optimal regularity of metrics with bounded curvature

Local analysis and possibly large curvature
Construction of a "canonical" CMC foliation
Existence of local CMC-spatially harmonic coordinates

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- Dynamics of self-gravitating massless/massive matter fields

  Global analysis near Minkowski spacetime
  Construction of Euclidian-Hyperboloidal foliations
  Small data global-in-time existence

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  – Weighted Sobolev spaces: regularity of the metric, decay conditions

  – Key challenge: quantitative estimates, uniform with respect to the relevant parameter (curvature scale, time variable)

1. Canonical foliations of Einstein spacetimes with bounded curvature

Joint work with Binglong Chen (Guangzhou)

1.1 Objective

Minimal regularity required to control the *geometry of the spacetime*

- Solely a bound on the curvature
- Fully geometric estimates

1. Canonical foliations of Einstein spacetimes with bounded curvature

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1.1 Objective

Minimal regularity required to control the *geometry of the spacetime*

- Solely a bound on the curvature
- Fully geometric estimates
- Three steps in our analysis:
  - injectivity radius of an observer
  - construction of a canonical CMC foliation
  - local canonical foliations and coordinates of an observer
- Optimal regularity theory in $W^{2,p}$ for all $p < +\infty$

1.2 Injectivity radius of an observer

(M, g, p, T_p) : *time-oriented, pointed, Lorentzian manifold*
(p, T_p) : (infinitesimal) *observer*
T_p future-oriented, unit time-like vector

Exponential map

- Exponential map: $\exp_p : B_{g_p}(0, i_0) \subset T_pM \to \mathcal{B}_g(p, i_0) \subset M$
- Defined in a neighborhood of $0 \in T_pM$

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Reference Riemannian metric

- *Positive definite inner product* $g_{T_p,p}$ at p
  - Orthonormal frame e_α at p with $e_0 := T_p$
  - From $$g_p = -e^0 \otimes e^0 + e^1 \otimes e^1 + \ldots + e^n \otimes e^n$$ we define $$g_{T_p,p} := \quad e^0 \otimes e^0 + e^1 \otimes e^1 + \ldots + e^n \otimes e^n$$
- *Reference Riemannian metric* g_T, once a field of observers T is prescribed
- By g-parallel transporting T_p, we define a vector field T_γ along any radial geodesic $\gamma : [0, r] \to M$ from p.

Norm of the curvature

- Using $g_{T_p,p}$, we can compute the norm $|A|_{g_{T_p,p}}$ of a tensor at p
- Using the associated Riemannian metric g_{T_γ}, we can compute the norm

$$\sup_{[0,r]} |Rm_g|_{g_{T_\gamma}}$$

- Finally the *Riemann curvature norm* associated with the observer is

$$\mathbf{Riem}_r(p, T_p) := \sup_{\gamma} \sup_{[0,r]} |Rm_g|_{g_{T_\gamma}}$$

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Lorentzian notion of injectivity radius

- *Injectivity radius of the observer* (p, T_p)

$$\mathrm{Inj}(M, g, p, T_p)$$

supremum of all radii r such that $\exp_p$ is a global diffeomorphism from $B_{g_T,p}(0, r)$ to its image $\mathcal{B}_T(p, r) \subset M$

Classical result fo Riemannian manifolds

- A complete Riemannian n-manifold (M, g) such that, in the unit geodesic ball $\mathcal{B}_g(p, 1)$ centered at some $p \in M$,

$$\textbf{Riem}_1 := \sup_{\mathcal{B}_g(p,1)} |\textbf{Rm}_g| \leqslant K_0$$

- Cheeger, Gromov, and Taylor: there exists a constant $c_0(K_0, n) > 0$ such that

$$\mathrm{Inj}(M, g) \geqslant c_0(K_0, n)\, \textbf{Vol}_g(\mathcal{B}_g(p, 1))$$

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We establish a Lorentzian version

- Local and geometric estimate "a la Cheeger-Gromov-Taylor"
- No a priori prescription of a foliation or coordinate chart
- No assumption on the derivative of the curvature

Our Lorentzian version.

Recall our definition

$$\mathbf{Riem}_r(M, g, p, T_p) := \sup_{\gamma} \sup_{[0,r]} |Rm_g|_{g_{T_\gamma}}$$

Theorem.
Lower bound on the Lorentzian injectivity radius (BL Chen & PLF)

There exists a universal constant $c(n) > 0$ such that, if (M, g, p, T_p) is a pointed Lorentzian $(n+1)$-manifold satisfying the curvature bound

$$\mathbf{Riem}_r(M, g, p, T_p) \leqslant \frac{1}{r^2}$$

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then

$$\frac{\mathrm{Inj}(M, g, p, T_p)}{r} \geqslant c(n) \, \frac{\mathbf{Vol}_g\left(\mathcal{B}_{M,g}(p, c(n)r)\right)}{r^{n+1}}.$$

Proof based on a study the geometry of the covering

$$\exp_p : B_{g_{T_p,p}}(0, r) \to \mathcal{B}_{g_{T_p}}(p, r) \subset M$$

1.3 Local CMC foliation of an observer

Objective

- Given an observer (p, T_p), define and construct a *canonical* CMC (constant mean curvature) foliation by spacelike hypersurfaces
- Defined *locally* in a neighborhood of p
- Quantitative estimates involving curvature and injectivity bounds

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Earlier works

- Riemannian manifolds: De Turck and Kazdan, Jost and Karcher:
  - There exists $i_1 = i_1(\mathrm{Inj}, K_0)$ such that, given $\varepsilon > 0$, one can cover $\mathcal{B}_g(p, i_1)$ by harmonic coordinates and get the optimal regularity of the metric coefficients

$$e^{-\varepsilon} g_E \leqslant g \leqslant e^{\varepsilon} g_E \qquad g_E : \text{ Euclidian}$$

$$\|g\|_{W^{2,a}(\mathcal{B}_g(p,i_0))} \leqslant C_{\varepsilon,a} \qquad a \in [1, \infty)$$

- Lorentzian manifolds: ∇**Rm** bounded (or even more regularity)
  - Bartnik-Simon, Gerhardt, etc.

Local canonical foliations

Definition

Given $\theta \in (0,1)$ (close to 1, say), a *local canonical CMC foliation* for the observer (p, T_p):

- a foliation by n-dimensional spacelike hypersurfaces Σ_t of constant mean curvature t

$$\Big(\bigcup_{\underline{t} \leqslant t \leqslant \overline{t}} \Sigma_t \Big) \ni p$$

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- the range of t of order $1/r$, specified by some constant $s \in [\theta, 2\theta]$

$$\underline{t} := (1 - \theta)\frac{n}{sr}, \qquad \overline{t} := (1 + \theta)\frac{n}{sr}$$

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- the unit normal N, the lapse function $\lambda := \big(-g(\nabla t, \nabla t)\big)^{1/2}$ and the second fundamental form h satisfy (pointwise bounds)

$$-g(N,T) \leqslant \theta^{-1}, \qquad \theta \leqslant -r^{-2}\lambda \leqslant \theta^{-1}, \qquad r\,|h| \leqslant \theta^{-1}$$

T being defined by parallel translating T_p along radial geodesics

Theorem. Existence of canonical foliations

There exist universal constants $c(n), \theta(n) > 0$ such that, if (M, g, p, T_p) is a pointed Lorentzian manifold satisfying at some scale $r > 0$

$$\mathbf{Riem}_r(M, g, p, T_p) \leqslant r^{-2}, \qquad \mathrm{Inj}(M, g, p, T_p) \geqslant r,$$

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- Search for CMC graphs over Lorentzian geodesic spheres
- Prescribed mean curvature problem: nonlinear elliptic problem for the level set function
- Barrier functions: Lorentzian and Riemannian geodesic spheres
- Uniform control of the geometry of these graphs in terms of the curvature and injectivity radius:
  - low regularity of the metric, loss of derivatives
  - estimates derived with Nash-Moser iterations

1.4 Local CMC-harmonic coordinates of an observer

- (M, g, p, T_p): an $(n+1)$-dimensional, pointed Einstein vacuum spacetime $R_{\alpha\beta} = 0$
- Satisfying the curvature and injectivity bounds at the scale $r > 0$

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Then, for some small constants $0 < \underline{c} < c << 1$ and $r_1 \in [\underline{c}r, cr]$ there exist local coordinates

$$x = (x^0, x^1, \ldots, x^n) = (t, x^1, \ldots, x^n)$$
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$$x(p) = (r_1, 0, \ldots, 0)$$
$$|t - r_1| < c^2 r$$
$$\left((x^1)^2 + \ldots + (x^n)^2\right)^{1/2} < c^2 r$$

so that the following properties hold:

- Σ_t (constant t) **spacelike hypersurfaces with constant CMC** equal to $c^{-1}r^{-2}t$

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- The Lorentzian metric, decomposed as

$$g = -\lambda(x)^2 (dt)^2 + g_{ij}(x)\big(dx^i + \xi^i(x)\, dt\big)\big(dx^j + \xi^j(x)\, dt\big),$$

remains **uniformly close to the Minkowski metric**

$$e^{-C} \leqslant \lambda \leqslant e^C, \qquad e^{-C}\delta_{ij} \leqslant g_{ij} \leqslant e^C \delta_{ij}, \qquad |\xi|_g^2 := g_{ij}\xi^i\xi^j \leqslant e^C$$

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- **Optimal regularity property**: for all $q \in [1, +\infty)$ and a constant $C(n, q) > 0$

$$r^{-n+q} \int_{\Sigma_t} |\partial g|^q \, dv_{\Sigma_t} + r^{-n+2q} \int_{\Sigma_t} |\partial^2 g|^q dv_{\Sigma_t} \leqslant C(n, q)$$

low regularity: only up to 2 derivatives of the metric

ADM formulation of the Einstein equations $R_{\alpha\beta} = 0$.

- Since $x^1, \ldots, x^n$ are harmonic coordinates on Σ_t, we have the **elliptic equations** $g^{kl} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + Q_{ij}(\partial g, \partial g) = -2R_{ij}$, where $Q_{ij}(\partial g, \partial g)$ is quadratic in ∂g.

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- Denote by $\mathbf{g}$ the Lorentzian metric. The second fundamental form $k_{ij} = \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, N \right\rangle$ satisfies **Einstein constraint equations**

$$R^{\Sigma}_{ijkl} + k_{ik} k_{jl} - k_{il} k_{kj} = \mathbf{R}_{ijkl}$$
$$\nabla_l k_{ij} - \nabla_i k_{lj} = \mathbf{R}_{liNj}$$

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- The induced metric g_{ij} and the second fundamental form k_{ij} satisfy **Einstein evolution equations**

$$\frac{\partial g_{ij}}{\partial x^0} = -2\lambda k_{ij} + \mathcal{L}_\xi g_{ij}$$
$$\frac{\partial k_{ij}}{\partial x^0} = -\nabla_i \nabla_j \lambda + \mathcal{L}_\xi k_{ij} - \lambda\, g^{pq} k_{ip} k_{qj} + \lambda\, \mathbf{R}_{iNjN}$$

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- We deduce an **elliptic equation for the shift vector** ξ

  differentiating the spatially harmonic condition $\Delta x^k = 0$ with respect to x^0 and using the CMC condition:
  $\Delta \xi^k = -g^{ki} R_{ij} \xi^j - (trk) g^{kl} \nabla_l \lambda + 2 g^{kl} g^{ij} k_{li} \nabla_j \lambda - 2\lambda\, g^{kl} \mathbf{R}_{lN}$.
- And an **elliptic equation for the lapse function** λ

1.5 Construction of the local canonical foliation

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1. **Lorentzian geodesic foliation** of the observer (p, T_p).
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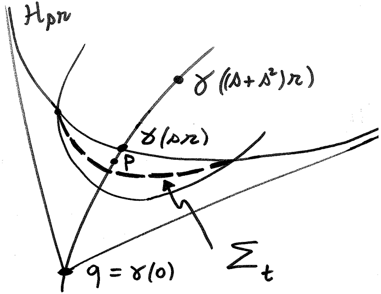
   - $\bigcup_\tau \mathcal{H}_\tau$: a neighborhood of p foliated by Lorentzian geodesic spheres centered at q in the past of p
   - $y = (y^\alpha) = (\tau, y^j)$: normal coordinates associated with radial geodesics from the point q



Three families of hypersurfaces:

- Lorentzian geodesic spheres
- Riemannian geodesic spheres
- CMC hypersurfaces

2. Distance Hessian comparison.

- on the orthogonal hyperplane $E := \left(\nabla\tau\right)^{\perp}$

$$\underline{k}(\tau,r)\, g_{ij} \leqslant (-\nabla^2\tau)|_{E,ij} \leqslant \overline{k}(\tau,r)\, g_{ij}$$

in which $\underline{k}(\tau,r) := \frac{r^{-1}C}{\tan\left(\tau\, r^{-1}C\right)}$ and $\overline{k}(\tau,r) := \frac{r^{-1}C}{\tanh\left(\tau\, r^{-1}C\right)}$

- constant C depending on the sup-norm of the curvature, only
- In particular, for the mean curvature we obtain the uniform control $n\,\underline{k}(\tau,r) \leqslant H_{\mathcal{H}_\tau} \leqslant n\,\overline{k}(\tau,r)$ solely in terms of our curvature norm.

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3. **Riemannian geodesic foliation** of the observer.

- Take $p' = \gamma(\tau)$ with $\tau \in [\underline{c}r, \overline{c}r]$
- For each $a \in [\underline{c}r, \overline{c}r]$, consider the Riemannian slice

$$\mathcal{A}(p', a) := S_{g_T}(p', a) \cap \mathcal{I}^+(q)$$

  determined by the reference metric g_T associated with T (parallel transport from T_p)

- For the mean curvature we obtain the uniform control $n\,\underline{k}(a, r) \leqslant H_{\mathcal{A}(p',a)} \leqslant n\,\overline{k}(a, r)$ solely in terms of our curvature norm.

4. Equations for the CMC foliation $\bigcup_t \Sigma_t$.

- The unknown hypersurfaces $\Sigma_t = \big\{(u^t(y), y)\big\}$ (with second fundamental form h_{ij}) are sought for
- as graphs over a given geodesic slice $\mathcal{H}_\tau$ (with second fundamental form A_{ij}) for a given τ
- Mean curvature equation

$$\mathcal{M}u := h_{ij}g^{ij} = \frac{1}{\sqrt{1 + |\nabla_\Sigma u|^2}} \Big(\Delta_\Sigma u + A_j{}^j\Big)$$

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- Nonlinear elliptic Partial Differential Equation
    - Barriers provided by the Lorentzian and Riemannian slices
    - Existence by the method of continuation, <u>provided Σ remains spacelike</u>

Expression of the mean curvature operator

- Setting $u_j := \partial u / \partial y^j$, the induced metric and its inverse read

$$g_{ij} = \mathbf{g}_{ij} - u_i u_j, \qquad g^{ij} = \mathbf{g}^{ij} + \frac{\mathbf{g}^{ik}\mathbf{g}^{jl} u_k u_l}{1 - |\nabla u|^2}.$$

- The hypersurface Σ_t is spacelike iff

$$|\nabla u|^2 = \mathbf{g}^{ij}(u, \cdot) u_i u_j < 1.$$

- ∇: covariant derivative associated with the induced metric g_{ij}:

$$|\nabla u|^2 = g^{ij} u_i u_j := \frac{|\nabla u|^2}{1 - |\nabla u|^2}$$

- Future-oriented unit normal

$$\mathbf{N} = -\sqrt{1 + |\nabla u|^2}\,(1, \nabla u)$$

- Second fundamental form of the slice Σ

$$h_{ij} = \frac{1}{\sqrt{1+|\nabla u|^2}}\left(\nabla_i \nabla_j u + A_{ij}\right)$$

- Mean curvature

$$\mathcal{M}u := h_{ij}g^{ij} = \frac{1}{\sqrt{1+|\nabla u|^2}}\left(\Delta_\Sigma u + A_j{}^j\right)$$

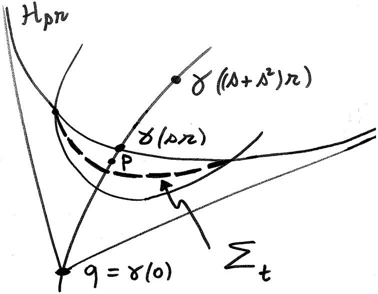
- In local coordinates

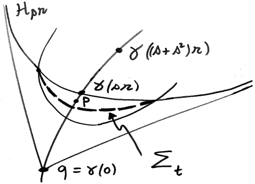
$$\begin{aligned}\mathcal{M}u =& \frac{1}{\sqrt{\mathbf{g}(u,\cdot)}}\frac{\partial}{\partial y^i}\left(\sqrt{\mathbf{g}(u,\cdot)}\,\nu(\nabla u)\,\mathbf{g}^{ij}(u,\cdot)\frac{\partial u}{\partial y^j}\right)\\ &+\left(\nu(\nabla u)^{-1}\mathbf{g}^{ij}(u,\cdot)+\nu(\nabla u)\,\mathbf{g}^{ik}(u,\cdot)\mathbf{g}^{jl}(u,\cdot)u_k u_l\right)\frac{1}{2}\frac{\partial \mathbf{g}_{ij}}{\partial \tau}(u,\cdot)\end{aligned}$$

with $\nu(\nabla u) := \frac{1}{\sqrt{1-|\nabla u|^2}} = \sqrt{1+|\nabla u|^2} = \nu(\nabla u)$

5. Localization of the CMC slices and existence of the foliation

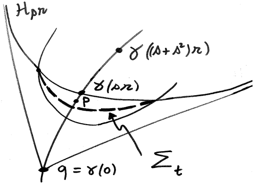
"Quantitative estimate": we must make sure that our parameters depend only on the assumed curvature bound.





- Fix $s \in [c, 2c]$ (small parameter) and consider the following two points in the future of p

$$p_{sr} := \gamma(sr), \qquad p'_{sr} := \gamma(s'r) \qquad\qquad s' = s + s^2$$



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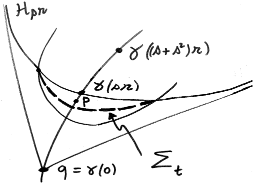
$$\partial\Omega_{sr} := \mathcal{A}\big(p'_s, s'sr\big) \cap \mathcal{H}_{\tau=sr} \qquad\qquad s's = s^2 + s^3$$



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$$p_{sr} := \gamma(sr), \qquad p'_{sr} := \gamma(s'r) \qquad\qquad s' = s + s^2$$

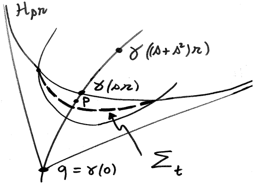
- Consider the subset $\Omega_{sr} \subset \mathcal{H}_{T=sr}$ of the geodesic slice, bounded by its intersection with a Riemannian 3-sphere, defined as follows:

$$\partial\Omega_{sr} := \mathcal{A}\big(p'_s, s'sr\big) \cap \mathcal{H}_{T=sr} \qquad\qquad s's = s^2 + s^3$$

- Our CMC hypersurface Σ_t: graph of the function u given by the *Dirichlet problem*

$$\mathcal{M}u = t \quad \text{in } \Omega_{sr} \qquad\qquad u = sr \quad \text{in } \partial\Omega_{sr}$$

for any chosen mean curvature value $t \in [n\,\overline{k}(sr, r), n\,\underline{k}(2s^2r, r)]$



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for any chosen mean curvature value $t \in [n\,\overline{k}(sr, r), n\,\underline{k}(2s^2 r, r)]$

- Finally, for each s, we specifically choose the slice corresponding to $t := 2k(sr, r) \in [k(sr, r), k(2s^2 r, r)]$.

1.6 Further ingredients for the proof

- Rely on the geometric structure of the problem / properties of the prescribed curvature problem
  - Simons identity : second fundamental form h_{ij} controled in terms of the ambiant curvature of the Lorentzian space

$$\begin{aligned}\Delta_\Sigma h_{ij} &= \Delta_\Sigma h_{ij} - (tr\, h)_{ij} \\ &= |h|^2 h_{ij} - (tr\, h) h_{ik} h_{lj} g^{kl} - R_{ipjq} h_{kl} g^{pk} g^{ql} + R_{jplq} h_{ik} g^{pq} g^{kl} \\ &\quad + \nabla_p (R_{qjNi}) g^{pq} - \nabla_j (R_{iN})\end{aligned}$$

  - Weizenbock identity : Global gradient estimate ensuring that the prescribed mean curvature equation is uniformly elliptic (cf. more details below)
- Quantitative estimates involving the curvature $\mathbf{Rm}_g$, only / Nash-Moser type technique

Spacelike nature of the CMC hypersurfaces

Lemma

Weitzenböck's identity and the prescribed CMC equation imply the following inequality satisfied by the Laplacian of $|\nabla u|^2$ on the hypersurface Σ_t

$$\Delta|\nabla u|^2 - 2|\nabla^2 u|^2 \gtrsim \langle \nabla u, \nabla \Delta u \rangle - \left(1 + |\nabla u|^2\right)^3$$

with, moreover,

$$|\Delta u| \lesssim 1 + |\nabla u|^2 =: \nu(\nabla u)^2$$

- At this stage, the operator Δ on Σ_t has possibly unbounded coefficients, since we do not control $|\nabla u|$ yet.

Proposition

The CMC hypersurfaces are spacelike:

$$\|\nabla u\|_{L^\infty} = \sup_{\Omega_{sr}} |\nabla u| \lesssim 1$$

Step 1. Estimate $\|\nabla u\|_{L^\infty}$ in term of $\|\nabla u\|_{L^{p_0}}$ for some finite p_0

- We set
$$v = (\nu^2 - k)_+ := (1 + |\nabla u|^2 - k)_+$$
with k so large that $v = 0$ on $\partial\Sigma$
- Choosing such a k is possible, since the desired gradient estimate *near* the boundary follows from the maximum principle

- Given $q \geqslant 1$, we multiply by v^q our Weitzenböck's inequality

$$\Delta|\nabla u|^2 - 2|\nabla^2 u|^2 \gtrsim \langle \nabla u, \nabla \Delta u \rangle - \left(1 + |\nabla u|^2\right)^3$$

  and then we integrate over the hypersurface Σ

- Using also that $|\Delta u| \lesssim 1 + |\nabla u|^2$, we obtain

$$\int_\Sigma \left(q\, v^{q-1} |\nabla v|^2 + v^q\, |\nabla^2 u|^2 \right) dv_\Sigma$$
$$\lesssim \int_\Sigma \left(q\, v^{q-1} \langle \nabla v, \nabla u \rangle + v^{q+3} + v^q \right) dv_\Sigma$$

- Setting $q =: 2m - 1$ we obtain (for all $m \geqslant 1$)

$$\|\nabla v^m\|^2_{L^2(\Sigma)} \lesssim m^2 \, \|v^{2m+2} + v^{2m-2}\|_{L^1(\Sigma)}$$

- Rewritting this in the coordinates y^j *in the geodesic slice* and applying the Sobolev inequality (in a fixed compact domain)

$$\|w\|^2_{L^{2n/(n-1)}(\Omega_{sr})} \lesssim \|g^{ij}\partial_i w \partial_j w + w^2\|_{L^1(\Omega_{sr})}$$

with the function $w := v^{p/2}$ with now $p := 2m - 1/2$, we deduce

$$\|v\|_{L^{pn/(n-1)}(\Omega_{sr})} \lesssim p^{2/p} \|v^{p+2} + v^{p-2}\|^{1/p}_{L^1(\Omega_{sr})}, \qquad p > 2$$

- Control the $L^{pn/(n-1)}$ norm of v in terms of its L^p norm

- Since $pn/(n-1) < p$, an iteration procedure allows us to control the sup norm of v
- Namely, without loss of generality, assume that $\|v\|_{L^\infty(\Omega_{sr})} \geqslant 1$ (for otherwise the result is immediate) so our main estimate reads

$$\max\left(1, \|v\|_{L^{pn/(n-1)}(\Omega_{rs})}\right) \lesssim p^{2/p} \|v\|^{2/p}_{L^\infty(\Omega_{rs})} \max\left(1, \|v\|_{L^p(\Omega_{rs})}\right)$$

and after iteration

$$\|v\|_{L^\infty(\Omega_{rs})} \lesssim \|v\|^{\alpha}_{L^\infty(\Omega_{rs})} \|v\|_{L^{p_0}(\Omega_{rs})}$$
$$\alpha := \frac{2}{p_0} \sum_{k=0}^{\infty} (1-1/n)^k = \frac{2n}{p_0}$$

- It suffices to take $p_0 > 2n$

Step 2. Uniform gradient estimate in a fixed L^{p_0} norm

- From $|\Delta u| \lesssim |\nu(\nabla u)|^2$ and for all $\lambda > 0$, we find

$$\begin{aligned}\Delta(e^{\lambda u}) &= \lambda^2 e^{\lambda u}|\nabla u|^2 + \lambda e^{\lambda u}\Delta u \\ &\gtrsim \lambda^2 e^{\lambda u}|\nabla u|^2 - \lambda\, e^{\lambda\, u}|\nu(\nabla u)|^2\end{aligned}$$

- From this and our Weitzenböck's inequality, we deduce

$$\begin{aligned}\Delta\big(v^{p_0} e^{\lambda u}\big) \gtrsim & - v^{p_0-1}\, e^{\lambda u}\big(\nu^2(\nu^4 + \lambda v) - \lambda^2(\nu - 1)\big) \\ & + \lambda p_0 v^{p_0-1} e^{\lambda u}\big\langle \nabla u, \nabla v\big\rangle \\ & + p_0 v^{p_0-1} e^{\lambda u}\big\langle \nabla u, \nabla(\Delta u)\big\rangle + p_0(p_0 - 1)v^{p_0-2} e^{\lambda u}|\nabla v|^2\end{aligned}$$

- $\nu^2(\nu^4 + \lambda v) - \lambda^2(\nu - 1) \lesssim v^3$, provided $k > 1$ is fixed and λ is arbitrarily large
- Integrate over Σ and proceed as in Step 1 (with large λ)

$$\int_\Sigma |\nabla u|^{p_0}\, dv_\Sigma \lesssim C_{p_0}$$

2. Euclidian-Hyperboloidal Foliations of Matter Spacetimes

With Yue Ma (Xi'an Jiaotong)

2.1 Field equations: Einstein and f(R)-modified gravity

Massive matter

- Einstein-Klein-Gordon equations
- main challenge: no invariance under scaling
- energy-momentum tensor
  $T_{\alpha\beta} := \nabla_\alpha \phi \nabla_\beta \phi - \Big(\frac{1}{2} g^{\alpha'\beta'} \nabla_{\alpha'} \phi \nabla_{\beta'} \phi + U(\phi) \Big) g_{\alpha\beta}$

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Einstein-Klein-Gordon system: typically $U(\phi) = c^2 \phi^2 / 2$

$$R_{\alpha\beta} - 8\pi \Big(\nabla_\alpha \phi \nabla_\beta \phi + U(\phi)\, g_{\alpha\beta}\Big) = 0$$

$$\Box_g \phi - U'(\phi) = 0$$

- nonlinear system of coupled wave and Klein-Gordon equations in wave (harmonic, De Donder) gauge

Generalized Hilbert-Einstein functional

$$\int_M \Big(f(R) + 16\pi L[\phi, g]\Big)\, dV_g \qquad f(R) = R + \frac{\kappa}{2} R^2 + \kappa^2 \mathcal{O}(R^3)$$

"mass parameter" $1/\kappa > 0$

The well-posed formulation of the f(R)-gravity theory

ArXiv gr-qc: 1412.8151
The mathematical validity of the f(R) theory of modified gravity,

PLF &, Y. Ma, Mémoires Société Math. France

Field equations of modified gravity $M_{\alpha\beta} = 8\pi T_{\alpha\beta}$

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- *fourth-order* field equations, well-posed Cauchy formulation
- vacuum Einstein solutions are vacuum f(R)-solutions

Conformal transformation $g^\dagger_{\alpha\beta} = e^{\kappa\rho} g_{\alpha\beta}$ with $\rho = \frac{1}{\kappa}\ln(f'(R))$

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- third-order in the (unphysical) conformal metric $g^\dagger$
- scaling with $\kappa \to 0$ so that $\rho \to R$

  $\rho = \rho(R)$ still referred to as the *scalar curvature field*

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Evolution of ρ trace of the field equations

Klein-Gordon equation for the spacetime scalar curvature

$$3\kappa \Box_{g^\dagger} \rho - \rho = 8\pi e^{-\kappa\rho} g^{\dagger\alpha\beta} T_{\alpha\beta} + f_2(\rho) \qquad |f_2(\rho)| \lesssim \kappa\rho^2$$

Field equations in the conformal metric

$$R^{\dagger}{}_{\alpha\beta} - 6\kappa^2 \nabla^{\dagger}{}_{\alpha}\rho\nabla^{\dagger}{}_{\beta}\rho - \frac{1}{2}e^{-2\kappa\rho} f_1(\rho)\, g^{\dagger}_{\alpha\beta}$$
$$= 8\pi e^{-2\kappa\rho}\left(T_{\alpha\beta} - \frac{1}{2}g^{\dagger}_{\alpha\beta}\left(g^{\dagger\alpha'\beta'} T_{\alpha'\beta'}\right)\right)$$

$$|f_1(\rho)| \lesssim \kappa\rho^2$$

- no fourth-order derivatives in the conformal metric $g^{\dagger}$
- only Ricci curvature, first-order derivatives of ρ

Field equations in the conformal metric

$$R^{\dagger}{}_{\alpha\beta} - 6\kappa^2 \nabla^{\dagger}{}_{\alpha}\rho\nabla^{\dagger}{}_{\beta}\rho - \frac{1}{2}e^{-2\kappa\rho} f_1(\rho)\, g^{\dagger}_{\alpha\beta}$$

$$= 8\pi e^{-2\kappa\rho}\left(T_{\alpha\beta} - \frac{1}{2}g^{\dagger}_{\alpha\beta}(g^{\dagger\alpha'\beta'} T_{\alpha'\beta'})\right)$$

$$|f_1(\rho)| \lesssim \kappa\rho^2$$

- no fourth-order derivatives in the conformal metric $g^{\dagger}$
- only Ricci curvature, first-order derivatives of ρ

We regard ρ as an independent unknown.

Klein-Gordon equation for the curvature field

$$3\kappa \Box_{g^{\dagger}} \rho - \rho = 8\pi e^{-\kappa\rho} g^{\dagger\alpha\beta} T_{\alpha\beta} + f_2(\rho)$$

Defining relations

$$g^{\dagger}_{\alpha\beta} = e^{\kappa\rho} g_{\alpha\beta} \qquad\qquad \rho = \frac{1}{\kappa}\ln(f'(R))$$

2.2 The Euclidian-Hyperboloidal Foliation Method

2015 ArXiv gr-qc: 1507.01143 The global nonlinear stability of Minkowski space for self-gravitating massive fields. The wave-Klein-Gordon model, Communications in Math. Phys. 346 (2016)

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Our foliation in conformal wave gauge $\Box_{g^\dagger} x^\alpha = 0$

- asympt. hyperbol. surfaces in the interior of the light cone (boost fields)
- asympt. flat hypersurfaces in the exterior (translation fields)
- a transition region connecting them near the light cone

[Sobolev inequalities, hyperboloidal-euclidian energy functional, etc.]

Some earlier related work for massless matter.

- hyperboloidal foliations for wave equations
  Friedrich 1981, Klainerman 1986, Hormander 1997
- wave gauge $\Box_g x^\alpha = 0$ Lindblad & Rodnianski (2010)
  Einstein-massless fields

f(R)-gravity for a self-gravitating massive field

$$\widetilde{\Box}_{g^\dagger} g^\dagger_{\alpha\beta} = F_{\alpha\beta}(g^\dagger, \partial g^\dagger) + 8\pi \Big(-2e^{-\kappa\rho} \partial_\alpha\phi \partial_\beta\phi + c^2 \phi^2 e^{-2\kappa\rho} {g^\dagger}_{\alpha\beta} \Big) - 3\kappa^2 \partial_\alpha\rho \partial_\beta\rho + \kappa\, \mathcal{O}(\rho^2) g^\dagger_{\alpha\beta}$$

$$\widetilde{\Box}_{g^\dagger} \phi - c^2\phi = c^2 \big(e^{-\kappa\rho} - 1\big)\phi + \kappa {g^\dagger}^{\alpha\beta} \partial_\alpha\phi \partial_\beta\rho$$

$$3\kappa \widetilde{\Box}_{g^\dagger} \rho - \rho = \kappa\, \mathcal{O}(\rho^2) - 8\pi e^{-\kappa\rho} \Big({g^\dagger}^{\alpha\beta} \partial_\alpha\phi \partial_\beta\phi + 2c^2\, e^{-\kappa\rho} \phi^2 \Big)$$

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- wave gauge conditions ${g^\dagger}^{\alpha\beta}{\Gamma^\dagger}^\lambda_{\alpha\beta} = 0$
- curvature compatibility $e^{\kappa\rho} = f'(R_{e^{-\kappa\rho} g^\dagger})$
- Hamiltonian and momentum constraints of modified gravity (propagate from any given Cauchy hypersurface)

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- Hamiltonian and momentum constraints of modified gravity (propagate from any given Cauchy hypersurface)

Taking the limit $\kappa \to 0$

Einstein system for a self-gravitating massive field

$$\tilde{\Box}_g g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g) + 8\pi \big(-2\partial_\alpha\phi\partial_\beta\phi + c^2\phi^2\, g_{\alpha\beta}\big)$$

$$\tilde{\Box}_g\phi - c^2\phi = 0$$

$$g^\dagger \to g \qquad \rho \to 8\pi\big(g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + 2c^2\phi^2\big)$$

Constructing the interior/exterior spacetime foliation

Global coordinate chart (t, x^a) $s^2 = t^2 - r^2$ and $r^2 := \sum (x^a)^2$

Asymptotically Killing fields

- translations ∂_α (tangent fields in the exterior)
- boosts $L_a = x_a \partial_t + t \partial_a$ (tangent fields in the interior)
- rotation fields $\Omega_{ab} = x_a \partial_b - x_b \partial_a$ (tangent fields exterior/interior)

but not on the scaling field $S = t\partial_t + r\partial_r$

Constructing the interior/exterior spacetime foliation

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but not on the scaling field $S = t\partial_t + r\partial_r$

We combine two foliations together:

- **Interior:** (asymptotically) hyperboloidal slices $\{t^2 - r^2 = s^2\} \subset \mathbb{R}^{3+1}$ with hyperbolic radius $s \geqslant s_0 > 0$ wave cone propagation

- **Exterior:** (asymptotically) Euclidian slices $\{t = c\} \subset \mathbb{R}^{3+1}$ of constant time t

asymptotic flatness

Asymptotically Euclidian-Hyperboloidal hypersurfaces $\mathcal{M}_s = \{t = T(s,r)\}$

- **Transition function** $\xi(s,r) = 1 - \chi(r + 1 - s^2/2) \in [0,1]$
  - based on a cut-off function χ
  - $\chi(y) = 0$ for $y \leqslant 0$ while $\chi(y) = 1$ for $y \geqslant 1$.
  - "transition" around $2r \simeq s^2 = t^2 - r^2$
- **Foliation parameter** s defined by $\partial_r T(s,r) := \frac{\xi(s,r)\, r}{\sqrt{s^2+r^2}}$ with $T(s,0) = s$
  - in the interior $T^2 = s^2 + r^2$
  - in the exterior $T = T(s) \simeq s^2$ independent of r, slow time

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Properties (tangent vector, deformation, etc.)

- tangent vectors boosts in the interior
- interpolation in the intermediate region
- translations in the exterior

2.3 Further ingredients in the method

The Euclidian-hyperboloidal energy

Weight function $\omega_\gamma = \chi(r-t)(r-t)^\gamma$ for some $\gamma \geqslant 0$

Weighted wave/Klein-Gordon energy for $\Box v - c^2 v$ with $c \geqslant 0$

$$E_c^\gamma(s, v) := \int_{\mathcal{F}_s} (1+\omega_\gamma)\left(\left(1-\xi^2\frac{r^2}{t^2}\right)(\partial_t v)^2 + \sum_a \left(\frac{\xi}{t}x^a\partial_t v + \partial_a v\right)^2 + c^2 v^2\right) dx$$

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Energy balance law

$$\begin{aligned} E_c^\gamma(s,v)^{1/2} \lesssim E_c^\gamma(s_0,v)^{1/2} &+ \int_{s_0}^{s} \|\Box v - c^2 v\|_{L^2(\mathcal{H}_{s'})}\,ds' \\ &+ \int_{s_0}^{s} \|\big(1+s(1-\xi^2)\big)^{1/2}(\Box v - c^2 v)\|_{L^2(\mathcal{T}_{s'})}\,ds' \\ &+ \int_{s_0}^{s} s'\,\|(1+\omega_\gamma)(\Box v - c^2 v)\|_{L^2(\mathcal{E}_{s'})}\,ds' \end{aligned}$$

Notation $\mathcal{M}_s := \mathcal{H}_s \cup \mathcal{M}_s \cup \mathcal{E}_s$

$$\mathcal{H}_s := \{t^2 = s^2 + r^2, \quad r \leqslant -1 + s^2/2\} \qquad \text{hyperboloidal interior region}$$

$$\mathcal{T}_s := \{-1 + s^2/2 \leqslant r \leqslant s^2/2, \quad t = T(s,r)\} \qquad \text{transition region}$$

$$\mathcal{E}_s := \{t = T(s), \quad r \geqslant s^2/2\} \qquad \text{Euclidian exterior region}$$

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Controled norms $s^2 = t^2 - r^2$ and $\xi = \xi(s,r) \in [0,1]$

$$\|\frac{s}{t}\partial_t u\|_{L^2(\mathcal{H}_s)} + \|\frac{1}{t}L_a u\|_{L^2(\mathcal{H}_s)} + c\|u\|_{L^2(\mathcal{H}_s)}$$

$$\Big\|\sqrt{t^2 - \xi^2 r^2}\frac{1}{t}\partial_t u\Big\|_{L^2(\mathcal{T}_s)} + \|\overline{\partial}_a u\|_{L^2(\mathcal{T}_s)} + c\|u\|_{L^2(\mathcal{T}_s)}$$

$$\Big\|(1+\omega_\gamma)\partial_t u\Big\|_{L^2(\mathcal{E}_s)} + \|(1+\omega_\gamma)\partial_a u\|_{L^2(\mathcal{E}_s)} + c\|(1+\omega_\gamma)u\|_{L^2(\mathcal{E}_s)}$$

Higher-order energies:

- based on the Killing fields of Minkowski
- we establish “good” commutator properties for our foliation

Functional inequalities

Define the Euclidian-hyperboloidal frame to be:

$$\overline{\partial}_s = \partial_s T \partial_t, \quad \overline{\partial}_a = \frac{\xi(s,r)}{t} x^a \partial_t + \partial_a$$

Translations ∂_α in the exterior / boosts $L_a = x^a \partial_t + t\partial_a$ in the interior

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Proposition. Sobolev inequalities without scaling field

For arbitrary functions u defined on the Euclidian-Hyperboloidal foliation one has

$$|u(x)| \lesssim t^{-3/2} \sum_{|I|+|J|\leqslant 2} \|\partial^I L^J u\|_{L^2(\mathcal{H}_s)} \qquad \text{hyperboloidal interior region}$$

$$|u(x)| \lesssim (1+r+t)^{-1} \sum_{|I|+|J|\leqslant 2} \|\overline{\partial}^I \Omega^J u\|_{L^2(\mathcal{T}_s)} \qquad \text{transition region}$$

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Many more ingredients

- Quasi-null structure of the Einstein equations in wave gauge
- Huyghens-Kirchhoff formula, etc.
- Hierarchy of energy bounds, bootstrap argument

2.4 Stability statements in wave gauge

Theorem. Stability of Minkowski space for massive fields (PLF-YM 2015–2017)

Consider the Einstein-massive field system in wave coordinates and initial data with Schwarzschild-like decay $g_{ab} \simeq \delta_{ab} + O(1/r)$ and $k_{ab} = O(1/r^2)$ satisfying Einstein's Hamiltonian and momentum constraints.

Then, there exist constants $\epsilon, \eta > 0$ (small) and $C_0 > 0$ (large) such that for any data satisfying

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$$E^{\gamma}(s_0, \partial^I L^J \Omega^K h_{\alpha\beta})^{1/2} \leqslant \epsilon \qquad P \leqslant N + 2$$

$$E_c^{\gamma+1/2}(s_0, \partial^I L^J \Omega^K \phi)^{1/2} \leqslant \epsilon, \qquad P \leqslant N + 2$$

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a global solution (g, ϕ) exists with a Euclidian-hyperb. foliation $\bigcup_{s \geqslant s_0} \mathcal{M}_s$

$$E^{\gamma}(s, \partial^I L^J \Omega^K h_{\alpha\beta})^{1/2} \leqslant C_0 \epsilon s^{\delta}, \qquad P \leqslant N$$

$$E_c^{\gamma+1/2}(s, \partial^I L^J \Omega^K \phi)^{1/2} \leqslant C_0 \epsilon s^{\delta+1/2}, \qquad P \leqslant N$$

$$E_c^{\gamma+1/2}(s, \partial^I L^J \Omega^K \phi)^{1/2} \leqslant C_0 \epsilon s^{\delta}, \qquad P \leqslant N - 4$$

In summary

1. **CMC foliations and spatially harmonic coordinates**
   - Local behavior, quantitative bounds
   - Notion of CMC–harmonic radius of an observer
   - Main result established with this method :

     "Bounded curvature" implies "controled Lorentzian geometry"

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1. **CMC foliations and spatially harmonic coordinates**
   - Local behavior, quantitative bounds
   - Notion of CMC–harmonic radius of an observer
   - Main result established with this method :

     "Bounded curvature" implies "controled Lorentzian geometry"

2. **Euclidian-hyperboloidal foliations and wave coordinates**
   - Global construction, weighted Sobolev norms
   - Control the decay of solutions at time-like and space-like infinity
   - Main result established with this method :

     global nonlinear stability of massive fields
     (under smallness conditions)