Optimal transportation in Lorentzian geometry

Stefan Suhr (Ruhr-Universität Bochum)

June 21, 2018

Transport Problem: Given two probability measures

$$\mu, \nu$$

on a manifold M.

What is the **optimal** fashion of **transferring** μ to ν?

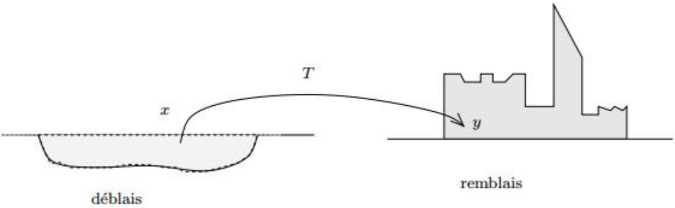

Fig. 3.1. Monge's problem of déblais and remblais

(Source: C. Villani. Optimal Transport, Old and New. Springer (2009))

Introduction by Monge (1781) and (in a relaxed form) by Kantorovich (1942)

Transfer according to Kantorovich: By means of a p-measure π on $M \times M$ such that

$$\pi(A \times M) = \mu(A) \text{ and } \pi(M \times B) = \nu(B)$$

for all $A, B \subset M$. π is called a **coupling of μ and ν**. μ and ν are the **martingales** of π.

Optimality according to Kantorovich: Transferring mass is subject to a cost function

$$c \colon M \times M \to \mathbb{R} \cup \{\infty\},$$

usually related to the distance with respect to a Riemannian metric. One minimizes the **cost**

$$\pi \mapsto \int c \, d\pi$$

of the coupling π with martingales μ and ν. If a coupling realizes the infimum, then it is **optimal**.

$\rightsquigarrow$ Resembles a variational problem with fixed boundary values.

Transfer according to Monge: By means of a Borel map $F \colon M \to M$ such that

$$\nu = F_\sharp(\mu),$$

where $F_\sharp(\mu)(A) := \mu(F^{-1}(A))$ is the **push forward of μ under F**.

Optimality according to Monge: For the cost function

$$c \colon M \times M \to \mathbb{R} \cup \{\infty\}$$

one minimizes the **cost**

$$F \mapsto \int c(x, F(x))\, d\mu(x)$$

among the maps F which push μ forward to $\nu = F_\sharp(\mu)$. A map realizing the infimum is **optimal**. Again this resembles a variational problem with fixed boundary values.

Relation between Kantorovich transport and Monge transport:

- If F pushes μ forward to ν then $\pi := (\mathrm{id}, F)_\sharp \mu$ is a coupling of μ and ν, i.e.

$$\inf \left\{ \int c \, d\pi \right\} \leq \inf \left\{ \int c(x, F(x)) \, d\mu(x) \right\}$$

- A map $F\colon M \to M$ with $F_\sharp(\mu) = \nu$ does not exist in general, e.g.

$$\mu = \delta_x, \ \nu = \frac{1}{2}(\delta_y + \delta_z).$$

$\rightsquigarrow$ Transport has to "split" mass.

- Existence of optimal couplings/maps for real valued cost functions by
  - Kantorovich '42 (Kantorovich optimality) and
  - Brenier '89 (Monge optimality) for $\mu \ll \mathcal{L}$ with equality of both infima.

A "logistics problem" at heart. E.g. μ describes a distribution of mines and ν describes a distribution of factories. The cost function $c(x, y)$ measures the transport cost from x to y.
BUT in the case of

$$c = \text{dist}^p \text{ with } p \geq 1$$

the convexity properties of certain functionals defined via optimal transportation are equivalent to bounds on the Ricci curvature.
Compare

Myer's Theorem

(M, g) complete with $Ric_g \geq (dimM - 1)\varepsilon \Rightarrow diam(M, g) \leq \pi/\sqrt{\varepsilon}$.

Optimal transport is more flexible than (smooth) Riemannian geometry. $\rightsquigarrow$ "synthetic Ricci curvature" for "metric measure spaces". MMS with Ricci curvature bounded from below have good compactness properties with respect to measured GH-convergence.

distant goal or beacon: a similar theory in Lorentzian geometry.

Lorentzian Formulation: Let (M, g) be globally hyperbolic. Define

$$\begin{aligned} c_g \colon M \times M &\to \mathbb{R} \cup \{\infty\} \\ (x, y) &\mapsto \begin{cases} -d_g(x, y), & (x, y) \in J^+ \\ \infty, & \text{else.} \end{cases} \end{aligned}$$

Recall that

$$d_g(x, y) := \sup\{L^g(\gamma) | \, \gamma \text{ future pointing from } x \text{ to } y\}.$$

Attention: Changed sign convention! c_g is "convex" in this formulation.

Lorentzian Transport Problem: Given two probability measures μ and ν on M. Does there exist a coupling π of μ and ν minimizing the **Lorentzian cost**

$$\sigma \mapsto \int c_g \, d\sigma$$

among all p-measures σ on $M \times M$ with martingales μ and ν?

Work so far on Lorentzian transportation:

- **Brenier '92:** $\mathbb{R}^{n+1}_1$ and the cost as above with μ concentrated on $\{0\} \times \mathbb{R}^n$ and ν concentrated on $\{t\} \times \mathbb{R}^n$. Studied by Bertrand/Puel, Bertrand/Pratelli/Puel, Louet/Pratelli/Zeisler among others. Considered costs are **relativistic costs functions**.
- **Frisch et al. '02, Brenier et al. '03:**

  early universe reconstruction problem

  Question: What is the genesis of the mass distribution in the universe from the Big-Bang to what we see today?
  Modeled on a FLRW-spacetime $(\mathbb{R} \times \Sigma, g)$. On large scales (galaxy cluster) and with a "semi-Newtonian limit" the problem is described by optimal transportation.
- **Eckstein/Miller '17, Miller '17/'18:**

  Causal evolution of measures
- **Kunzinger/Sämann '18:**

  Lorentzian length spaces

Existence of optimal couplings:

Theorem (Bernard/-'18)

Let (M, g) be globally hyperbolic and h a Riemannian metric. Then there exists a smooth function $\tau \colon M \to \mathbb{R}$ with

$$d\tau(v) \geq \|v\|_h$$

for all future pointing $v \in TM$. Especially τ is temporal.

Remark

Interesting when h is complete with $h(v, v) \geq |g(v, v)|$ for all future pointing $v \in TM$. $\rightsquigarrow$ **steep Lyapunov functions**
Related results by Müller/Sanchez '11 and Minguzzi '16.

Proposition (Lorentzian Kantorovich Problem)

Let μ, ν be p-measures on M such that $\tau \in L^1(\mu) \cap L^1(\nu)$. Then there exists a optimal coupling π of μ and ν with

$$\int c_g \, d\pi \in \mathbb{R} \cup \{\infty\}.$$

Sketch of proof:

Lemma

The set of couplings of μ and ν is compact with respect to the weak topology on measures.

The proof of the lemma uses Prokhorov's Theorem.

$$d\tau(v) \geq \sqrt{|g(v,v)|} \text{ for all } v \in TM \text{ future pointing}$$

$$\Rightarrow c_g(x,y) \geq \tau(y) - \tau(x)$$

$$\Rightarrow \int c_g \, d\pi \geq \int [\tau(y) - \tau(x)] \, d\pi(x,y) = \int \tau \, d\nu - \int \tau \, d\mu > -\infty$$

With this lower bound one proves the **lower semicontinuity** of the cost:

Lemma

Assume that $\tau \in L^1(\mu) \cap L^1(\nu)$. If a sequence $\{\pi_k\}_{k\in\mathbb{N}}$ of couplings of μ and ν converges weakly to π, then

$$\int c_g \, d\pi \leq \liminf_{k\to\infty} \int c_g \, d\pi_k.$$

When is the costs of a coupling finite?

Observation:

(1)

$$\int c_g \, d\pi \in \mathbb{R} \Rightarrow \text{supp } \pi \subset J^+$$

(2) If supp $\pi \subset J^+$, then

$$\begin{aligned}\mu(A) = \pi(A \times M) &= \pi(J^+ \cap (A \times M)) \\ &\leq \pi(A \times J^+(A)) \\ &\leq \pi(M \times J^+(A)) = \nu(J^+(A))\end{aligned}$$

for all $A \subset M$ Borel and analogously

$$\nu(B) \leq \mu(J^-(B))$$

for all $B \subset M$ Borel.

Question: If $\mu(A) \leq \nu(J^+(B))$ and $\nu(B) \leq \mu(J^-(B))$ for all $A, B \subset M$ Borel, does there exist a coupling π with supp $\pi \subset J^+$?

Abstract formulation: Let $(\mathcal{X}, d_\mathcal{X})$ and $(\mathcal{Y}, d_\mathcal{Y})$ be Polish spaces (complete and separable). Given $\mathcal{J} \subset \mathcal{X} \times \mathcal{Y}$ define for $A \subset \mathcal{X}$ and $B \subset \mathcal{Y}$

$$\mathcal{J}^+(A) := p_\mathcal{Y}((A \times \mathcal{Y}) \cap \mathcal{J}) \subset \mathcal{Y}$$

and

$$\mathcal{J}^-(B) := p_\mathcal{X}((\mathcal{X} \times B) \cap \mathcal{J}) \subset \mathcal{X}$$

for the canonical projections $p_{\mathcal{X},\mathcal{Y}} \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{X}, \mathcal{Y}$. Note that $\mathcal{J}$ is completely general, especially not necessarily a causal structure.

Definition

Two p-measures μ,ν on $\mathcal{X}$ and $\mathcal{Y}$, respectively, are $\mathcal{J}$-**related** if there exists a coupling π with supp $\pi \subset \mathcal{J}$.

Theorem (-'18)

Let $\mathcal{J} \subset \mathcal{X} \times \mathcal{Y}$ be closed. Then two p-measure μ and ν are $\mathcal{J}$-related if and only if $\nu(\mathcal{J}^+(A)) \geq \mu(A)$ and $\mu(\mathcal{J}^-(B)) \geq \nu(B)$ for all $A \subset \mathcal{X}$, $B \subset \mathcal{Y}$ Borel.

Remark

Similar results by Eckstein/Miller'17 for spacetimes.

Idea of the proof: Approximate both measures by finite measures

$$\mu_{app} = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i},\ \nu_{app} = \frac{1}{n}\sum_{j=1}^{n}\delta_{y_j}$$

with $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$.

Lemma

There exists a permutation $\sigma \in S(n)$ with $(x_i, y_{\sigma(i)}) \in \mathcal{J}$ for all $i \in \{1, \ldots, n\}$ if and only if

$$\sharp\{j|\ (x_i, y_j) \in \mathcal{J} \text{ for an } i \in A\} \geq \sharp A$$

and

$$\sharp\{i|\ (x_i, y_j) \in \mathcal{J} \text{ for an } j \in B\} \geq \sharp B$$

for all $A, B \subset \{1, \ldots, n\}$

Proof of the Lemma is by induction over n. σ induces a coupling of μ_{app} and ν_{app}. The closedness of $\mathcal{J}$ ensures that one can pass to the limit in the approximation and obtain a coupling π with supp $\pi \subset \mathcal{J}$.

Lorentzian Monge problem: Under what conditions is mass not split up during the transport? More precisely:

(1) Under what assumptions does there exists an optimal Borel map $F\colon M \to M$ in the Lorentzian transport problem à la Monge?

(2) Under what assumptions is

$$\pi := (\mathrm{id}, F)_\sharp \mu$$

an optimal coupling? [Recall $F_\sharp \mu(A) := \mu(F^{-1}(A))$]

Definition

A pair of p-measures (μ, ν) belongs to $\mathcal{P}_\tau^+(M)$ if μ and ν are J^+-related and $\tau \in L^1(\mu) \cap L^1(\nu)$.

Remark

For a pair $(\mu, \nu) \in \mathcal{P}_\tau^+(M)$ one has

$$\inf \left\{ \int c_g \, d\pi \,\middle|\, \pi \textit{ is a coupling of } \mu \textit{ and } \nu \right\} \in \mathbb{R}.$$

Theorem A (Kell/-'18, solution to the Monge problem)

Let $(\mu, \nu) \in \mathcal{P}_\tau^+(M)$ with $\mu \ll \mathcal{L}$. Then there exists a Borel map $F\colon M \to M$ such that

$$\pi := (id, F)_\sharp \mu$$

is an optimal coupling of μ and ν.

Theorem B (-'18, unique solution to the Monge problem)

Let $(\mu, \nu) \in \mathcal{P}_\tau^+(M)$ with $\mu \ll \mathcal{L}$ or $\mu \ll \mathcal{L}_A$ for a spacelike hyper surface $A \subset M$ and ν concentrated on an achronal set. Then there exists an **unique** *optimal coupling π of μ and ν and a Borel map $F\colon M \to M$ such that*

$$\pi := (id, F)_\sharp \mu.$$

Remark

Theorem B generalizes the early universe reconstruction problem for FLRW-spacetimes.

Problem of the proof of Theorem B: Let π be an optimal coupling. Show that

$$\{x \in M \mid \exists y \neq z \in M \text{ with } (x,y),(x,z) \in \text{supp}(\pi)\}$$

is $\mathcal{L}$-negligible (if $\mu \ll \mathcal{L}$) or $\mathcal{L}_A$-negligible (if $\mu \ll \mathcal{L}_A$).
Observation: If $(x,y),(x,z) \in \text{supp}(\pi)$, then x, y, z lie on a common maximal geodesic.

Proposition

Let $A \subset M$ be a spacelike hyper surface and $B \subset M$ achronal. Further let Γ_B be the set of maximal causal geodesics γ that intersect B more than once. Then

$$\mathcal{L}_A(\{x \in A \mid x \in \gamma \in \Gamma_B\}) = 0.$$

Since an achronal set is the graph of a locally Lipschitz function over a part of a Cauchy hyper surface, it has a well-defined tangent space almost everywhere.
$\rightsquigarrow$ **Question:** Does the previous proposition holds also for maximal geodesics tangent to B?

The dynamical picture: A **dynamical coupling** is a Borel p-measure Π on $C^0([0,1], M)$. Let

$$\mathrm{ev}_t \colon C^0([0,1], M) \to M,\ \eta \mapsto \eta(t)$$

be the evaluation map at $t \in [0,1]$. Then $\pi := (\mathrm{ev}_0, \mathrm{ev}_1)_\sharp \Pi$ is a coupling of $\mu := (\mathrm{ev}_0)_\sharp \Pi$ and $\nu := (\mathrm{ev}_1)_\sharp \Pi$.
Consider:

$$\Gamma := \{\gamma \colon [0,1] \to M |\ \gamma \text{ maximizes } L^g \text{ between its endpoints and } d\tau(\dot\gamma) = \text{const}\}$$

$\Rightarrow \gamma \in \Gamma$ is a pregeodesic and

$$\Gamma_{x \to y} := \{\gamma \in \Gamma |\ \gamma(0) = x,\ \gamma(1) = y\}$$

is compact (independent of the C^k-topology on Γ).

Definition

A p-measure Π on Γ is a **dynamical optimal coupling** if $(\mathrm{ev}_0, \mathrm{ev}_1)_\sharp \Pi$ is an optimal coupling of $(\mathrm{ev}_0)_\sharp \Pi$ and $(\mathrm{ev}_1)_\sharp \Pi$.

Proposition

For every pair $(\mu, \nu) \in \mathcal{P}_\tau^+(M)$ there exists a dynamical optimal coupling Π of μ and ν.

Sketch of proof: $\Gamma_{x \to y}$ is compact and nonempty for all $(x, y) \in J^+$.

Proposition

There exists a Borel map $S\colon J^+ \to \Gamma$ with $S(x, y) \in \Gamma_{x \to y}$.

Remark

S is called a **selection***, i.e. a right-inverse of $(ev_0, ev_1)\colon \Gamma \to J^+$, i.e.*

$$(ev_0, ev_1) \circ S = id_{J^+}.$$

Let π be an optimal coupling of μ and ν. $\Rightarrow \Pi := S_\sharp \pi$ is the desired dynamical optimal coupling since

$$(p_1 \circ (\mathrm{ev}_0, \mathrm{ev}_1))_\sharp \Pi = \mu \text{ and } (p_2 \circ (\mathrm{ev}_0, \mathrm{ev}_1))_\sharp \Pi = \nu$$

for the canonical projections $p_{1,2}\colon M \times M \to M$, $(x, y) \mapsto x$ and y, respectively.

Intermediate measures and regularity: Define the map

$$[\partial_t \mathrm{ev}]\colon \Gamma \times [0,1] \to PTM,\ (\gamma, t) \mapsto [\dot\gamma(t)] \in PTM_{\gamma(t)}$$

where PTM denotes the projective tangent bundle.

Theorem (-'18)

Let $(\mu, \nu) \in \mathcal{P}^+_\tau(M)$ with supp$(\mu) \cap$ supp$(\nu) = \emptyset$. Then every dynamical optimal coupling Π of μ and ν has the following property: The canonical projection $P\colon PTM \to M$ restricted to the image of $T := [\partial_t ev](supp\Pi \times]0,1[)$ is injective. Further the inverse $(P|_T)^{-1}$ is locally Hölder continuous with exponent $1/2$.

Remark

- *The result is optimal as formulated.*
- *For $T \subset \{$timelike vectors$\}$, the map $(P|_T)^{-1}$ is locally Lipschitz.*
- *The theorem implies that measures are transported along a Hölder continuous geodesic vector field.*

Open Problems:

"You name it, we've got it!"

- smoothness of the optimal transportation
  $\rightsquigarrow$ Lorentzian Ma-Trudinger-Wang condition to be found
- "Lorentzian" Wasserstein spaces (tentative definition, $p \in (0,1]$)

$$P_L^p(M) := \left\{ \mu \,\middle|\, \int_M |c_g(x, \{\tau = 0\})|^p d\mu < \infty \right\}$$

  $\rightsquigarrow$ Correct frame for the variational analysis; basic to the advancement of the theory; Which exponent p is best?
- Displacement convexity, Boltzman's H-functional and its relation with Ricci curvature
- Lorentzian measure spaces, synthetic Ricci curvature, measured Gromov-Hausdorff convergence

References:

- Bernard, P., Suhr, S., *Lyapounov Functions of closed Cone Fields: from Conley Theory to Time Functions*, Comm. Math. Phys. **359** (2018), 467–498.
- Suhr, S., *Theory of Optimal Transport for Lorentzian Cost Functions*, to appear in Münster J. Math.
- Kell, M., Suhr, S. *On the Existence of Dual Solutions for Lorentzian Cost Functions*, preprint (2018).