Asymptotic Behavior of Massless Fields in Arbitrary Dimensions and the Gravitational Memory Effect

Robert M. Wald

work in progress with Gautam Satishchandran

The Nature of This Work

This work will be concerned with the asymptotic behavior of massless fields in asymptotically flat spacetimes of arbitrary dimension. The focus will be on the relations between behavior at *different powers* of fall-off in an assumed $1/r$ type expansion (an assumption that is slightly weaker than smoothness at scri in even dimensional spacetimes). No attempt will be made to investigate issues like convergence of the expansion.

Although the primary application of this work will be to the memory effect in $d > 4$ spacetime dimensions, all of this talk will focus on linear scalar, electromagnetic, and linearized gravitational fields on Minkowski spacetime.

Asymptotic Behavior of a Scalar Field

Consider a scalar field, ϕ, in d-dimensional Minkowski spacetime, $d \geq 4$, satisfying the massless Klein-Gordon equation

$$\Box\phi = f\,.$$

Introduce retarded null coordinates $(u \equiv t - r,\, r,\, x^A)$, where x^A are coordinates on the $(d-2)$-sphere. Assume as an *ansatz* that ϕ admits an expansion of the form

$$\phi = \sum_n \frac{\phi^{(n)}(u, x^A)}{r^n}$$

and that f admits a similar expansion. (More generally, in these expansions, one could replace n by $n + \alpha$ for any

$\alpha \in [0, 1)$, but we will soon restrict to $\alpha = 0$ in any case. For $\alpha = 0$, this ansatz is equivalent to smoothness at scri.) Then the Klein-Gordon equation becomes the following sequence of equations

$$\left[\mathcal{D}^2 + (n-1)(n-d+2)\right]\phi^{(n-1)} + (2n-d+2)\partial_u\phi^{(n)} = f^{(n+1)}$$

where $\mathcal{D}^2$ denotes the unit sphere Laplacian. There are two special values of n that stand out in this equation:

- $n = d-3$ ("Coulombic order"): At this value of n and all smaller values of n, the term in square brackets is a negative definite operator.
- $n = d/2 - 1$ ("radiative order"): At this value of n the coefficient of the second term vanishes.

Homogeneous Wave Equation (Even Dimensions)

Set $f = 0$ and consider the case where d is even. Can "solve" the homogeneous wave equation as follows: At Coulombic order ($n = d - 3$), specify $\phi^{(d-3)}(u, x^A)$ arbitrarily. Obtain the unique solution at one order slower fall-off

$$\phi^{(d-4)} = -(d-4)\left[\mathcal{D}^2 - 2(d-4)\right]^{-1}\partial_u \phi^{(d-3)}$$

Iterate to get $\phi^{(d-5)}$, etc. The sequence terminates at radiative order, $n = d/2 - 1$.

To obtain the solution at one power faster fall-off than Coulombic, integrate the equation

$$\partial_u \phi^{(d-2)} = -\frac{1}{d-2}\mathcal{D}^2 \phi^{(d-3)}$$

which has a unique solution up to adding a u-independent term, which can be fixed by initial conditions at $u = u_0$ (or $u \to -\infty$). Iterate to get successively faster fall-off terms. For any lth spherical harmonic dependence of ϕ, the right side will vanish at $n = l + d - 2$, so the sequence may be terminated at that order.

Remarks:

- The above results show that the radiative order term is related to the Coulombic order term by a formula of the form

$$\phi^{(d/2-1)} = \partial_u^{(d/2-2)}\left[(\mathcal{D}_1^2)^{-1}\dots(\mathcal{D}^2_{(d/2-2)})^{-1}\phi^{(d-3)}\right]$$

- If instead we had tried the ansatz

$$\phi = \sum_n \frac{\phi^{(n)}(u,x^A)}{r^{(\alpha+n)}}$$

with $0 < \alpha < 1$, then "radiative order" would never be attained. One can choose $\phi^{(n)}(u,x^A)$ at any order and uniquely solve for all slower fall-off orders. Unless $\phi^{(n)}(u,x^A)$ is a polynomial in u, the sequence never terminates. Thus, one must use integral powers of $1/r$ to get general solutions in even dimensions.

Odd Dimensions

In odd dimensions, a similar analysis shows that if we specify $\phi^{(n)}(u, x^A)$ at any order, we can uniquely solve for all slower fall-offs. For a non-polynomial u-dependence, the slower fall-off sequence never terminates unless we choose $\alpha = 1/2$. Thus, we must expand ϕ in half-integral powers of $1/r$. Otherwise, the analysis is completely analogous to even dimensions, except that "Coulombic order" is never attained and the faster fall-off sequence does not terminate for any spherical harmonic.

In addition to the half-integral sequence, one can additionally have integral powers of $1/r$ (e.g., static multipoles), but these must have polynomial dependence in u in order to avoid blowing up at infinity.

For the remainder of this talk, I will restrict consideration to even dimensions.

Scalar Wave Equation with Source

For a source, f, it would be physically reasonable to demand that its integral over a sphere at large r be finite, i.e., $f^{(n)} = 0$ for all $n < d - 2$. Thus, the slowest fall-off appearance of f is in the equation

$$\left[\mathcal{D}^2 - (d-4)\right]\phi^{(d-4)} + (d-4)\partial_u \phi^{(d-3)} = f^{(d-2)}$$

For $d > 4$, we may again specify $\phi^{(d-3)}(u, x^A)$ arbitrarily and solve this equation for $\phi^{(d-4)}$, and then successively solve for the slower fall-offs as before. We also may solve for the faster fall-offs as before, so the presence of f makes no significant difference. However, when $d = 4$ we must solve

$$\mathcal{D}^2\phi^{(0)} = f^{(2)}$$

If $f^{(2)}$ has no $l = 0$ part, this can be solved straightforwardly. If $f^{(2)}$ has an $l = 0$ part, then this equation cannot be solved within the ansatz. However, if this $l = 0$ part is u-independent, then it can be solved by adding the single term $\ln r$ to ϕ. On the other hand, if the $l = 0$ part of $f^{(2)}$ has general u-dependence, one would need to include an infinite series of new terms to ϕ involving $\ln r/r^n$.

Maxwell's Equations

We now consider Maxwell's equations

$$\nabla^a(\nabla_a A_b - \nabla_b A_a) = -4\pi j_b$$

in d-dimensional Minkowski spacetime with $d \geq 4$ and even. Assume the ansatz

$$A_\mu = \sum_{n=1}^{\infty} \frac{A_\mu^{(n)}(u, x^A)}{r^n}$$

(Cartesian components). This is slightly weaker than assuming smoothness at scri. Can A_a be put in Lorenz gauge, $\nabla^a A_a = 0$? This requires solving the scalar wave equation for the gauge function ϕ

$$\Box\phi = \nabla^a A_a$$

This is exactly the issue that we just analyzed, except the source is not "physical" and one can, in principle, have contributions starting at order $1/r$ in all dimensions. However, if A_a satisfies the *source free* Maxwell equations, one can show that there is a unique solution of the equations for ϕ^n for all $1 \leq n \leq d/2 - 2$ which satisfies,

$$\partial_u \phi^{(n)} = A_u^{(n)}$$

Since the equations can be trivially solved for $n \geq d/2 - 1$, the issue of putting A_a in Lorenz gauge boils down to solving the equation for $\phi^{(0)}$

$$\mathcal{D}^2\phi^{(0)} = (\nabla^a A_a)^{(2)} + (d-4)A_u^{(1)}$$

Using the source free Maxwell equations again, the right side can be shown to have vanishing u-derivative, so this equation can also be solved. Thus, under the above ansatz for A_a, every source free solution to Maxwell's equation can be put in the Lorenz gauge, preserving the ansatz.

If a charge-current source j_a is allowed but is required to fall-off as $1/r^{(d-2)}$, then for $d > 4$, j_a does not enter the relevant part of the above analysis, and this conclusion continues to hold.

However, when $d = 4$, if one has a nonvanishing flux of charge to infinity, the last step of the analysis fails, and one cannot impose the Lorenz gauge condition in $d = 4$.

Maxwell's Equations in Lorenz Gauge

Maxwell's equations in Lorenz gauge are simply

$$\Box A_b = -4\pi j_b$$

Since each Cartesian component satisfies the scalar wave equation, all of the fall-off results of the scalar case continue to hold.

It is much more convenient to work with the components A_u, A_r, A_A of the coordinates that we have introduced. Our conventions on the assignment of "$1/r$ orders" is that the superscript "(n)" will denote that the "physical components" are falling as $1/r^n$. Thus, the (angular) coordinate components of $A_A^{(n)}$ as $1/r^{(n-1)}$. Maxwell's equations are

$$\left[\mathcal{D}^2 + (n-1)(n-d+2)\right] A_u^{(n-1)} + (2n-d+2)\partial_u A_u^{(n)} = -4\pi j_u^{(n+1)}$$

$$\left[\mathcal{D}^2 + n(n-d+1)\right] A_r^{(n-1)} + (d-2) A_u^{(n-1)} + (2n-d+2)\partial_u A_r^{(n)} - 2\mathcal{D}^A A_A^{(n-1)} = -4\pi j_r^{(n+1)}$$

$$\left[\mathcal{D}^2 + (n-1)(n-d+2) - 1\right] A_A^{(n-1)} - 2\mathcal{D}_A (A_u^{(n-1)} - A_r^{(n-1)}) - (2n-d+2)\partial_u A_A^{(n)} = -4\pi j_A^{(n+1)}$$

In addition, the Maxwell field has to satisfy $\nabla^a A_a = 0$. This condition, in conjunction with the wave equation, gives rise to the constraints

$$\begin{aligned}[\mathcal{D}^2 &- (n-d+2)(n-d+3)]A_r^{(n)} + (2n-d+2)\mathcal{D}^A A_A^{(n)} \\ &+ (2n-d+2)(n-d+3)A_u^{(n)} = -4\pi j_r^{(n+2)}\end{aligned}$$

If one wishes to solve Maxwell's equations by choosing data at Coulombic order ($n = d-3$), then one must impose this constraint at $n = d-3$

$$\mathcal{D}^2 A_r^{(d-3)} + (d-4)\mathcal{D}^A A_A^{(d-3)} = -4\pi j_r^{(d-1)}$$

It turns out that to automatically ensure that the constraints are satisfied at all $n < d-3$, it is only necessary to impose the following additional constraint on the $l = 0$ part of the Coulombic order data

$$\partial_u\left[(d-4)A_u^{(d-3)}|_{l=0} + A_r^{(d-3)}|_{l=0}\right] = -4\pi(d-4)j_u^{(d-2)}$$

In addition, one must impose the above constraints at an initial time $u = u_0$ for all $n > d-3$.

Linearized Gravity

Now consider the linearized Einstein equation

$$G_{ab}^{(1)} = 8\pi T_{ab}$$

for the metric perturbation h_{ab} in d-dimensional Minkowski spacetime, with d even and $d \geq 4$. Assume the ansatz

$$h_{\mu\nu} = \sum_{n=1}^{\infty} \frac{h_{\mu\nu}^{(n)}(u, x^A)}{r^n}$$

(Cartesian components). This is slightly weaker than assuming smoothness at scri. Can h_{ab} be put in harmonic gauge, $\nabla^a \bar{h}_{ab} = 0$, where $\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab} h$? This requires solving the vector equation for the gauge function ψ_a

$$\Box\psi_b = \nabla^a \bar{h}_{ab}$$

Assuming $T_{\mu\nu} = O(1/r^{(d-2)})$, the result is : (i) For $d > 4$, the harmonic gauge condition can be imposed within the ansatz. (ii) For $d = 4$, the harmonic gauge condition can be imposed in linearized gravity, if there is no flux of energy to infinity. It *cannot* be imposed within the ansatz if there is an energy flux to infinity. Nonlinear gravity in radiative spacetimes is like linearized gravity with a stress-energy flux to infinity. The (nonlinear) harmonic gauge condition cannot be imposed in $d = 4$, but there should not be a problem with imposing it in $d > 4$.

The linearized Einstein equation in harmonic gauge takes a form analogous to the Maxwell equations equations when written in terms of the components in coordinates (u, r, x^A) takes a form analogous to the Maxwell equations (but more complicated). The constraints are now d in number and also take a similar form. The equations can again be solved by (i) specifying the metric perturbation at Coulombic order subject to the Coulombic order constraints and an additional constraint on the $l = 0$ part of the perturbation, (ii) uniquely solving for the slower than Coulombic fall-off orders, and (iii) solving for the faster fall-off orders, imposing the constraints on the solutions at $u = u_0$.

The Memory Effect

The memory effect is the permanent displacement of the relative displacement of test particles after the passage of a gravitational wave. These relative displacements are governed by the geodesic deviation equation

$$\frac{d^2\xi^a}{d\tau^2} = -R_{cbd}{}^{a} u^c u^d \xi^b$$

Geodesic deviation at large distances at fall-off order $n \leq d-3$ is thus determined by the electric part of the Weyl tensor, $\mathcal{E}_{ab} = C_{acbd} u^c u^d$. For $n \leq d-3$, if the double u-integral of $\mathcal{E}_{ab}^{(n)}(u, x^A)$ is non-zero, there will be a memory effect at order $1/r^n$. Using our procedure for solving Einstein's equation starting with a Coulombic order perturbation $h_{ab}^{(d-3)}(u, x^A)$, it follows that for all $d-3 \geq n \geq d/2-1$, $\mathcal{E}_{ab}^{(n)}$ can be expressed in terms of $h_{ab}^{(d-3)}$ as well as $T_{ab}^{(d-2)}$. Many u-derivatives enter this expression, both from the formula for the Weyl tensor and from the procedure for solving for $h_{ab}^{(n)}$ for $n < d-3$. Assuming that the Coulombic order perturbation is stationary at early and late times, then (i) No memory effect can occur for any $n < d-3$. (ii) At Coulombic order $n = d-3$ a memory effect can occur. This was first found by Pate, Raclariu, and Strominger. This memory effect can be due either to a flux of energy to infinity at order $d-2$ ("null memory") or the nonstationarity of the metric at order $d-2$ ("ordinary memory").

Conclusions

A surprisingly (to me, at least) large amount of information about the asymptotic behavior of scalar, electromagnetic, and gravitational fields can be learned by studying the order-by-order field equations arising in a $1/r$ expansion. In particular, the key features of the gravitational memory effect in all dimensions can be understood.