Non-self-adjoint graphs

based on joint work with
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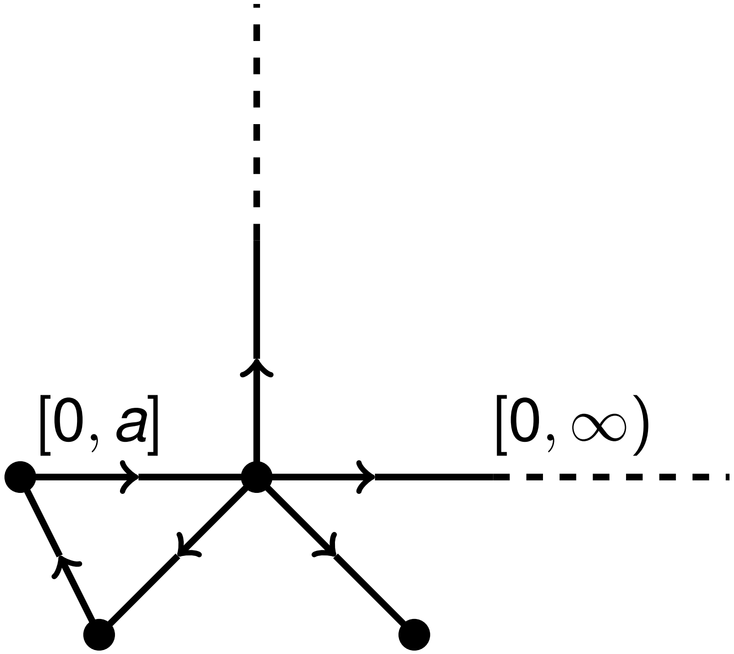
Finite metric graphs



Metric graph

- edges $[0, a_i]$ or $[0, \infty)$
- vertices

Non-self-adjoint operators

$$-\Delta \neq -\Delta^*$$

- $-\frac{d^2}{dx^2}$ on the edges
- $\mathcal{H} = \bigoplus_{e \in \text{edges}} L^2(e)$
- **Matching conditions** on the vertices, e.g.

$$\varphi(0) + i\varphi'(0) = 0$$

on $[0, \infty)$

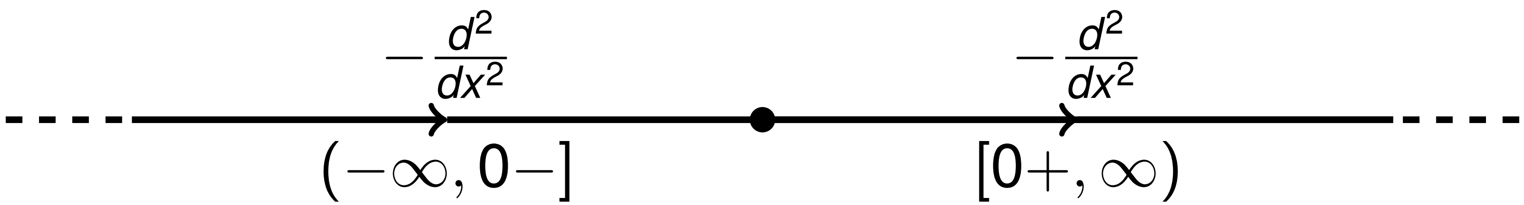
Non-self-adjointess – intervals, systems, graphs

- Spectral operators – cf. Birkhoff (1908), Dunford-Schwartz Part III (1971) [...]
- Sectorial operators – cf. Mugnolo (2014): Semigroup methods for evolution equations on networks and references therein [...]
- $\mathcal{PT}$-symmetry, similarity to self-adjoint operators – cf. Bender, Mostafazadeh ($\geq$ 2007) [...]

Example

Define $-\Delta_\tau$ by

$$\psi(0+) = e^{i\tau}\psi(0-) \quad \text{and} \quad \psi'(0+) = e^{-i\tau}\psi'(0-), \quad \tau \in [0, \pi/2]$$



Studied as example for $\mathcal{PT}$-symmetry, e.g. by

- S. Albeverio, S.-M. Fei and P. Kurasov (2002)
- S. Albeverio and S. Kuzhel (2005)
- P. Siegl (2009)
- and others

Example: $-\frac{d^2}{dx^2}$ on $(-\infty, 0-] \cup [0+, \infty)$

'the good' $\tau = 0$

$$\psi(0+) = \psi(0-)$$
$$\psi'(0+) = \psi'(0-)$$

- $-\Delta_0 = -\Delta_0^*$
- $\mathcal{N}(-\Delta_0) = [0, \infty)$

'the ugly' $\tau \in (0, \pi/2)$

$$\psi(0+) = e^{i\tau}\psi(0-)$$
$$\psi'(0+) = e^{-i\tau}\psi'(0-)$$

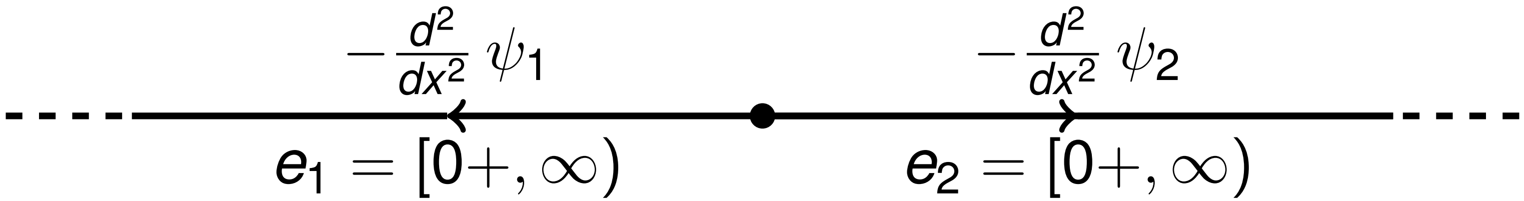
- $-\Delta_\tau \neq -\Delta_\tau^*$
- $\langle -\Delta_\tau \psi, \psi \rangle = \int_{\mathbb{R}} |\psi'|^2 + (1 - e^{2i\tau})\psi(+0)\overline{\psi'(0+)}$

'the bad' $\tau = \pi/2$

$$\psi(0+) = i\psi(0-)$$
$$\psi'(0+) = -i\psi'(0-)$$

- $-\Delta_{\pi/2} \neq -\Delta_{\pi/2}^*$
- $\sigma_p(-\Delta_{\pi/2}) = \mathbb{C} \setminus [0, \infty)$

General matching conditions



Define $-\Delta(A, B)$ by $\psi \in \bigoplus_{e \in \text{edges}} H^2(e)$ and for $A, B \in \text{Mat}(2 \times 2; \mathbb{C})$

$$\psi \in D(-\Delta(A, B)) :\Leftrightarrow \quad A \underbrace{\begin{bmatrix} \psi_1(0) \\ \psi_2(0) \end{bmatrix}}_{=:\underline{\psi}} + B \underbrace{\begin{bmatrix} \psi_1'(0) \\ \psi_2'(0) \end{bmatrix}}_{=:\underline{\psi}'} = 0$$

$$\Leftrightarrow [\underline{\psi}] = \begin{bmatrix} \underline{\psi} \\ \underline{\psi}' \end{bmatrix} \in \mathcal{M} = \text{Ker}(A\ B) \subset \mathbb{C}^{2 \cdot 2}$$

How many matching conditions?

- internal edges $i_j = [0, a_j]$, $1 \le j \le n$
- external edges $e_j = [0, \infty)$, $1 \le j \le m$
- deficiency space $[\underline{\psi}] \in \mathbb{C}^{2(2n+m)}$
  (one trace $\psi(0)$ and one trace of derivative $\psi'(0)$ per endpoint)

$$[\underline{\psi}] = \left[\frac{\underline{\psi}}{\underline{\psi'}} \right] \in \mathcal{M} \subset \mathbb{C}^{2(2n+m)}$$

$\dim \mathcal{M} < (2n+m)$ – **too little!**

- under-determined
- $\sigma(\Delta_{\mathcal{M}}) = \mathbb{C}$
- $\sigma_r(\Delta_{\mathcal{M}}) = \mathbb{C} \setminus [0, \infty)$

$\dim \mathcal{M} < (2n+m)$ – **too much!**

- over-determined
- $\sigma(\Delta_{\mathcal{M}}) = \mathbb{C}$
- $\sigma_p(\Delta_{\mathcal{M}}) = \mathbb{C} \setminus [0, \infty)$

$\Rightarrow$ Consider $\dim \mathcal{M} = 2n + m$ – **just right?**

Self-adjoint matching conditions

$-\Delta(A, B)$ with $A\underline{\psi} + B\underline{\psi}' = 0$ self-adjoint if and only if

(1.) $\dim \mathcal{M} = \dim \operatorname{Ker}(A\ B) = 2n + m$

(2.) $AB^* = BA^*$

Then invertibility of

$$A \pm ikB, \quad k > 0,$$

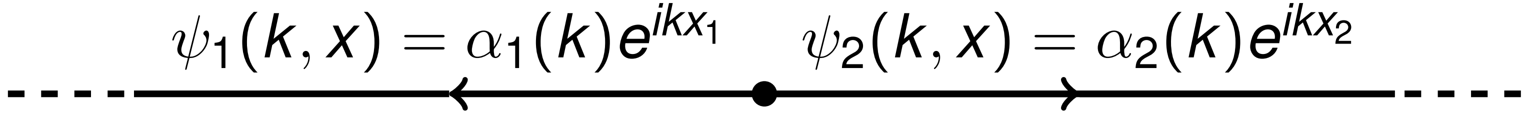
Cayley-transform well-defined

$$\mathfrak{S}(k, A, B) := -(A + ikB)^{-1}(A - ikB), \quad k > 0$$

if no internal edges, $\mathfrak{S}(k, A, B)$ scattering matrix, in particular $\mathfrak{S}(k, A, B)$, $k > 0$, unitary

Cayley transform $\mathfrak{S}(k, A, B) := -(A + ikB)^{-1}(A - ikB)$

- Eigenvalue equation if $\mathcal{I} = \emptyset$



$$\operatorname{Im} k > 0, k^2 \in \sigma_p(-\Delta(A, B)) \Leftrightarrow (A + ikB)\begin{bmatrix}\alpha_1(k)\\ \alpha_2(k)\end{bmatrix} = 0$$

- Integral kernel of the resolvent for $\mathcal{I} = \emptyset$, i.e., only external edges,

$$r_{\mathcal{M}}(x, y; k) =$$
$$\frac{i}{2k}\mathrm{diag}\{e^{ik|x_j - y_j|}\}_{j\in\mathcal{E}} + \frac{i}{2k}\mathrm{diag}\{e^{ikx_j}\}_{j\in\mathcal{E}}\mathfrak{S}(k, A, B)\mathrm{diag}\{e^{iky_j}\}_{j\in\mathcal{E}}$$

Regular and irregular matching conditions

Consider $-\Delta(A, B)$ with $A\underline{\psi} + B\underline{\psi}' = 0$ and $\mathcal{M} = \ker(A\ B)$

$-\Delta(A, B)$ **regular** if

(R1) $\dim \mathcal{M} = 2n + m$

(R2) $A \pm ikB$ invertible for some $k \in \mathbb{C}$

- Then $A \pm ikB$ not invertible for only finitely many $k \in \mathbb{C}$
- Cayley-transform well-defined

$-\Delta(A, B)$ **irregular** if

(R1) $\dim \mathcal{M} = 2n + m$

$\neg$(R2) $A \pm ikB$ not invertible for any $k \in \mathbb{C}$

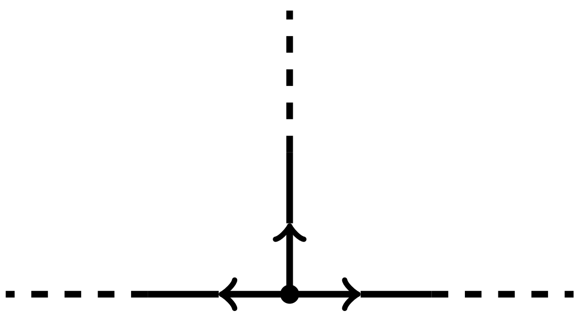
- $-\Delta(A, B)$ irregular $\Leftrightarrow$ $\ker A \cap \ker(B) \neq \{0\}$

Spectra of irregular graphs

Let $-\Delta(A, B)$ be **irregular**

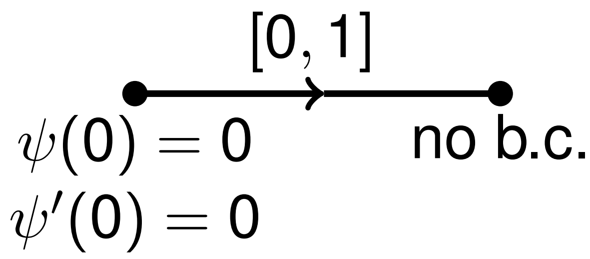
If only **external edges** then

$$\mathrm{Im} k > 0, k^2 \in \sigma_p(-\Delta(A, B))$$
$$\Leftrightarrow \det(A + ikB) = 0$$
$$\Rightarrow \sigma(-\Delta(A, B)) = \mathbb{C}$$



If only **internal edges** one can have

$$\sigma(-\Delta(A, B))) = \emptyset$$

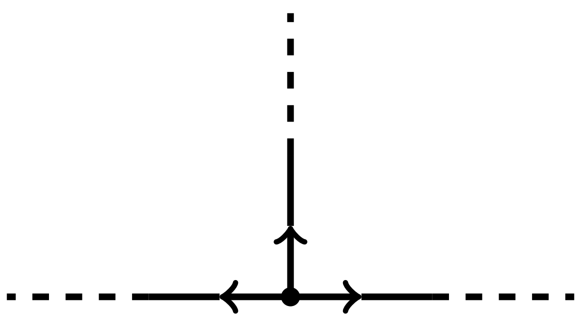
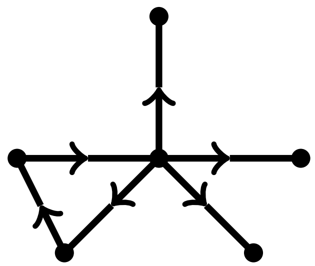
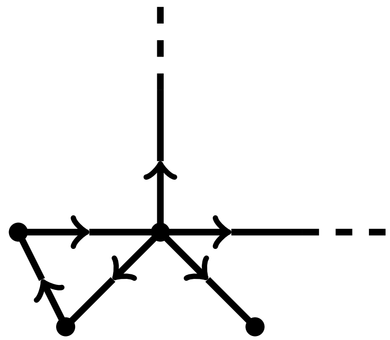


$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

cf. Dunford-Schwartz, Sec. XIX.6(b)

Spectra of regular graphs

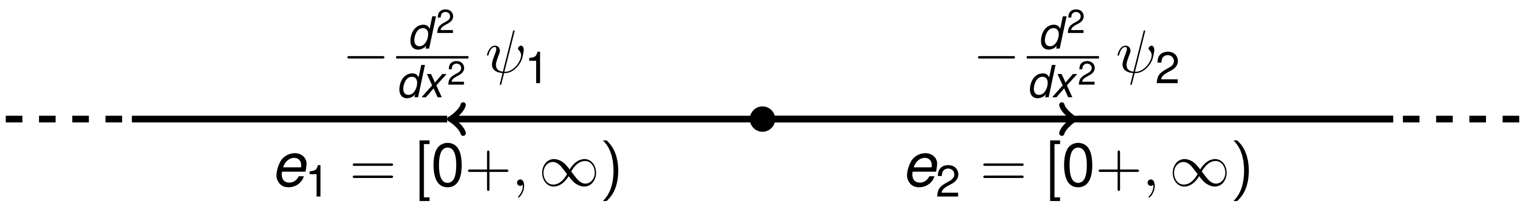
Let $-\Delta(A, B)$ **regular**, then we have a resolvent formula involving Cayley transform and $\sigma(-\Delta(A, B)) \neq \mathbb{C}$

external edges only	**internal edges only**	**int. and ext. edges**
$\sigma_p(-\Delta(A, B)) \subset \mathbb{C} \setminus [0, \infty)$ and finite	$\sigma(-\Delta(A, B))$ discrete	$\sigma_p(-\Delta(A, B))$ discrete
$\sigma_r(-\Delta(A, B)) = \emptyset$	$\sigma_r(-\Delta(A, B)) = \emptyset$	$\sigma_r(-\Delta(A, B)) \subset [0, \infty)$ possible
$\sigma_e(-\Delta(A, B)) = [0, \infty)$[1]	$\sigma_e(-\Delta(A, B)) = \emptyset$	$\sigma_e(-\Delta(A, B)) = [0, \infty)$[1]
		

[1] $\sigma_e(-\Delta(A, B)) = \sigma_{ei}(-\Delta(A, B))$, $i, j \in \{1, \ldots, 5\}$

Example

$$-\Delta(A,B) \text{ for } A_\tau = \begin{pmatrix} 1 & -e^{i\tau} \\ 0 & 0 \end{pmatrix} \text{ and } B_\tau = \begin{pmatrix} 0 & 0 \\ 1 & e^{-i\tau} \end{pmatrix}, \quad \tau \in [0, \pi/2]$$



corresponds to $\psi(0+) = e^{i\tau}\psi(0-)$ and $\psi'(0+) = e^{-i\tau}\psi'(0-)$

$$\det(A_\tau + ikB_\tau) = 2ik\cos(\tau), \quad k \in \mathbb{C}$$

$\Rightarrow \Delta_{\pi/2}$ **irregular**
$\Rightarrow \Delta_\tau$ for $\tau \in [0, \pi/2)$ **regular**

Example

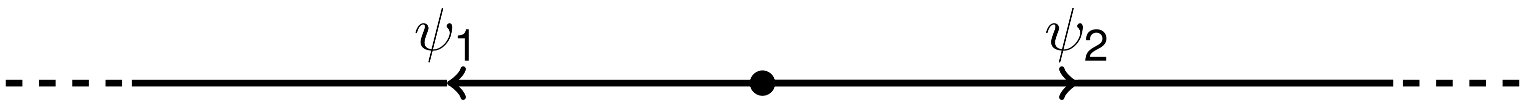
Cayley-transform for $\tau \in [0, \pi/2)$

$$\mathfrak{S}(A_\tau, B_\tau, k) = \frac{1}{\cos(\tau)} \begin{bmatrix} i\sin(\tau) & 1 \\ 1 & -i\sin(\tau) \end{bmatrix}$$
$$= \frac{-1}{2\cos(\tau)} \begin{bmatrix} 1 & 1 \\ e^{-i\tau} & -e^{i\tau} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -e^{i\tau} & -1 \\ -e^{-i\tau} & 1 \end{bmatrix}.$$

Similarity $\mathfrak{S}(A_\tau, B_\tau, k) = G_\tau \mathfrak{S}(A_0, B_0, k) G_\tau^{-1}$ induces via

$$G_\tau \colon L^2([0,\infty))^2 \to L^2([0,\infty))^2, \quad \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \mapsto G_\tau \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

similarity of operators



Example

’the good’

$\tau = 0$ – regular

- self-adjoint
- $\sigma(-\Delta_0) = [0, \infty)$
- defined by form $\langle -\Delta_0 \psi, \psi \rangle = \int_{\mathbb{R}} |\psi'|^2$

’the ugly’

$\tau \in (0, \pi/2)$ – regular

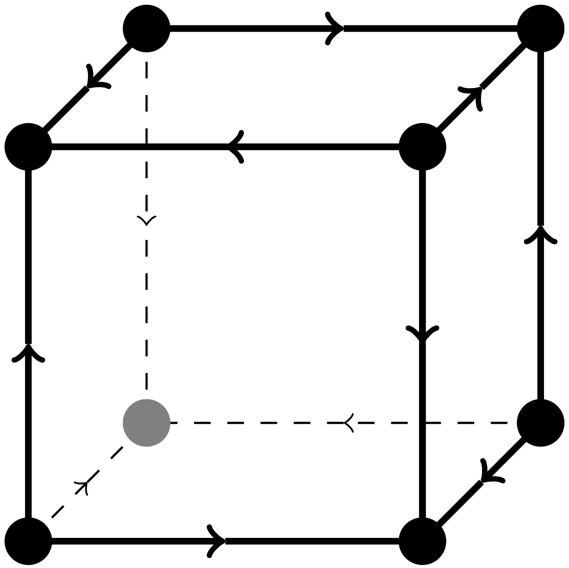
- non-self-adjoint
- but similar to self-adjoint $-\Delta_0$
- not defined by form $\langle -\Delta_\tau \psi, \psi \rangle = \int_{\mathbb{R}} |\psi'|^2 + (1 - e^{2i\tau})\psi(+0)\overline{\psi'(0+)}$
- $\mathcal{N}(-\Delta_\tau) = \mathbb{C}$

’the bad’

$\tau = \pi/2$ – irregular

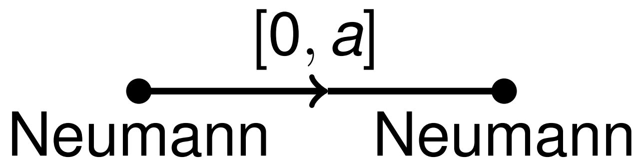
- similar to $-\Delta_{\mathrm{min}} \oplus -\Delta_{\mathrm{max}}$
- Δ_{max} on $[0, \infty)$ no boundary conditions
- Δ_{min} on $[0, \infty)$ with $\psi(0) = \psi'(0) = 0$

Similarity transforms for self-adjoint graphs

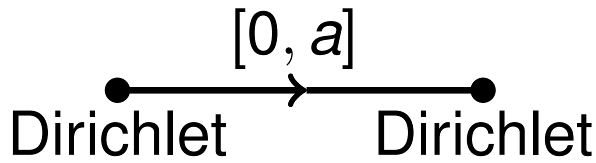


Equivalent to

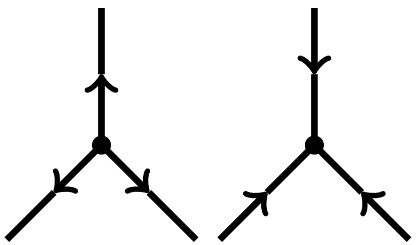
- 4 times



- 8 times



via similarity transform

- At each vertex 

$$\psi_1(v) = \psi_2(v) = \psi_3(v)$$
$$\psi_1'(v) + \psi_2'(v) + \psi_3'(v) = 0$$

- $a_j = a$ for $j = 1, \ldots, 12$

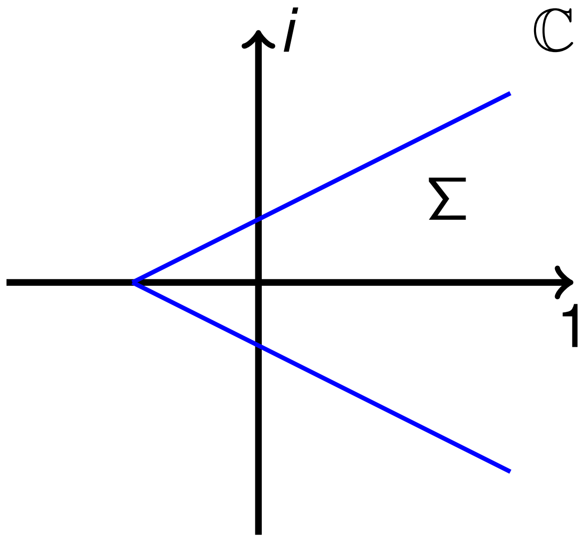
Two conjectures on resolvent estimates

$-\Delta(A,B)$ **regular**, then

- $\sigma(-\Delta(A,B)) \subset \Sigma$
- for $k^2 \in \mathbb{C} \setminus \Sigma$

$$\|(-\Delta(A,B)-k^2)^{-1}\| \lesssim |p(k)|/|k|^2$$

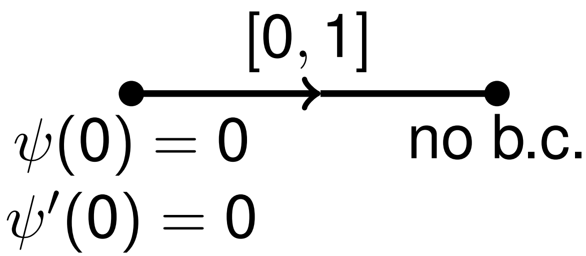
for some polynomial p



$-\Delta(A,B)$ **irregular**, then

- pseudo-spectrum in $\mathbb{C} \setminus \Sigma$
- for $k^2 \in \mathbb{C} \setminus \Sigma \cap \rho(-\Delta(A,B))$

$$C \leq \|(-\Delta(A,B) - k^2)^{-1}\|$$



$$e^{ck}/|k^2| \lesssim \|(-\Delta(A,B) + k^2)^{-1}\|$$

for all $k > 0$ and some $c > 0$

References

- Hussein, A., Krejčiřík, D., & Siegl, P. (2015): Non-self-adjoint graphs. *Trans. of the AMS, 367*(4), 2921 – 2957.
- Hussein, A. (2014): Maximal quasi-accretive Laplacians on finite metric graphs. *J. Evol. Eq., 14* (2), 477 – 497.
- Hussein, A., Krejčiřík, D., & Siegl, P.: Ongoing work