Comparison principles for parabolic equations and applications to PDEs on networks

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Aronson 1967

Let A be a 2nd order elliptic operator in divergence form with Dirichlet b.c. on $\Omega \subset \mathbb{R}^n$ (+ technical assumptions). Then for $t \in (0,1]$, $x,y \in \mathbb{R}^n$:

$$0 \le k^A(t,x,y) \le \gamma k^{\Delta_{\mathbb{R}^n}}(t,x,y), \qquad .$$

When is $\gamma \le 1$?

Simon 1977

Domination of a semigroup $(e^{tA})_{t\geq 0}$ acting on a Hilbert lattice H by another positive semigroup $(e^{tB})_{t\geq 0}$ means that

$$|e^{tA}f| \leq e^{tB}|f| \quad \text{for all } t \geq 0 \text{ and all } f \in H.$$

Short: $e^{tA} \leq e^{tB}$

Special case #1

Let $a \sim A$, $b \sim B$ be Dirichlet forms on $H = L^2(X, \mu; \mathbb{C})$ with $D(a) = D(b)$.
Then TFAE:

- $e^{tA} \leq e^{tB}$
- $b(u,v) \leq a(u,v)$ for all $0 \leq u, v \in D(a)$

Example

- $e^{t\Delta^R} \leq e^{t\Delta^N}$ *on arbitrary open sets* $\Omega \subset \mathbb{R}^d$
- *If* $A = (\alpha_{ij})$ *and* $B = (\beta_{ij})$ *with* $\alpha_{ii}, \beta_{ii} \in \mathbb{R}$ *and* $\alpha_{ij}, \beta_{ij} \geq 0$ $\forall i \neq j$, *then* $e^{tA} \leq e^{tB} \Leftrightarrow \alpha_{ij} \leq \beta_{ij}$ $\forall i, j$
  - *In particular: if* G *is a finite graph with adjacency matrix* $\mathcal{A}_{\mathsf{G}}$ *and Laplacian* $\mathcal{L}_{\mathsf{G}}$, *then for all subgraphs* G': $e^{t\mathcal{A}_{\mathsf{G}'}} \leq e^{t\mathcal{A}_{\mathsf{G}}}$ *but* $e^{-t\mathcal{L}_{\mathsf{G}'}} \nleq e^{-t\mathcal{L}_{\mathsf{G}}} \nleq e^{-t\mathcal{L}_{\mathsf{G}'}}$

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Special case #2

Let $a \sim A$, $b \sim B$ be Dirichlet forms on $H = L^2(X,\mu;\mathbb{C})$ *and* restrictions of a Dirichlet form s.
Then TFAE:

- $e^{tA} \leq e^{tB}$
- $0 \leq v \leq u$ with $v \in D(b), u \in D(a)$ implies $v \in D(a)$

Example

- $e^{t\Delta^D} \leq e^{t\Delta^N}$ *on arbitrary open sets* $\Omega \subset \mathbb{R}^d$

In this case: $-B \leq -A$ *and* $e^{tA} \leq e^{tB}$.

A counterexample

Spectral conditions alone can **not** characterize domination:

On $(0,\pi)$

$\sigma(-\Delta^{per}) = \{0, 4, 4, 16, 16, \ldots\}, \quad \sigma(-\Delta^{N}) = \{0, 1, 4, 9, 16, \ldots\}$

hence $-\Delta^N \le -\Delta^{per}$.

But: $H^1_{per}(0,\pi)$ is not an ideal of $H^1(0,\pi)$, hence $e^{t\Delta^{per}} \not\le e^{t\Delta^N}$.

The general case

Theorem (Ouhabaz 1996)

Let $a \sim A$, $b \sim B$ be densely defined, accretive, continuous, and closed with $e^{tB} \geq 0$. TFAE:

- $e^{tA} \leq e^{tB}$;
- (i) $b(|u|,|v|) \leq \operatorname{Re} a(u,v)$ *for all* $u, v \in D(a)$ *s.t.* $u\bar{v} \geq 0$
  (ii) $u \in D(a)$ *implies* $|u| \in D(b)$
  (iii) $u \in D(a)$, $v \in D(b)$ *and* $|v| \leq |u|$ *imply* $v \operatorname{sgn} u \in D(a)$

(Partial results in: Simon 1977; Hess-Schrader-Uhlenbrock 1977; Arendt 1984; Stollmann-Voigt 1996)

What if $e^{tA} \not\geq 0$?

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Example

- $e^{-t(\Delta^D)^2} \not\leq e^{t\Delta^D}$ *on any open set* $\Omega \subset \mathbb{R}^d$ *because* $D(a) = H^2 \cap H^1_0$, $D(b) = H^1$ *do not satisfy (iii).*
- $e^{-t(\Delta^D)^2} \not\geq e^{t\Delta^D}$ *on any open set* $\Omega \subset \mathbb{R}^d$ *because* $D(b) = H^2 \cap H^1_0$, $D(a) = H^1$ *do not satisfy (ii).*

Daners–Glück–Kennedy 2016

Let $H = L^2(X, \mu; \mathbb{C})$ be a Hilbert lattice with $\mu(X) < \infty$.
$(T(t))_{t \geq 0}$ on H is

- *eventually positive* if

$$\exists t_0 > 0 \text{ s.t. } \left[f \geq 0,\ f \neq 0 \Rightarrow T(t)f \gg 0 \quad \forall t \geq t_0 \right].$$

- *eventually Markovian* if additionally

$$\exists t_0 > 0 \text{ s.t. } T(t)1 = 1 \quad \forall t \geq t_0.$$

Eventually positive/eventually (sub)Markovian systems have been observed often:

- open quantum systems: Suarez–Silbey–Oppenheim 1992, Gnutzmann–Haake 1996
- biological stochastic petri nets: Hufton–Lin–Galla 2018
- polyharmonic heat equations: Gazzola–Grunau 2008, Gregorio–M. 2018

Theorem (Daners–Kennedy–Glück 2016)

Let A be a self-adjoint operator on $L^2(X,\mu;\mathbb{C})$, $\mu(X) < \infty$, s.t. e^{tA} is real and $e^{t_0 A}L^2(X) \subset L^\infty(X)$ for some $t_0 > 0$.

Then $(e^{tA})_{t\geq 0}$ is eventually positive iff $s(A)$ is a simple eigenvalue and the associated eigenspace contains a vector v such that $v \gg 0$.

If additionally $A1 = 0$, then e^{tA} is eventually Markovian.

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Theorem (Glück-M. 2018)

Let A, B be distinct self-adjoint operators on $L^2(X,\mu;\mathbb{C})$, $\mu(X)<\infty$, s.t.

(i) $e^{t_0 A}L^2 \subset L^\infty$ *and* $e^{t_0 B}L^2 \subset L^\infty$ *for some* $t_0 > 0$;

(ii) e^{tA} *is (eventually) positive,* e^{tB} *is irreducible.*

Then TFAE:

- *For all* $0 < f \in L^2$ *there exists a time* $t_1 \geq 0$ *such that* $e^{tB}f \geq e^{tA}f \geq 0$ *for all* $t \geq t_1$.
- *There exists a time* $t_1 \geq 0$ *and a number* $\delta > 0$ *such that* $e^{tB} \geq e^{tA} + \delta(1 \otimes 1) \geq e^{tA}$ *for all* $t \geq t_1$.
- $s(B) > s(A)$.

Example

If $\mu(X)$ *small enough, then* $e^{t(-\Delta^D)^2} \leq e^{t\Delta^D}$ *for some* $t_1 > 0$ *and all* $t \geq t_1$.

"$s(B) > s(A) \Rightarrow e^{tB} \geq e^{tA} + \delta(1 \otimes 1)$"

- e^{tA}, e^{tB} are compact for all $t \geq 0$ and Hilbert-Schmidt for all $t \geq t_0$.
- WLOG: $s(B) = 0 > s(A)$.
- Let $(\lambda_n, e_n)_{n=1}^{\infty}$, $(\mu_n, f_n)_{n=0}^{\infty}$ sequences of eigenpairs of A, B; then $|f_n| \leq Me^{t_0\mu_n}$, $|e_n| \leq Me^{t_0\lambda_n}$ for some $M \geq 1$.
- Let $0 \leq g \in L^2$ and consider

$$e^{tB}g - e^{tA}g = \langle g, f_0\rangle f_0 + \sum_{k=1}^{\infty}\left(e^{-t\mu_k}\langle g, f_k\rangle f_k - e^{-t\lambda_k}\langle g, e_k\rangle e_k\right)$$

- The series on the RHS is abs. convergent in L^∞ with

$$\left\|\sum_{k=1}^{\infty}\left(e^{-t\mu_k}\langle g, f_k\rangle f_k - e^{-t\lambda_k}\langle g, e_k\rangle e_k\right)\right\|_\infty \leq \frac{(\text{ess inf } f_0)^2}{2}\langle g, 1\rangle$$

  for some $t_1 \geq 4t_0$ and all $t \geq t_1$.
- Therefore, $e^{tB}g - e^{tA}g \geq c\langle g, 1\rangle 1$

Interwoven semigroups

Definition

e^{tA}, e^{tB} are *interwoven* if for all $t \ge 0$

$$e^{t_1 B} \not\geq e^{t_1 A} \quad \text{and} \quad e^{t_2 A} \not\geq e^{t_2 B}.$$

for some $t_1, t_2 \ge t$.

Corollary

Let A, B be distinct self-adjoint operators on $L^2(X, \mu; \mathbb{C})$, $\mu(X) < \infty$, s.t.

(i) $e^{t_0 A} L^2 \subset L^\infty$ *and* $e^{t_0 B} L^2 \subset L^\infty$ *for some* $t_0 \ge 0$*;*

(ii) e^{tA} *is (eventually) positive,* e^{tB} *is irreducible.*

If $s(A) = s(B)$, then e^{tA}, e^{tB} are interwoven.

Remark

An extension to general Banach lattices is available.

In particular: if e^{tA}, e^{tB} are bounded, positive semigroups on $L^p(X, \mu; \mathbb{C})$, μ σ-finite and A, B have compact resolvent, then their spectral bounds $s(A), s(B)$ are dominant eigenvalues.
If furthermore the associated eigenspaces are spanned by the same vector, then

$$\|e^{tA} - e^{tB}\| \leq 2e^{\lambda t} \stackrel{t\to\infty}{\to} 0$$

where $\lambda := \min\{s(A), s(B)\}$.

Comparison principles for heat equations on networks

A *network* (or *metric graph*) $\mathcal{G}$ is obtained by associating an interval $(0, \ell_{\mathrm{e}})$ of length ℓ_{e} with each edge e of $\mathrm{G} = (\mathrm{V}, \mathrm{E})$.

$\mathcal{G}$ is a metric measure space: consider

- $C(\mathcal{G})$
- $L^2(\mathcal{G})$

$$H^1(\mathcal{G}) := \{f = (f_e)_{e \in E} \in C(\mathcal{G}) : f_e \in H^1(0, \ell_e) \ \forall e \in E\}$$

$\Delta_{\mathcal{G}}$ is the self-adjoint, positive semidefinite operator on $L^2(\mathcal{G})$ associated with

$$a(f) := \sum_{e \in E} \int_0^{\ell_e} |f'|^2, \quad f \in H^1(\mathcal{G})$$

Diffusion on graphs and networks

Laplacians on graphs and quantum graphs are associated with Dirichlet forms:

- Beurling-Deny 1959 (graphs)
- von Below 1991, Kramar Fijavž-M-Sikolya 2007 (networks)

$\Rightarrow e^{-t\mathcal{L}_{\mathsf{G}}}$, $e^{t\Delta_{\mathcal{G}}}$ are positive (Markovian) semigroups.

Modifying the connectivity of $\mathcal{G}$

If $\mathcal{G}'$ arises by gluing two vertices of $\mathcal{G}$, then

$$H^1(\mathcal{G}') \neq H^1(\mathcal{G}) \quad \text{but} \quad L^2(\mathcal{G}') \approx L^2(\mathcal{G}).$$

Proposition

If $\mathcal{G}'$ is obtained from $\mathcal{G}$ by gluing two distinct vertices, then $e^{t\Delta_{\mathcal{G}}} \nleq e^{t\Delta_{\mathcal{G}'}} \nleq e^{t\Delta_{\mathcal{G}}}$.

Proof.

Neither is $H^1(\mathcal{G})$ a generalized ideal of $H^1(\mathcal{G}')$, nor is $H^1(\mathcal{G}')$ a generalized ideal of $H^1(\mathcal{G})$. □

Proposition (Glück-M. 2018)

- *If* G, G' *are any two graphs on the finite vertex set* V*, then* $e^{-t\mathcal{L}_{\mathsf{G}}}, e^{-t\mathcal{L}_{\mathsf{G}'}}$ *are interwoven.*
- *If* $\mathcal{G}, \mathcal{G}'$ *are two networks of same finite total length, then* $e^{t\Delta_{\mathcal{G}}}, e^{t\Delta_{\mathcal{G}'}}$ *are interwoven (regardless of the chosen isomorphism* $L^2(\mathcal{G}) \simeq L^2(\mathcal{G}')$*).*

Proof.

$s(-\mathcal{L}_{\mathsf{G}}) = s(-\mathcal{L}_{\mathsf{G}'}) = 0$, $s(\Delta_{\mathcal{G}}) = s(\Delta_{\mathcal{G}'}) = 0$.

□

All in all:

$$e^{-t\mathcal{L}_{\mathsf{G}}} \approx e^{-t\mathcal{L}_{\mathsf{G}'}}$$

and

$$e^{t\Delta_{\mathcal{G}}} \approx e^{t\Delta_{\mathcal{G}'}}$$

Convenient because semi-explicit formulae for the heat kernels of $e^{-t\mathcal{L}_{\mathsf{G}'}}$ and $e^{t\Delta_{\mathcal{G}'}}$ are known if $\mathsf{G}', \mathcal{G}'$ are cycles (Chinta–Jorgensen–Karlsson 2015)

Spectral estimates for quantum graphs

Proposition (Nicaise 1987)

Let $\mathcal{G}$ be a network of total length L, $V_D \neq \emptyset$. Then

$$s(\Delta_{\mathcal{G}}^D) \leq -\frac{\pi^2}{4L^2}$$

with equality iff $\mathcal{G}$ is an interval.

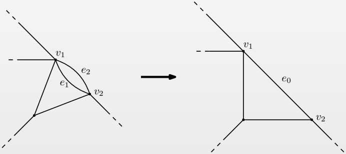
Corollary

Let $\mathcal{G}$ be a network of total length L, $V_D(\mathcal{G}) \neq \emptyset$, and $\mathcal{G}' = (0, L)$ with Dirichlet b.c.
Then $e^{t\Delta_{\mathcal{G}'}^D}$ eventually dominates $e^{t\Delta_{\mathcal{G}}^D}$ (regardless of the chosen isomorphism $L^2(\mathcal{G}) \simeq L^2(\mathcal{G}')$).

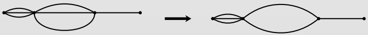
Theorem (Berkolaiko-Kennedy-Kurasov-M. 2018)

Let $V_D(\mathcal{G}) \neq \emptyset$ and let $\mathcal{G}'$ be obtained from $\mathcal{G}$ by

- *replacing edges "in parallel" by edges "in series";*



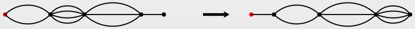
- *replacing m different edges in parallel by $\tilde{m} \leq m$ identical edges in parallel (+technical assumptions);*



- *"transplanting" regions of $\mathcal{G}$ where the ground state is positive but small to regions where the ground state is larger;*

Theorem (continuation)

- *swapping "thick regions" close to Dirichlet vertices with "thin regions" close to the maxima of ground states;*



then

$$s(\Delta_{\mathcal{G}}^{D}) \leq s(\Delta_{\mathcal{G}'}^{D}).$$

Equality can be characterized in terms of $\mathcal{G}$ and the eigenfunction associated with $s(\Delta_{\mathcal{G}}^{D})$.

Thank you for your attention!