Sharp growth rates for semigroups using resolvent bounds

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Semigroups of Operators: Theory and Applications

Joint work with Mark Veraar (Delft University of Technology)

Based on

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Unstable wave equation

Consider the perturbed wave equation

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + e^{-iy} u_x & \text{on } \mathbb{T}^2 \times [0,\infty), \\ (u, \partial_t u)\restriction_{t=0} = (f, g) & \text{on } \mathbb{T}^2. \end{cases}$$

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It can be formulated as an ACP on $X = H^1(\mathbb{T}^2) \times L^2(\mathbb{T}^2)$:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} + A\begin{pmatrix} u \\ v \end{pmatrix} = 0,$$

where

$$A = \begin{pmatrix} 0 & -1 \\ -\Delta & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -e^{iy}\frac{\partial}{\partial_x} & 0 \end{pmatrix}.$$

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Then $-A$ generates a C_0-group $(T(t))_{t\in\mathbb{R}}$ on X.
Renardy (1994): $\sigma(A) \subseteq i\mathbb{R}$ and $\omega_0(T) \geq \frac{1}{2}$.

Problem

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Analyze the growth behavior of $(T(t))_{t\in\mathbb{R}}$ in detail (not just exponential behavior).

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More generally:

Goal

Relate delicate growth behavior of a semigroup to the resolvent growth of its generator.

Notation

Fix a Banach space X, and $p, q \in [1, \infty]$.

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For $m : \mathbb{R} \to \mathcal{L}(X)$ measurable and of polynomial growth, set

$$T_m(f) := \mathcal{F}^{-1}(m \cdot \mathcal{F}(f))$$

for $f : \mathbb{R} \to X$ a Schwartz function.

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$$\mathcal{M}_{p,q}(X) := \{m \mid T_m : L^p(\mathbb{R}; X) \to L^q(\mathbb{R}; X) \text{ bounded}\}$$

and

$$\|m\|_{\mathcal{M}_{p,q}(X)} := \|T_m\|_{\mathcal{L}(L^p(\mathbb{R};X),L^q(\mathbb{R};X))}.$$

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Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. By rescaling, we may assume throughout that $\mathbb{C}_- \subseteq \rho(A)$ (not necessarily $\mathrm{i}\mathbb{R} \subseteq \rho(A)$).

Main abstract result

Theorem

Let $\alpha \geq 0$. Then the following are equivalent:

1. $\|T(t)\|_{\mathcal{L}(X)} = O(t^{\alpha})$ *as* $t \to \infty$;
2. *There exist* $p, q \in [1, \infty]$ *such that* $(a + \mathrm{i} \cdot + A)^{-1} \in \mathcal{M}_{p,q}(X)$ *for all* $a > 0$, *and*

$$\|(a + \mathrm{i} \cdot + A)^{-1}\|_{\mathcal{M}_{p,q}(X)} = O(a^{-\alpha})$$

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as $a \downarrow 0$.

There is also a version for general semigroup growth, and for fractional domains.

Characterization of uniformly bounded semigroups

Corollary

The following are equivalent:

1. $\sup_{t\geq 0}\|T(t)\|_{\mathcal{L}(X)}<\infty$;
2. *There exist* $p,q\in[1,\infty]$ *such that* $(a+\mathrm{i}\cdot+A)^{-1}\in\mathcal{M}_{p,q}(X)$ *for all* $a>0$, *and*

$$\sup_{a>0}\|(a+\mathrm{i}\cdot+A)^{-1}\|_{\mathcal{M}_{p,q}(X)}<\infty.$$

Characterization of uniformly bounded semigroups

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2. *There exist* $p, q \in [1,\infty]$ *such that* $(a + \mathrm{i}\cdot + A)^{-1} \in \mathcal{M}_{p,q}(X)$ *for all* $a > 0$, *and*

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Implies known characterizations of exponential stability.

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Implies known characterizations of exponential stability.
The theory of (L^p, L^p) multipliers does not suffice (these are bounded).

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as $a \downarrow 0$.

Fourier multipliers

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If X is a Hilbert space: $p = q = 2$ and $\sup_{\xi \in \mathbb{R}} \|m(\xi)\|_{\mathcal{L}(X)} < \infty$.

Hilbert spaces

Corollary

Let X *be a Hilbert space, and let* $\alpha \geq 0$. *Suppose that*

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} = O(\mathrm{Re}(\lambda)^{-\alpha})$$

as $\mathrm{Re}(\lambda) \downarrow 0$. *Then*

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For $\alpha \in \mathbb{Z}_+$: optimal up to possible arbitrarily small polynomial loss.

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For $\alpha = 0$: the Gearhart–Prüss Theorem.

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For $\alpha = 1$: Eisner–Zwart (2007).

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For $\alpha \in \mathbb{Z}_+$: optimal up to possible arbitrarily small polynomial loss.
For $\alpha = 0$: the Gearhart–Prüss Theorem.
For $\alpha = 1$: Eisner–Zwart (2007).
A partial converse holds.

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What are sufficient conditions on m such that $m \in \mathcal{M}_{p,q}(X)$?

Positive kernels

Theorem (R., Veraar (J. Fourier Anal. Appl. 2017))

Let $X = L^p(\Omega)$ for $1 \leq p < \infty$ and Ω a measure space. Let $m, K : \mathbb{R} \to \mathcal{L}(X)$ be such that:

1. *$K(t)$ is positive for all $t \in \mathbb{R}$;*
2. *$K(\cdot)x \in L^1(\mathbb{R}; X)$ for all $x \in X$;*
3. *$\mathcal{F}(K(\cdot)x)(\xi) = m(\xi)x$ for all $x \in X$ and $\xi \in \mathbb{R}$.*

Then $m \in \mathcal{M}_{p,p}(X)$ and

$$\|m\|_{\mathcal{M}_p(X)} = \|m(0)\|_{\mathcal{L}(X)}.$$

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Condition (2) can usually be dealt with using approximation arguments. Also holds for $p = \infty$ if X is e.g. a suitable space of continuous functions.

Positive semigroups

Corollary

Let $X = L^p(\Omega)$ *for* $1 \leq p < \infty$ *and* Ω *a measure space, and let* $\alpha \geq 0$*. Suppose that* $T(t)$ *is positive for all* $t \geq 0$*, and that*

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as $a \downarrow 0$*. Then*

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Also holds if X is a suitable space of continuous functions.
For $\alpha = 0$: exponential stability result by Weis (1995).

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Let $\mathcal{H}(\mathcal{L}(X))$ be the set of all $S : (0, \infty) \to \mathcal{L}(X)$ that extend to holomorphic, exponentially bounded functions on a sector around $(0, \infty)$. Set

$$\zeta(T) := \inf\{\omega_0(T - S) \mid S \in \mathcal{H}(\mathcal{L}(X))\}.$$

If $\zeta(T) < 0$ then $(T(t))_{t \geq 0}$ is *asymptotically analytic*.

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If $\zeta(T)<0$ then $(T(t))_{t\geq 0}$ is *asymptotically analytic*.
Eventually differentiable (in particular analytic) semigroups are asymptotically analytic.

Asymptotically analytic semigroups

Theorem (Batty, Srivastava (J. Differential Equations 2003))

Let $\alpha \geq 0$. *Suppose that* $(T(t))_{t\geq 0}$ *is asymptotically analytic, and that*

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} = O(\mathrm{Re}(\lambda)^{-\alpha})$$

as $\mathrm{Re}(\lambda) \downarrow 0$. *Then* $(a + \mathrm{i}\cdot + A)^{-1} \in \mathcal{M}_{1,\infty}(X)$ *for all* $a > 0$, *and*

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Proof also uses that $L^1(\mathbb{R}; \mathcal{L}(X)) \subseteq \mathcal{M}_{1,\infty}(X)$.

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as $a \downarrow 0$.

Proof also uses that $L^1(\mathbb{R}; \mathcal{L}(X)) \subseteq \mathcal{M}_{1,\infty}(X)$.
Here we need to consider $\mathcal{M}_{p,q}(X)$ for $p \neq q$.

Growth for asymptotically analytic semigroups and multipliers

Corollary

Let $\alpha \geq 0$*. Suppose that* $(T(t))_{t\geq 0}$ *is asymptotically analytic, and that*

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Applies in particular to eventually differentiable (and analytic) semigroups.

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Applies in particular to eventually differentiable (and analytic) semigroups.
For $\alpha = 1$: extends results by Eisner and Zwart (2007).

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X is UMD, $p = q \in (1, \infty)$, $m \in C^1(\mathbb{R}; \mathcal{L}(X))$, and

$$\{m(\xi) \mid \xi \in \mathbb{R}\} \quad \text{and} \quad \{\xi m'(\xi) \mid \xi \in \mathbb{R}\}$$

are R-bounded in $\mathcal{L}(X)$.

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So far not useful. Requires (too) fast decay of m'.

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are R-bounded in $\mathcal{L}(X)$.
So far not useful. Requires (too) fast decay of m'.
However, there are useful (L^p, L^q) Fourier multiplier theorems for $p \neq q$ which use R-boundedness.

Fourier type

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X has *Fourier type* $p \in [1,2]$ if $\mathcal{F} : L^p(\mathbb{R}; X) \to L^{p'}(\mathbb{R}; X)$ is bounded.
There is also an $(L^p, L^{p'})$ Fourier multiplier theorem using Fourier type.

Stability using Fourier type

Let $X_\gamma := D((\omega + A)^\gamma)$ for $\gamma \geq 0$ and ω large.

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Corollary

Let X have Fourier type $p \in [1,2]$, and let $\alpha \geq 0$. Suppose that

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} = O(\mathrm{Re}(\lambda)^{-\alpha})$$

as $\mathrm{Re}(\lambda) \downarrow 0$. Then, for each $\gamma > \frac{1}{p} - \frac{1}{p'}$,

$$\|T(t)\|_{\mathcal{L}(X_\gamma, X)} = O(t^\alpha)$$

as $t \to \infty$.

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For $p = 1$: general Banach spaces.

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For $p = 1$: general Banach spaces.
Eisner–Zwart (2006): need to consider $\gamma > 0$ for $p = 1$.

Unstable wave equation

Theorem

Consider the perturbed wave equation

$$u_{tt} = u_{xx} + u_{yy} + e^{-iy} u_x$$

on $\mathbb{T}^2$*, formulated as an ACP on* $X = H^1(\mathbb{T}^2) \times L^2(\mathbb{T}^2)$*. Let* $(T(t))_{t \geq 0}$ *be the associated group. Then*

$$\|T(t)\|_{\mathcal{L}(X)} = O(|t|e^{|t|/2}) \text{ as } |t| \to \infty.$$

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Sharp up to possible polynomial loss.

Take-home message

1. Various types of (asymptotic) behavior of semigroups can be *characterized* using (L^p, L^q) Fourier multiplier properties of the resolvent (consider also $p \neq q$).
2. Then (L^p, L^q) Fourier multiplier theorems yield semigroup results (consider also $p \neq q$).

References

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Thank you for your attention