Convergence of Positive Semigroups and Hyper-Bounded Operators

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Semigroups of Operators: Theory and Applications
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Convergence of positive semigroups

Theorem (Lotz, 1986 [Lot86])

Let $(T_t)_{t\in(0,\infty)}$ *be a positive and bounded operator semigroup on* L^p *(or, more generally, on a Banach lattice).*

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Let $(T_t)_{t\in(0,\infty)}$ be a positive and bounded operator semigroup on L^p (or, more generally, on a Banach lattice).
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Question

Let T be a positive operator on L^p. What are non-trivial criteria to ensure that the essential spectral radius of T fulfils $r_e(T) < 1$?

Hyper-bounded operators

Let (Ω, μ) be a finite measure space.

Definition

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Question (Simon & Høegh-Krohn, 1972 [SH72])

Let T be a self-adjoint operator on $L^2(\Omega, \mu)$ with spectral radius 1. Assume that T is positive (in the sense of Banach lattices) and that its fixed space consists of the constant functions.

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Let T be a self-adjoint operator on $L^2(\Omega, \mu)$ with spectral radius 1. Assume that T is positive (in the sense of Banach lattices) and that its fixed space consists of the constant functions.
If T is hyper-bounded, does it follow that 1 is an isolated spectral value of T?

Miclo's theorem

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We actually have $r_e(T) < 1$ *(by a bit of Perron–Frobenius theory).*

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Miclo's proof relies on an approximation procedure and on certain dimension-independent estimates on finite graphs.

Weakening the assumptions

Theorem (Miclo, 2015)

Let (Ω, μ) *be a finite measure space.*

- *Let* T *be a positive operator on* $L^2(\Omega)$*.*
- *Assume that* T *is self-adjoint and let* $r(T) = 1$*.*
- *Assume that* $\ker(1 - T)$ *consists of the constant functions.*

If T *is hyper-bounded, then* $r_e(T) < 1$*.*

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Theorem (G., 2018 [Glu18])

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Theorem (G., 2018 [Glu18])

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$\Rightarrow$ $\ker(1 - T^{\mathcal{U}})$ is isomorphic to an L^p-space and to an L^q-space and thus finite-dimensional.

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Let S be a linear contraction on a Banach space. If $0 \neq \ker(1 - S^{\mathcal{U}})$ is finite dimensional, then 1 is a pole of the resolvent $\mathcal{R}(\cdot, S)$.

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$\Rightarrow$ Theorem of Niiro–Sawashima: $r_e(T) < 1$. □

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- Positivity of T can be replaced with the assumption $\|T\|_{L^q \to L^q} \leq 1$.
- One can also prove a version of the theorem over non-finite measure spaces.
- For $p = 1$ a more general result is true: if $T : L^1 \to L^1$ is a hyper-bounded operator, then T^2 is compact (this follows from Dunford–Pettis theory).

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Open Problem

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2. If not, can we replace it at least with $\sup_{n\in\mathbb{N}_0} \|T^n\| < \infty$?

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