Semigroups of Operators: Theory and Applications
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Substochastic semigroups and positive perturbations of boundary conditions

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Setting

- $(E, \mathcal{E}, m)$ and $(E_\partial, \mathcal{E}_\partial, m_\partial)$ - two σ-finite measure spaces
- $L^1 = L^1(E, m)$ and $L^1_\partial = L^1(E_\partial, m_\partial)$
- $\mathcal{D}$ - a linear dense subspace of L^1
- $A\colon \mathcal{D} \to L^1$ and $\Psi_0\colon \mathcal{D} \to L^1_\partial$ - linear operators
- $A_0 f = Af, \ f \in \mathcal{D}(A_0) = \mathcal{N}(\Psi_0) = \{f \in \mathcal{D} : \Psi_0 f = 0\}$ - generator of a substochastic (positive and contractive C_0) semigroup on L^1

Problem

Consider the operator $(A_\Psi, \mathcal{D}(A_\Psi))$ defined by

$$A_\Psi f = Af, \qquad f \in \mathcal{D}(A_\Psi) = \{f \in \mathcal{D} : \Psi_0(f) = \Psi(f)\},$$

where $\Psi\colon \mathcal{D} \to L^1_\partial$ is a linear operator.

When is $(A_\Psi, \mathcal{D}(A_\Psi))$ the generator of

- ◇ a C_0-semigroup?
- ◇ a positive C_0-semigroup?
- ◇ a (sub)stochastic semigroup?

Greiner's theorem

$$A_\Psi f = Af, \quad f \in \mathcal{D}(A_\Psi) = \{f \in \mathcal{D} : \Psi_0(f) = \Psi(f)\},$$

is the generator of a C_0-semigroup provided that

(a) $A\colon \mathcal{D} \to L^1$ and $\Psi_0\colon \mathcal{D} \to L^1_\partial$ are closed, Ψ_0 is onto

(b) $(A_0, \mathcal{D}(A_0))$ generates a C_0-semigroup

(c) there exist constants $\gamma > 0$ and λ_0 such that

$$\|\Psi_0 f\| \geq \lambda\gamma\|f\|, \quad f \in \mathcal{N}(\lambda I - A), \lambda > \lambda_0$$

(d) $\Psi\colon L^1 \to L^1_\partial$ is bounded

G. Greiner. Perturbing the boundary conditions of a generator. Houston J. Math., 13(2):213–229, 1987.

Greiner's approach

Condition

(a) $A\colon \mathcal{D} \to L^1$ and $\Psi_0\colon \mathcal{D} \to L^1_\partial$ are closed, Ψ_0 is onto

implies that $\mathcal{D} = \mathcal{N}(\lambda I - A) \oplus \mathcal{N}(\Psi_0)$,

$\Psi_0|_{\mathcal{N}(\lambda I - A)}$ is invertible and the Dirichlet operator

$$\Psi(\lambda) := \left(\Psi_0|_{\mathcal{N}(\lambda I - A)}\right)^{-1} : L^1_\partial \to L^1$$

is bounded for each $\lambda \in \rho(A_0)$

$$A_0 f = Af, \quad f \in \mathcal{D}(A_0) = \mathcal{N}(\Psi_0) = \{f \in \mathcal{D} : \Psi_0(f) = 0\}$$

Greiner's approach

Conditions

(a) $A\colon \mathcal{D} \to L^1$ and $\Psi_0\colon \mathcal{D} \to L^1_\partial$ are closed, Ψ_0 is onto

(c) there exist constants $\gamma > 0$ and λ_0 such that

$$\|\Psi_0 f\| \geq \lambda\gamma\|f\|, \qquad f \in \mathcal{N}(\lambda I - A), \lambda > \lambda_0$$

imply that $\Psi(\lambda) = \left(\Psi_0|_{\mathcal{N}(\lambda I - A)}\right)^{-1}$ satisfies

$$\|\Psi(\lambda)\| \leq \frac{1}{\lambda\gamma}$$

Take $f = \Psi(\lambda)f_\partial$, $f_\partial \in L^1_\partial$ in (c).

Greiner's approach

If Ψ is bounded then

$$\|\Psi(\lambda)\Psi\| \leq \frac{\|\Psi\|}{\lambda\gamma},$$

$I - \Psi(\lambda)\Psi \colon L^1 \to L^1$ is invertible with bounded inverse and

$$R(\lambda, A_\Psi) = (I - \Psi(\lambda)\Psi)^{-1} R(\lambda, A_0)$$

for $\lambda \in \rho(A_0)$, $\lambda > \max\{\lambda_0, \|\Psi\|/\gamma\}$

$$A_\Psi f = Af, \qquad f \in \mathcal{D}(A_\Psi) = \{f \in \mathcal{D} : \Psi_0(f) = \Psi(f)\},$$

$$R(\lambda, A_\Psi) := (\lambda I - A_\Psi)^{-1}$$

Nickel's extension to unbounded Ψ

(a) $A\colon \mathcal{D} \to L^1$ and $\Psi_0\colon \mathcal{D} \to L^1_\partial$ are closed, Ψ_0 is onto

(b) $(A_0, \mathcal{D}(A_0))$ generates a C_0-semigroup

(cd) $\Psi\colon \mathcal{D} \to L^1_\partial$ and there are constants $C, \omega > 0$:
$\Psi(\lambda)\Psi\colon \mathcal{D} \to L^1$ extends to a bounded operator

$$\|\Psi(\lambda)\Psi\| \leq \frac{C}{\lambda}, \quad \lambda > \omega.$$

Then $(A_\Psi, \mathcal{D}(A_\Psi))$ generates a C_0-semigroup.

G. Nickel. A new look at boundary perturbations of generators. Electron. J. Differential Equations 2004(95):1-14, 2004.

Extension to positive semigroups

(i) $A\colon \mathcal{D} \to L^1$ and $\Psi_0\colon \mathcal{D} \to L^1_\partial$ are closed, Ψ_0 is onto and positive

(ii) $(A_0, \mathcal{D}(A_0))$ generates a positive C_0-semigroup

(iii) for each nonnegative $f \in \mathcal{D}$

$$\int_E Af\, dm - \int_{E_\partial} \Psi_0 f\, dm_\partial \leq 0$$

imply: $\Psi(\lambda) = \left(\Psi_0|_{\mathcal{N}(\lambda I - A)}\right)^{-1}$ is positive and $\|\Psi(\lambda)\| \leq \dfrac{1}{\lambda}$

P. Gwiżdż, M. Tyran-Kamińska, Positive semigroups and perturbations of boundary conditions, arXiv: 1807.06992

Extension to positive unbounded Ψ

(i) $A\colon \mathcal{D} \to L^1$ and $\Psi_0\colon \mathcal{D} \to L^1_\partial$ are closed, Ψ_0 is onto and positive

(ii) $(A_0, \mathcal{D}(A_0))$ generates a positive C_0-semigroup

(iii) for each nonnegative $f \in \mathcal{D}$

$$\int_E Af\,dm - \int_{E_\partial} \Psi_0 f\,dm_\partial \le 0$$

(iv) $\Psi\colon \mathcal{D} \to L^1_\partial$ is positive and $I_\partial - \Psi\Psi(\lambda)\colon L^1_\partial \to L^1_\partial$ is invertible with positive inverse, $\lambda > \omega$, $I_\partial = \mathrm{Id}_{L^1_\partial}$

Then $(A_\Psi, \mathcal{D}(A_\Psi))$ generates a positive C_0-semigroup.

Proof

Consider $\mathcal{X} = L^1 \times L^1_\partial$ with norm

$$\|(f, f_\partial)\| = \int_E |f|\, dm + \int_{E_\partial} |f_\partial|\, dm_\partial, \qquad (f, f_\partial) \in L^1 \times L^1_\partial.$$

Define $\mathcal{A}, \mathcal{B} \colon \mathcal{D}(\mathcal{A}) \to L^1 \times L^1_\partial$, $\mathcal{D}(\mathcal{A}) = \mathcal{D} \times \{0\}$, by

$\mathcal{A}(f, 0) = (Af, -\Psi_0 f)$ and $\mathcal{B}(f, 0) = (0, \Psi f)$ for $f \in \mathcal{D}$.

We have $\overline{\mathcal{D}(\mathcal{A})} = L^1 \times \{0\}$, $\|R(\lambda, \mathcal{A})\| \leq \dfrac{1}{\lambda}$, $R(\lambda, \mathcal{A}) \geq 0$, $\lambda > 0$, and $\mathrm{spr}(\mathcal{B}R(\lambda, \mathcal{A})) < 1$, $\lambda > \max\{0, \omega\}$

$$R(\lambda, \mathcal{A} + \mathcal{B}) = R(\lambda, \mathcal{A})(\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}))^{-1}, \qquad \mathcal{I} = \mathrm{Id}_{\mathcal{X}}$$

Proof

$$(\mathcal{A}+\mathcal{B})(f,0) = (Af, \Psi f - \Psi_0 f),\ (f,0) \in \mathcal{D}(\mathcal{A}) = \mathcal{D} \times \{0\},$$

its part in $L^1 \times \{0\}$ generates a positive semigroup there

$$(\mathcal{A}+\mathcal{B})_|(f,0) = (Af,0) \quad f \in \mathcal{D}, \Psi f - \Psi_0 f = 0$$

$\mathcal{D}((\mathcal{A}+\mathcal{B})_|) = \mathcal{D}(A_\Psi)\times\{0\}$, $(\mathcal{A}+\mathcal{B})_|(f,0) = (A_\Psi f, 0)$.

We have

$$R(\lambda, A_\Psi)f = (I + \Psi(\lambda)(I_\partial - \Psi\Psi(\lambda))^{-1}\Psi)R(\lambda, A_0)f, \quad f \in L^1$$

and if Ψ is bounded then

$$I + \Psi(\lambda)(I_\partial - \Psi\Psi(\lambda))^{-1}\Psi = \sum_{n=0}^{\infty}(\Psi(\lambda)\Psi)^n = (I - \Psi(\lambda)\Psi)^{-1}.$$

Greiner's and Kato's theorems

(i) $A\colon \mathcal{D} \to L^1$ and $\Psi_0\colon \mathcal{D} \to L^1_\partial$ are closed, Ψ_0 is onto and positive

(ii) $(A_0, \mathcal{D}(A_0))$ generates a positive C_0-semigroup

(iiiiv) $\Psi\colon \mathcal{D} \to L^1_\partial$ is positive and for nonnegative $f \in \mathcal{D}$

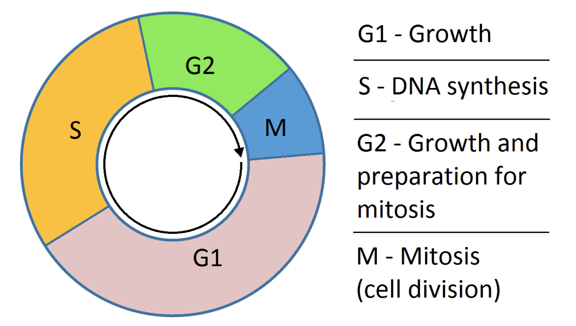
$$\int_E Af\, dm - \int_{E_\partial} \Psi_0 f\, dm_\partial + \int_{E_\partial} \Psi f\, dm_\partial = 0\ (\leq 0)$$

Then there is an extension of $(A_\Psi, \mathcal{D}(A_\Psi))$ that generates a smallest substochastic semigroup.

$(A_\Psi, \mathcal{D}(A_\Psi))$ is the generator of a (sub)stochastic semigroup iff $\operatorname{spr}(\Psi\Psi(\lambda)) < 1$ for some $\lambda > 0$.

Cell cycle

the period between two cell divisions, from the birth of a cell until its division into 2 daughter cells

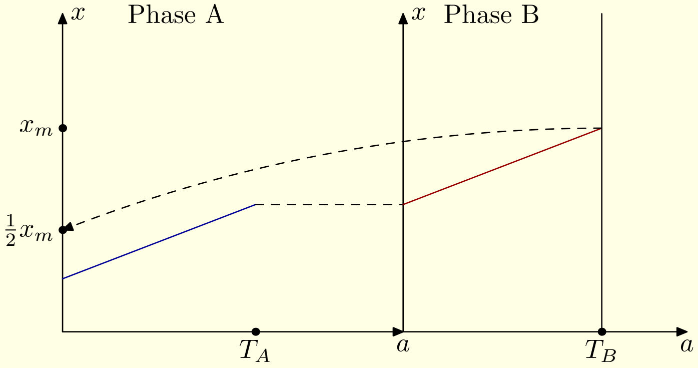


pl.wikipedia.org

Smith–Martin model - two phases: A (all or part of G_1) and B (the rest)

Two-phase cell cycle model

(a, x) - age and size of a cell in each phase
T_A - random length of phase A, distributed with density h
T_B - constant length of phase B



A piecewise deterministic Markov process

The process $X(t) = (a(t), x(t), i(t))$ satisfies

$$a'(t) = 1, \quad x'(t) = g(x(t)), \quad i'(t) = 0,$$

between consecutive times $t_0, s_0, t_1, s_1, \ldots$, at the time s_n the cell from the nth generation enters phase B

$$a(s_n) = 0, \quad x(s_n) = x(s_n^-), \quad i(s_n) = 2,$$

$t_{n+1} = s_n + T_B$ the cell divides into two daughter cells

$$a(t_{n+1}) = 0, \quad x(t_{n+1}) = \frac{1}{2}x(t_{n+1}^-), \quad i(t_{n+1}) = 1.$$

Two-phase cell cycle model

$p_1(t,a,x)$, $p_2(t,a,x)$ - densities of the age and size distribution of cell in A and B phases; $g(x)$ -size growth rate

$$\frac{\partial p_1(t,a,x)}{\partial t} = -\frac{\partial p_1(t,a,x)}{\partial a} - \frac{\partial (g(x)p_1(t,a,x))}{\partial x} - \rho(a)p_1(t,a,x),$$

$$\frac{\partial p_2(t,a,x)}{\partial t} = -\frac{\partial p_2(t,a,x)}{\partial a} - \frac{\partial (g(x)p_2(t,a,x))}{\partial x},$$

$$p_1(t,0,x) = 2p_2(t,T_B,2x), \quad x>0, t>0,$$

$$p_2(t,0,x) = \int_0^\infty \rho(a)p_1(t,a,x)da, \quad x>0, t>0$$

$$\rho(a) = -\frac{H'(a)}{H(a)}, \quad H(a) = \int_a^\infty h(r)dr$$

Stochastic semigroup

To simplify presentation assume $g(0) = 0$;
$E = E_1 \times \{1\} \cup E_2 \times \{2\},\ E_1 = E_2 = (0, \infty)^2$,
$L^1 = L^1(E_1) \times L^1(E_2),\ f(a, x) = (f_1(a, x), f_2(a, x))$
$E_\partial = \Gamma_1^- \times \{1\} \cup \Gamma_2^- \times \{2\}$, $\Gamma_1^- = \Gamma_2^- = \{0\} \times (0, \infty)$ - incoming boundaries for $a'(t) = 1, x'(t) = g(x(t))$

We have

$$\|\Psi\Psi(\lambda) f_\partial\| \le \max\left\{ e^{-\lambda T_B}, \int_0^\infty h(a) e^{-\lambda a} da \right\} \|f_\partial\|$$

$$\Psi f(0, x) = \left(2 f_2(T_B, 2x), \int_0^\infty \rho(a) f_1(a, x) da \right)$$

Evolution of densities of the process

There is a stochastic semigroup $\{S(t)\}_{t\geq 0}$ on L^1 which provides solutions of the cell cycle model equations.

Let $X(t) = (a(t), x(t), i(t))$ be the PDMP.

If the distribution of $X(0)$ has a density f ($\|f_1\| + \|f_2\| = 1$, $f_i \geq 0$) then $X(t)$ has a density $S(t)f$, i.e.,

$$\Pr(X(t) \in B_i \times \{i\}) = \int_{B_i} (S(t)f)_i(a, x)\, da\, dx$$

for any Borel set $B_i \subset E_i$

P. Gwiżdż, M. Tyran-Kamińska, Densities for piecewise deterministic Markov processes with boundary, in preparation

Greiner's and Kato's theorems

(i) $A\colon \mathcal{D} \to L^1$ and $\Psi_0\colon \mathcal{D} \to L^1_\partial$ are closed, Ψ_0 is onto and positive

(ii) $(A_0, \mathcal{D}(A_0))$ generates a positive C_0-semigroup

(iii) $B\colon \mathcal{D} \to L^1$ and $\Psi\colon \mathcal{D} \to L^1_\partial$ are positive

(iv) for nonnegative $f \in \mathcal{D}$

$$\int_E (Af + Bf)\, dm - \int_{E_\partial} \Psi_0 f\, dm_\partial + \int_{E_\partial} \Psi f\, dm_\partial = 0\ (\leq 0)$$

Then there is an extension of $(A_\Psi + B, \mathcal{D}(A_\Psi))$ that generates a smallest substochastic semigroup.

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