Chernoff approximation of evolution semigroups and beyond

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Chernoff approximation of Markov evolution

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- $(\xi_t)_{t\geqslant 0}$ is a time homogeneous Markov process ($\Rightarrow$ no memory)
  $\Rightarrow$ transition kernel $P(t,x,dy) := \mathbb{P}(\xi_t \in dy \mid \xi_0 = x)$

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- Then
$$f(t,x) := \int f_0(y)P(t,x,dy) \equiv \mathbb{E}\left[f_0(\xi_t) \mid \xi_0 = x\right]$$
solves the following evolution equation:
$$\begin{cases} \frac{\partial f}{\partial t}(t,x) & = & Lf(t,x), \\ f(0,x) & = & f_0(x), \end{cases}$$
where
$$T_t f_0(x) := \int f_0(y)P(t,x,dy) \equiv e^{tL} f_0.$$
$(T_t)_{t\geqslant 0}$ is an operator semigroup (i.e. $T_0 = \mathrm{Id}$, $T_t \circ T_s = T_{t+s}$).

Chernoff approximation of Markov evolution

Stochastics

To determine the transition kernel $P(t,x,dy)$ for a given process $(\xi_t)_{t\geqslant 0}$.

$\Updownarrow$

Functional Analysis

To construct the semigroup $T_t \equiv e^{tL}$ with a given generator L.

$\Updownarrow$

PDEs

To solve a (Cauchy problem for a) given PDE $\frac{\partial f}{\partial t} = Lf$.

Chernoff approximation of Markov evolution

Example:

- Heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f, \qquad x \in \mathbb{R}^d.$$

- Heat semigroup

$$T_t f_0(x) := (2\pi t)^{-d/2} \int\limits_{\mathbb{R}^d} f_0(y) \exp\left\{-\frac{|x-y|^2}{2t}\right\} dy.$$

- Transition kernel of Brownian motion

$$P(t,x,dy) = (2\pi t)^{-d/2} \exp\left\{-\frac{|x-y|^2}{2t}\right\} dy.$$

Chernoff approximation of Markov evolution

Chernoff approximation: To find $(F(t))_{t\geqslant 0}$ (not a SG!!!) such that

$$T_t f_0 = \lim_{n\to\infty} \left[F(t/n)\right]^n f_0.$$

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$\Rightarrow$ discrete time approximation to the solution $f(t,x)$:

$$u_0 := f_0, \qquad u_k := F(t/n)u_{k-1}, \quad k = 1, \dots, n, \qquad\qquad f(t,\cdot) \approx u_n.$$

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$\Rightarrow$ Markov chain approximation to $(\xi_t)_{t\geqslant 0}$ (e.g., Euler scheme),

$$(\xi^n_k)_{k=1,\dots,n}, : \qquad \mathbb{E}\left[f_0(\xi^n_k)\,|\,\xi^n_{k-1}\right] = F(t/n)f_0(\xi^n_{k-1})$$

$$\Rightarrow \qquad \mathbb{E}\left[f_0(\xi_t)\,|\,\xi_0 = x\right] = \lim_{n\to\infty} \mathbb{E}\left[f_0(\xi^n_n)\,|\,\xi_0 = x\right]$$

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$\Rightarrow$ approximation of path integrals in Feynman-Kac formulae.

Chernoff approximation of Markov evolution

Chernoff Theorem (1968): Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(L, \mathrm{Dom}(L))$. Let $(F(t))_{t\geqslant 0}$ be a family of bounded linear operators on X. Assume that

- $F(0) = \mathrm{Id}$,
- $\|F(t)\| \leqslant e^{wt}$ for some $w \in \mathbb{R}$, and all $t \geqslant 0$,
- $\lim\limits_{t\to 0} \dfrac{F(t)\varphi - \varphi}{t} = L\varphi$

  for all $\varphi \in D$, where D is a core for $(L, \mathrm{Dom}(L))$.

Then it holds

$$T_t\varphi = \lim_{n\to\infty} \left[F(t/n)\right]^n \varphi, \quad \forall\ \varphi \in X,$$

and the convergence is locally uniform with respect to $t \geqslant 0$.

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- $F(0) = \mathrm{Id}$, *(consistency)*
- $\|F(t)\| \leqslant e^{wt}$ for some $w \in \mathbb{R}$, and all $t \geqslant 0$, *(stability)*
- $\lim\limits_{t\to 0} \dfrac{F(t)\varphi - \varphi}{t} = L\varphi$ *(consistency)*

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Meta-theorem of Numerics: Consistency + stability $\Rightarrow$ convergence.

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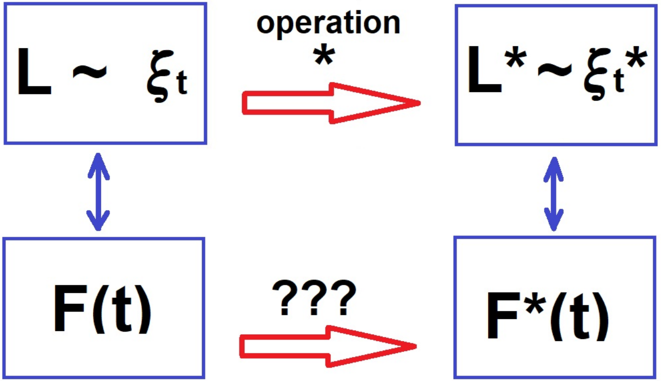

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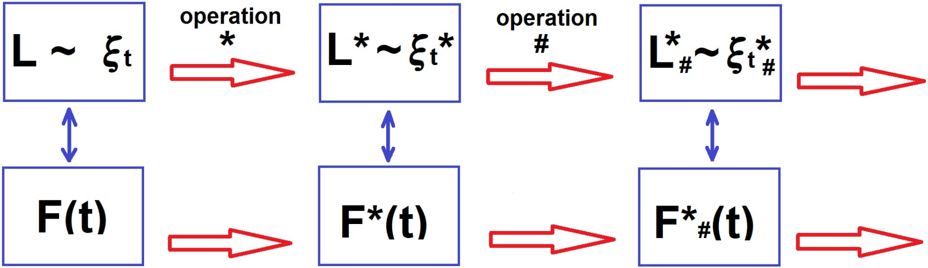

LEGO principle for Chernoff approximation:

Nice processes to start with:

- ξ_t with known $P(t, x, dy)$;
- Feller processes in $\mathbb{R}^d$;
- Brownian motion in a compact Riemannian manifold;
- ....

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If $F_k(t)$: $\quad F_k(0) = \mathrm{Id}, \quad \|F_k(t)\| \leqslant e^{tw_k}, \quad F_k'(0)\varphi = L_k\varphi \quad \forall\ \varphi \in D$
then

$$F^*(t) := F_1(t) \circ \ldots \circ F_m(t) \quad \sim \quad e^{tL^*}$$

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Corollary (Trotter formula):

$$e^{tL_1} \circ e^{tL_2} \sim e^{t(L_1+L_2)} \quad \text{i.e.} \quad e^{t(L_1+L_2)} = \lim_{n\to\infty} \left[e^{tL_1/n} \circ e^{tL_2/n} \right]^n$$

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Chernoff approximations are known for the following operations:

- operator splitting $\quad \Leftrightarrow \quad L^* := L_1 + \ldots + L_m$;
- multiplicative perturbations of $L \quad \Leftrightarrow \quad L^* := aL \quad \Leftrightarrow$ random time change of ξ_t via an additive functional;
- killing of ξ_t upon leaving a domain $G \subset \mathbb{R}^d \quad \Leftrightarrow$ $L^* := L +$ Dirichlet boundary / external conditions;
- subordination $\quad \Leftrightarrow \quad L^* := -f(-L), \quad \xi_t^* := \xi_{\eta_t}$;
- "rotation" $\Leftrightarrow$ $L^* := iL \quad (F(t) \sim e^{tL} \Rightarrow F^*(t) := e^{i(F(t)-\mathrm{Id})} \sim e^{itL})$;
- ......

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Feller process $\xi_t \quad \leftrightsquigarrow T_t \equiv e^{tL}$ on $C_\infty(\mathbb{R}^d)$ with $L \equiv -\widehat{H}$:

$$\widehat{H}\varphi(x) := (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip\cdot(x-q)} H(x,p)\varphi(q)\,dq\,dp,$$

where $H(x,\cdot)$ is given by the Lévy–Khintchine formula

$$H(x,p) = C(x) + iB(x)\cdot p + p\cdot A(x)p + \int\limits_{y\neq 0} \left(1 - e^{iy\cdot p} + \frac{iy\cdot p}{1+|y|^2}\right) N(x,dy).$$

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Remark: If $H = H(x,p)$ then

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Remark: Let $\mu_t^x : \mathcal{F}\left[\mu_t^x\right](p) = e^{-tH(x,-p)-ip\cdot x}$. Then

$$F(t)\varphi(x) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip\cdot(x-q)} e^{-tH(x,p)} \varphi(q)\,dqdp = \int\limits_{\mathbb{R}^d} \varphi(q)\mu_t^x(dq).$$

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Example: non-degenerate diffusion:

$$F(t)\varphi(x) = \frac{e^{-tC(x)}}{\sqrt{(4\pi t)^d \det A(x)}}\int\limits_{\mathbb{R}^d}\varphi(q)e^{-\frac{(x-q-tB(x))\cdot A^{-1}(x)(x-q-tB(x))}{4t}}dq$$

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Example: $H(x,p) := a(x)|p| \quad \Rightarrow \quad L : L\varphi(x) = a(x)\left(-(-\Delta)^{1/2}\right)\varphi(x)$

$$F(t)\varphi(x) = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ip\cdot(x-q)}e^{-ta(x)|p|}\varphi(q)\,dq\,dp$$

$$= \Gamma\left(\frac{d+1}{2}\right)\int_{\mathbb{R}^d}\varphi(q)\frac{a(x)t}{\left(\pi|x-q|^2+a^2(x)t^2\right)^{\frac{d+1}{2}}}dq.$$

Chernoff approximation for Killed Feller processes

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Feller process $\xi_t \quad \leadsto T_t \equiv e^{tL} \quad$ on $C_\infty(\mathbb{R}^d) \quad$ with $L \equiv -\widehat{H}$

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Feller process $\xi_t \quad \leftrightsquigarrow T_t \equiv e^{tL}$ on $C_\infty(\mathbb{R}^d)$ with $L \equiv -\widehat{H}$

Consider $G \subset \mathbb{R}^d$ bdd, regular, $\tau_G := \inf\{t > 0 \,:\, \xi_t \notin G\}$ and

$$\xi_t^o := \begin{cases} \xi_t, & t < \tau_G \\ \partial, & t \geqslant \tau_G. \end{cases}$$

Hence

$$T_t^o \;:\quad T_t^o \varphi(x) = \mathbb{E}\left[\varphi(\xi_t^o) \mid \xi_0^o = x\right]$$

is a Feller SG on $C_\infty(G)$; $\quad T_t^o \equiv e^{tL^o}$,

$$\mathrm{Dom}(L^o) := \{\varphi \in C_\infty(G) \,:\, L\varphi^o \in C_\infty(G)\}, \quad L^o\varphi(x) := L\varphi^o(x), \quad x \in G$$

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Hence **additional assumptions** on ξ_t.

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$$F^o(t) \ : \quad F^o(t)\varphi(x) = 1_G(x)\,(F(t)\varphi^o)\,(x)$$

Problems:

- $F^o(t) \ : \ C_\infty(G) \to C_\infty(G)$
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Hence **additional assumptions** on ξ_t.

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Example: non-deg. diffusion:

$$F^o(t)\varphi(x) = \frac{e^{-tC(x)}}{\sqrt{(4\pi t)^d \det A(x)}} \int_G \varphi(q) e^{-\frac{(x-q-tB(x))\cdot A^{-1}(x)(x-q-tB(x))}{4t}} dq$$

Approximation of non-Markovian evolution

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Some physical, chemical, biological phenomena can be modelled via evolution equations of the form

$$\partial_t^\beta f = Lf, \qquad \beta \in (0,1),$$

where ∂_t^β is the **Caputo derivative** of order β:

$$\partial_t^\beta u(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{u(r)}{(t-r)^\beta} dr - \frac{t^{-\beta}}{\Gamma(1-\beta)} u(0+),$$

and $(L, \mathrm{Dom}(L))$ generates a unif. bdd C_0-SG $(T_t)_{t \geqslant 0}$ (on a BS X).

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More general: We consider equations of the form:

$$\mathcal{D}_t^\mu f(t) = Lf(t),$$

where $\mathcal{D}_t^\mu$ is the **Caputo derivative of distributed order** given by a finite Borel measure μ on $(0,1)$:

$$\mathcal{D}_t^\mu u(t) := \int_0^1 \partial_t^\beta u(t) \mu(d\beta).$$

Approximation of non-Markovian evolution

Consider **"inverse subordinator"** $(E_t^\mu)_{t\geqslant 0}$ (inverse to $(\xi_t^\mu)_{t\geqslant 0}$):

$$E_t^\mu := \inf\left\{\tau \geqslant 0 \ : \ \xi_\tau^\mu > t\right\},$$

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Then E_t^μ

- has a.s. non-decreasing paths;
- is **non-Markovian!!!**;
- has a smooth PDF $p^\mu(t,x)$.

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Theorem (Hahn, Kobayashi, Umarov, Mijena, Nane, 2012-2014):
Consider the family $(\mathcal{T}_t)_{t\geqslant 0}$ of linear operators on X such that

$$\mathcal{T}_t\varphi := \int_0^\infty T_s\varphi\, p^\mu(t,s)ds, \qquad \varphi \in X.$$

Then $(\mathcal{T}_t)_{t\geqslant 0}$ is a strongly continuous family (**not a SG!!!**) and, for $\forall\, f_0 \in \mathrm{Dom}(L)$, the function

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$$f(t) := \mathcal{T}_t f_0 \quad = \mathbb{E}\left[f_0(\xi_{E_t^{\mu}}) \,\middle|\, \xi_{E_0^{\mu}} = x \right], \quad \xi_t \leftrightsquigarrow L$$

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Theorem: Let $F(t) \sim T_t$. Consider $f_n : [0,\infty) \to X$ such that

$$f_n(t) := \int_0^\infty F^n(t/n) f_0\, p^\mu(t,s)ds.$$

Then, for all $\forall\, f_0 \in \mathrm{Dom}(L)$,

$$\|f_n(t) - f(t)\|_X \to 0, \quad n \to \infty,$$

locally uniformly w.r.t. $t \geqslant 0$.

Approximation of non-Markovian evolution

Example: **time-space-fractional diffusion** with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$, $x \in \mathbb{R}^d$:

$$\partial_t^{1/2} f(t,x) = a(x)\left(-(-\Delta)^{1/2}\right) f(t,x), \qquad f(0,x) = f_0(x).$$

Then

$$p^{1/2}(t,\tau) = \frac{1}{\sqrt{\pi t}} e^{-\frac{\tau^2}{4t}},$$

and $\quad F(t) \sim T_t \equiv e^{t\left[a\left(-(-\Delta)^{1/2}\right)\right]} \quad$ where

$$F(t)\varphi(x) := \Gamma\left(\frac{d+1}{2}\right) \int_{\mathbb{R}^d} \varphi(q) \frac{a(x)t}{\left(\pi|x-q|^2 + a^2(x)t^2\right)^{\frac{d+1}{2}}} dq.$$

Hence for any $x_0 \in \mathbb{R}^d$:

$$f(t,x_0) = \lim_{n\to\infty} \Gamma^n\left(\frac{d+1}{2}\right) \times$$

$$\times \int_0^\infty \int_{\mathbb{R}^{nd}} \left[\prod_{k=1}^n \frac{a(x_{k-1})\tau/n}{\left(\pi|x_k - x_{k-1}|^2 + (a(x_{k-1})\tau/n)^2\right)^{\frac{d+1}{2}}}\right] \frac{f_0(x_n)}{\sqrt{\pi t}} e^{-\frac{\tau^2}{4t}} dx_1 \cdots dx_n d\tau.$$

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