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This is a joint work
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Rellich and Calderón-Zygmund inequalities for operators with discontinuous and singular coefficients

We give necessary and sufficient conditions for the validity of Rellich and Calderón-Zygmund inequalities in $L^p(\mathbb{R}^N)$ and in $L^p(B)$, where B is the unitary ball of $\mathbb{R}^N$, for the operator

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2}\cdot\nabla - b|x|^{-2},$$

with $a > 0$, $b, c \in \mathbb{R}$. Note that the condition $a > 0$ is equivalent to the ellipticity of the principal part of L.
More specifically, in the case of $L^p(\mathbb{R}^N)$, set

$$\gamma_p(a,c) := \left(\frac{N}{p} - 2\right)\left(N - 1 + c + a\Big(1 - \frac{N}{p}\Big)\right)$$

and considered the parabola

$$P_{p,a,c} := \left\{\lambda = -a\xi^2 + i\xi\left(N - 1 + c + a\left(3 - \frac{2N}{p}\right)\right) - \gamma_p(a,c)\ ;\ \xi \in \mathbb{R}\right\},$$

we prove that for $N \geq 2$, $a > 0$, b, $c \in \mathbb{R}$, $1 \leq p \leq \infty$, there exists a positive constant $C = C(N, a, p, c, b)$ such that the inequality

$$\int_{\mathbb{R}^N} |x|^{\alpha p}|Lu|^p\,dx \geq C\int_{\mathbb{R}^N} |x|^{p(\alpha-2)}|u|^p\,dx\|Lu\|_p \geq C\||x|^{-2}|u|\|_p$$

holds for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, if and only if $b + \lambda_j \notin P_{p,a,c}$ for every $j \in \mathbb{N}$, where λ_j are the eigenvalues of Laplace-Beltrami operator. The situation is more involved for the bounded domain B. However, also in this case, we give necessary and sufficient conditions for the validity of Rellich inequalities. As work in progress, we are investigating the validity of Rellich type inequalities with some correction terms in the cases where they fail in the form above written. Some partial results have been obtained.

Moreover we find necessary and sufficient conditions for the validity of the estimate

$$\int_{\mathbb{R}^N} |D^2u|^p\,dx \leq C\int_{\mathbb{R}^N} |Lu|^p\,dx,$$

where $u \in W^{2,p}(\mathbb{R}^N)$, $1 < p < \infty$.

We point out that Rellich and Calderón-Zygmund inequalities have been widely studied in literature for the Laplacian and more general operators. Firstly Rellich inequalities for the Laplacian in L^2 spaces (according to our notations $a = 1,\ b = c = 0$)

$$\left(\frac{N(N-4)}{4}\right)^2 \int_{\mathbb{R}^N} |x|^{-4}|u|^2\,dx \le \int_{\mathbb{R}^N} |\Delta u|^2\,dx,$$

for $N \neq 2$ and for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, have been proved by Rellich in 1956 and then extended to L^p-norms for $1 < p < \frac{N}{2}$.
Rellich inequalities with respect to the weight $|x|^\alpha$ have been also studied.