An Introduction to the Little Lip Function

Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ with $M_f(x,r) = \sup_{|x-y|\leq r} \dfrac{|f(x)-f(y)|}{r}$, the so-called "Big Lip" and "Little Lip" functions are defined as follows:

$$\text{Lip } f(x) = \limsup_{r\to 0^+} M_f(x,r) \qquad \text{lip } f(x) = \liminf_{r\to 0^+} M_f(x,r)$$

The big lip function has been around for over a hundred years and plays a prominent role in the Rademacher-Stepanov Theorem which tells us that f is differentiable almost everywhere on $L_f = \{x \,|\, \text{Lip } f(x) < \infty\}$. On the other hand, the little lip function has only recently come under study beginning with Balogh and Csörnyei's result (in 2006) showing that the Rademacher-Stepanov theorem no longer holds if we replace L_f with $l_f = \{x \,|\, \text{lip } f(x) < \infty\}$. There are many interesting questions concerning the little lip function, some of which have been answered and some of which are still open. In this talk, I consider the problems of characterizing the sets $E \subset \mathbb{R}$ for which there exist a continuous function f such that $l_f = E$ as well as characterizing the sets of non-differentiability for functions f with $l_f = \mathbb{R}$. I will also examine some additional questions about the relationship between L_f and l_f.