Linear Distortion and Rank-one Convexity

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Abstract

In 1998 Tadeusz Iwaniec gave an explicit example to show the failure of lower semicontinuity of the linear distortion in families of (quasiconformal) homeomorphisms $f : \Omega \to \mathbb{R}^n$, for $\Omega \subset \mathbb{R}^n$ a domain. Until that time lower semicontinuity was widely believed to be true. The linear distortion, or dilatation, is defined as

$$H(f,x) = \limsup_{r\to 0} \frac{\max_{|h|=r} |f(x+h)-f(x)|}{\min_{|h|=r} |f(x+h)-f(x)|} = \frac{\max_{|h|=1} |Df(x)|}{\min_{|h|=1} |Df(x)|}.$$

Then $H(f) = \text{ess sup}\{H(x,f) : x \in \Omega\}$. Iwaniec proved:

Theorem 0.1 *For each dimension $n \geq 3$ and dilatation $H \geq 1$, there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of H-quasiconformal mappings $f_j : \mathbb{R}^n \to \mathbb{R}^n$ converging uniformly to a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ whose dilatation is greater than H.*

He identified the key element being the fact that the linear dilatation function fails to be rank-one convex in dimensions greater than two.

A question here is how great the jump up can be. He and Fred Gehring gave an explicit conjectural bound.

Here we show that this property is completely generic; every linear mapping A of $\mathbb{R}^n$, $n \geq 3$ has a sequence approximants whose limit has strictly lower linear distortion as A has three distinct singular values (and does not otherwise). Further we identify explicitly the best rank-one direction to deform a linear mapping. We then test our results against Gehring and Iwaniec's conjecture.

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