A generalization of quasiconformal reflections on the Riemann sphere

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Let $\Omega \subset \hat{\mathbb{C}}$ be a Jordan domain. When it satisfies the so called Gehring-Osgood condition, i.e. for any two points $z_1,\, z_2 \in \Omega$, there exists a curve $\gamma \subset \Omega$ joining them so that

$$\int_\gamma \mathrm{dist}(z,\, \partial\Omega)^{-1}\, ds(z) \le C \log\left(1 + \frac{|z_1 - z_2|}{\min\{\mathrm{dist}(z_1,\, \partial\Omega),\, \mathrm{dist}(z_2,\, \partial\Omega)\}}\right),$$

then there exists a quasiconformal (even bi-Lipschitz) reflection between Ω and its complementary domains by the classical theory of Ahlfors on quasidisks. A natural generalization of the Gehring-Osgood condition is the following one: For the same pair of points, there exists a curve $\gamma \subset \Omega$ joining them so that

$$\int_\gamma \mathrm{dist}(z, \partial\Omega)^{1-p}\, ds(z) \le C|z_1 - z_2|^{2-p},$$

where $1 < p < 2$. Under this condition, jointly with P. Koskela and P. Pankka, we show that there exists a reflection f from Ω to its complementary domain so that

$$|Df|^p \le C J_f.$$

This generalizes the result of Ahlfors.