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Intermediate efficiency of tests for uniformity under heavy-tailed alternatives

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Testing for uniformity

$X_1, ..., X_n$ a sample from a distribution P on $[0, 1]$
P_0 the uniform distribution
$H_0 : P = P_0$

$T_1,\ T_2$ two test statistics of upper-tailed $\alpha-$level tests
compare test T_2 with respect to T_1
T_1 a benchmark procedure

$p_\theta(t) = p_{f,\theta}(t) = (1-\theta) + \theta f(t) = 1 + \theta(f(t) - 1)$ alternative
$f \neq 1$ fixed density
$f - 1$ "direction" of alternative, θ "distance" from P_0

N_{T_1}, N_{T_2} minimal sample sizes guaranteeing power $\beta \in (0, 1)$
under alternative p_θ

$ARE(T_2, T_1)$ – N_{T_1}/N_{T_2} after some limiting process

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ARE notions

Pitman ARE: $\theta = \theta_n = O(1/\sqrt{n})$, α fixed
Bahadur ARE: θ fixed, $\alpha = \alpha_n = O(e^{-cn})$

limitations:

Pitman	asymptotic normality under H_0 and under p_θ
Bahadur	large deviations under H_0

Kallenberg (intermediate) ARE
$\theta_n \to 0, \; n\theta_n^2 \to \infty$
$p_{\theta_n}(t) = 1 + \theta_n(f(t) - 1)$
$\alpha_n \to 0, \; (\log \alpha_n)/n \to 0$
rates of $\theta_n, \; \alpha_n$ related each other
some asymmetry

advantages:
– only moderate deviations under H_0
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Kallenberg ARE definition (sketch)

Kallenberg, AS, (1983)
Inglot & Ledwina, AS, (1996), Inglot, MMS, (1999)
Inglot, Ledwina & Ćmiel, ESAIM PS, (2019)

- there exists level α_n s.t. for sample size n test T_2 under p_{θ_n} attains a power β_n which is asymptotically nondegenerate
- $N_{T_1}(n)$ the **minimal** sample size for which T_1 on the same α_n and the same p_{θ_n} attains power **at least** β_n
- $N_{T_1}(n)/n \to \mathcal{E}(f) = \mathcal{E}_{T_2 T_1}(f) \in [0, \infty]$ as $n \to \infty$

  $\Rightarrow \mathcal{E}(f)$ Kallenberg ARE of T_2 with respect to T_1

Typically, ARE, if exists, strongly depends on the "direction" $f(t) - 1$

Assumptions

Moderate deviations (MD) for T

There exists $c_T \in (0, \infty)$ s.t.

$$-\lim_{n\to\infty} \frac{1}{nx_n^2} \log P(T_n \geq x_n\sqrt{n}) = c_T$$

for some (all) sequences $x_n > 0$, s.t. $x_n \to 0, \; nx_n^2 \to \infty$

Kallenberg ARE – crucial sufficient conditions

for benchmark test T_1

MD under H_0 for **all** x_n

for T_2

MD under H_0 possible for **narrow class of** x_n

often related to θ_n

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Neyman-Pearson test

natural benchmark test – the NP test for H_0 against p_{θ_n}

standardized NP test statistic

$$V_n = \frac{1}{\sqrt{n}\sigma_{0n}} \sum_{i=1}^{n} (\log p_{\theta_n}(X_i) - e_{0n}),$$

where

$$e_{0n} = \int_0^1 \log p_{\theta_n}(t)dt, \quad \sigma_{0n}^2 = \int_0^1 \log^2 p_{\theta_n}(t)dt - e_{0n}^2.$$

Inglot & Ledwina, AS, (1996)

Bounded "directions" $f(t) - 1 \Rightarrow$ MD for V_n for all x_n $(c_V = 1/2)$
So, NP test may be used as a banchmark test

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unbounded "directions"

testing goodness of fit $F = F_0$ on $\mathbb{R}$ against $F = F_1$ often leads to unbounded "directions" after transformation onto $[0,1]$ by F_0

- $F_0(x) = \Phi(x),\ F_1(x) = \Phi(x - \mu)$ Gaussian shift

  $f(t) = \varphi(\Phi^{-1}(t) - \mu)/\varphi(\Phi^{-1}(t)) \in L_q(0,1)$ for all $q > 1$

  f always unbounded

- $F_0(x) = \Phi(x),\ F_1(x) = \Phi(x/\sigma),\ \sigma > 1$ Gaussian scale

  $f(t) = \varphi(\Phi^{-1}(t)/\sigma)/\varphi(\Phi^{-1}(t)) \in L_q(0,1)$ for $q < \sigma^2/(\sigma^2 - 1)$

  f always unbounded

MD for the NP statistic for unbounded "directions"

Example

$f_r(t) = (1-r)t^{-r}, \; r \in (0, 1/2), \qquad f_r \in L_q(0,1)$ for all $q < 1/r$

Theorem 1. If $x_n/\theta_n^{r'} \to \infty$ and $x_n^{r/r'-1} \log \theta_n \to 0$ for some $r' < r$ ($x_n \to 0$ slower than θ_n^r) then MD for V_n corresponding to f_r degenerate i.e.

$$\lim_{n\to\infty} \frac{1}{nx_n^2} \log P_0(V_n \geq \sqrt{n}x_n) = 0$$

Theorem 2

If $f \in L_2(0,1)$ is unbounded and $x_n = o(\theta_n)$, $nx_n^2 \to \infty$, ($x_n \to 0$ faster than θ_n) then

$$-\lim_{n\to\infty} \frac{1}{nx_n^2} \log P_0(V_n \geq \sqrt{n}x_n) = \frac{1}{2} = c_V$$

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conclussion – disappointment

for unbounded $f \in L_2(0,1)$ (or even $\in L_q(0,1),\ q > 2$)
MD theorem for V_n may not hold in the full range of x_n

for unbounded f it may happen that the NP test cannot be a benchmark test

Kolmogorov-Smirnov as benchmark test

Theorem 3

Let

$$K_n = \sqrt{n} \sup_{t \in (0,1)} |\widehat{F}_n(t) - t|,$$

where $\widehat{F}_n$ is ecdf, be the unweighted KS test statistic. Then for every $x_n \to 0$, $nx_n^2 \to \infty$

$$-\lim_n \frac{1}{nx_n^2} \log P_0(K_n \geqslant \sqrt{n}x_n) = 2 = c_K$$

KS may always be used as a benchmark test in testing uniformity

Kolmogorov-Smirnov as benchmark test

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Kallenberg ARE for V_n with respect to K_n

Theorem 4

If $f \in L_2(0,1)$ (bounded or not) then Kallenberg ARE of V_n with respect to K_n exists and is equal to

$$\mathcal{E}(f) = \frac{||f-1||_2^2}{4||A||_\infty^2},$$

where $A(t) = \int_0^t (f(u)-1)du$

If f is bounded then Kallenberg ARE of K_n with respect to V_n is equal to (Inglot & Ledwina, JSPI, 2006)

$$\frac{1}{\mathcal{E}(f)} = \frac{4||A||_\infty^2}{||f-1||_2^2}$$

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heavy-tailed alternatives

Theorem 4 cannot be extended to $f \notin L_2(0,1)$

Suppose for some $r \in (1/2, 1)$ and $\delta > 0$ f satisfies

$(*)$ $f(t)t^r$ bounded from 0 and ∞ on $(0,\delta)$ and $f(t)$ bounded on $(\delta, 1)$

$$(*) \Rightarrow (f \notin L_{1/r}(0,1)) \Rightarrow (f \notin L_2(0,1))$$

Theorem 5

If f satisfies $(*)$ then Kallenberg ARE of V_n with respect to K_n is equal to ∞

$$\lim_{n\to\infty} \frac{N_K(n)}{n} = \infty$$

the same holds for all classical tests which have finite Kallenberg ARE with respect to KS

Empirical powers (in %) of KS and NP, alternative $f_r(t) = (1-r)t^{-r}$, $\alpha = \mathbf{0.05}$, small θ and several n

$r = 0.7$						$r = 0.3$					
$\theta = 0.05$			$\theta = 0.02$			$\theta = 0.1$			$\theta = 0.05$		
n	KS	NP	n	KS	NP	n	KS	NP	n	KS	NP
11	4	15	39	4	15	160	5	15	640	6	15
20	4	20	70	5	20	290	7	20	1200	7	20
42	5	**30**	155	5	**30**	610	10	**30**	2300	10	**30**
70	6	40	250	5	40	950	13	40	4800	15	47
105	7	50	3350	15	99	1200	15	47	6800	20	59
150	7	60	4900	20	100	1340	17	50	11100	**30**	77
540	15	94	7700	**30**	100	1650	20	58			
750	20	98	10100	40	100	1830	21	60	$\frac{2800}{610}$	$=$	4.6
1200	**30**	100				2800	**30**	76			
1600	40	100	$\frac{1200}{42}$	$=$	28.6	3880	40	86	$\frac{11100}{2300}$	$=$	4.8
2080	50	100				5050	50	93			
2500	60	100	$\frac{7700}{155}$	$=$	49.7	6350	60	97			

Ratios $\mathbf{N_K(n)/n}$ (n for V_n and $N_K(n)$ for K_n)
for the alternative $f_r(t) = (1-r)t^{-r}$, small θ, four values of r,
several powers separated from 0 and 1, $\alpha = \mathbf{0.01}$

r		power in %						
	θ	15	20	30	40	50	60	70
0.7	0.10	**30.4**	**23.7**	**18.2**	**15.0**	**12.9**	**11.0**	**9.7**
$\notin L_2$	0.05	**44.8**	**36.0**	**27.3**	**22.3**	**19.2**	**16.7**	
	0.02	**80.9**	**63.4**	**47.5**				
0.6	0.10	**17.9**	**14.5**	**11.2**	**9.7**	**8.6**	**7.8**	**7.0**
$\notin L_2$	0.05	**24.4**	**19.9**	**15.5**	**13.4**	**11.6**	**10.5**	
	0.02	**34.7**	**29.4**	**23.1**				
0.4	0.20	**5.8**	**5.0**	**4.4**	**4.0**	**3.8**	**3.6**	**3.4**
$\in L_2$	0.10	**6.2**	**5.8**	**5.0**	**4.6**	**4.3**	**4.0**	**3.8**
$\mathcal{E}$=5.79	0.05	**6.6**	**6.3**	**5.5**				
0.3	0.20	**4.3**	**3.9**	**3.4**	**3.2**	**3.0**	**2.9**	**2.9**
$\in L_2$	0.10	**4.3**	**4.0**	**3.6**	**3.4**	**3.3**	**3.1**	
$\mathcal{E}$=3.30	0.05	**4.4**	**4.0**	**3.7**				

PROOF OF THEOREM 5. Set P_n corresponding to p_{θ_n}, $n\theta_n^2 \to \infty$

Step 1 – asymptotic shift (calculation)

$b_n = E_{P_n} V_n \asymp \sqrt{n}\theta_n^{1/2r}$

Step 2 – asymptotic power (Liapunov's CLT)

$\forall x \quad P_n(V_n \geqslant x + b_n)$ bounded away from 0 and 1

$x + b_n$ critical value, $\quad \alpha_n = P_0(V_n \geqslant x + b_n)$

Step 3 – MD for V_n under H_0

if $x_n = O(\theta_n^{1/2r})$, $nx_n^2 \to \infty$ ($x_n \to 0$ sufficiently fast) then

$$-\limsup_n \frac{1}{nx_n^2} \log P_0(V_n \geqslant \sqrt{n}x_n) > 0$$

Step 4 – weak convergence and MD for K_n (Theorem 3) under H_0
(Step 3 – very weak version of MD for V_n)