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OPEN PROBLEM SESSION

As part of the 2007 Będlewo workshop, an open problem session was held on the morning of Saturday, May 26. Most of the following problems were presented at that session; in a few cases, problems have been modified or additional problems have been added.

1. Dimension of Čebyšev sets in hyperspaces (Robert Dawson). Most Čebyšev sets in hyperspaces over $\mathbb{R}^d$ are continuously parametrized by a set in $\mathbb{R}^d$ for some finite d, although some infinite-dimensional examples exist [1]. The example, also in [1], of the set of all balls and singletons in $\mathcal{K}^d$ has dimension $d+1$, as do the translational closures of strongly nested families in $\mathcal{Z}^d$ [2]. Can this bound be increased? (Note: Recent work has yielded 5-dimensional examples in $\mathcal{K}^2$ and $(2d-1)$-dimensional examples in $\mathcal{Z}^d$. Can these be improved?)

[1] A. Bogdewicz and M. Moszyńska, *Čebyšev sets in the space of convex bodies*, Rend. Circ. Mat. Palermo (2) Suppl. 77 (2006), 19–39.

[2] R. Dawson, *Some Čebyšev sets with dimension $d+1$ in hyperspaces over $\mathbf{R}^d$*, this volume, 89–110.

2. Continuously nested families of centrally symmetric bodies (Robert Dawson). We say that a family of bodies $(A_i|i \in [0,1])$ is *strongly nested* if for $i<j$ we have $A_i \subset \mathrm{int} A_j$. Suppose the nest is continuous; that is, for $j>0$ we have $A_j = \mathrm{cl} \cup_{i<j} A_i$ and for $i<1$ we have $A_i = \cap_{i<j} A_j$, and that each A_i is centrally symmetric with center c_i. Does it follow that the bodies $A_i - c_i$ are strongly nested? Without continuity it does not; see Fig. 1 of [1].

[1] R. Dawson, *Some Čebyšev sets with dimension $d+1$ in hyperspaces over $\mathbf{R}^d$*, this volume, 89–110.

3. Indecomposable bodies (Paul Goodey). A compact convex set $K \subset \mathbb{E}^d$ is said to be *indecomposable* if the equality $K = K_1 + K_2$ implies that K_1 and K_2 are homothetic to K. It is straightforward to show that, in dimension 2, the only indecomposable compact convex sets are the line segments and triangles. However, in higher dimensions the situation is rather more complicated. If all 2-dimensional faces of a polytope are triangles, then the polytope is indecomposable. From this, it can be shown that, for $d \geq 3$, the indecomposable bodies in $\mathbb{E}^d$ form a dense G_δ with respect to the usual topology (given by the Hausdorff metric). Furthermore, it is the case that the smooth strictly convex bodies also form a G_δ. Consequently, in the Baire sense, most convex bodies are smooth, strictly convex and indecomposable. Unfortunately, there are no known examples of such bodies. [A good description of this problem can be found in [1], pages 150–153.].

[1] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge University Press, 1993.

4. Constant surface area projections (Paul Goodey). We assume that K_1, K_2 are convex bodies in $\mathbb{E}^4$. We denote by $S\left(K|u^\perp\right)$ the surface area of the orthogonal projection of K onto the subspace orthogonal to $u \in S^3$ and by $A(K|E)$ the area of the orthogonal projection of K onto the 2-dimensional subspace E of $\mathbb{E}^4$. We ask, if $S\left(K_1|u^\perp\right) = S\left(K_2|u^\perp\right)$ for all $u \in S^3$, does it necessarily follow that $A\left(K_1|E\right) = A\left(K_2|E\right)$ for all 2-dimensional subspaces E of $\mathbb{E}^4$? The reverse implication is an easy consequence of the Cauchy-Kubota formulas. The question is known to have an affirmative answer if both bodies are bodies of revolution, if both are centrally symmetric, or if either body is a polytope.

This is a generalization of a problem of Firey who considered only the case that one of the bodies is a ball (in which case the surface areas of all 3-dimensional projections are the same).

If we denote by R the Radon transform from 3-dimensional subspaces of $\mathbb{E}^4$ to the 2-dimensional subspaces, then this question asks for the injectivity properties of the mapping $A(K|\cdot) \mapsto R\left(A(K|\cdot)\right)$. The mapping $R : C(G(4,2)) \to C(G(4,3))$ is known, from harmonic analysis, to not be injective. So this question has at its heart the study of the nature of those functions in $C(G(4,2))$ which are areas of projections of convex bodies.

The problem extends to all dimensions. Here we denote by $V_j(K)$ the j-th intrinsic volume of a convex body $K \subset \mathbb{E}^d$, for $1 \le j \le d-1$. If, for some $1 < j < d-1$, $V_j\left(K_1|E\right) = V_j\left(K_2|E\right)$ for all $E \in G(d,j)$, does it follow that $V_j\left(K_1|u^\perp\right) = V_j\left(K_2|u^\perp\right)$ for all $u \in S^{d-1}$? Again, the question is known to have an affirmative answer if both bodies are bodies of revolution, if both are centrally symmetric, or if either body is a polytope.

5. Minimal pairs of convex sets (Jerzy Grzybowski). We define pairs of compact convex sets in a topological vector space to be *equivalent* and write $(A,B) \sim (C,D)$ if $A+D = B+C$. The equivalence class containing (A,B) is denoted by $[A,B]$. It is known that $[A,B]$ always contains a minimal pair (C,D). If A and B are polytopes, is it true that we can always find a minimal pair of polytopes? (This has been answered in the affirmative for dimension ≤ 3.)

6. Minimal pairs representing continuous selections (Jerzy Grzybowski). For the two- and the three-dimensional case the minimal pairs of compact convex sets which correspond to the continuous selections of the coordinate functions and their negative sum are classified in section 8 of [1]. In four dimensions there are 111 types, including 7579 pairs. What happens in higher dimensions?

[1] J. Grzybowski, D. Pallaschke, and R. Urbański, *Minimal pairs of bounded closed convex sets as minimal representations of elements of the Minkowski–Rådström–Hörmander spaces*, this volume, 31–55.

7. Dual volumes and affine sections (Irmina Herburt). For a and b in a convex body A in $\mathbb{R}^d$ let $\mathcal{H}_{a,b}$ be the class of affine hyperplanes through a and b. Is it true that if, for every $H \in \mathcal{H}_{a,b}$,

$$\int_{S^{d-1}\cap(H-a)} \varrho_{(A\cap H)-a}(u)d\sigma(u) \le \int_{S^{d-1}\cap(H-b)} \varrho_{(A\cap H)-b}(u)d\sigma(u)$$

then

$$\int_{S^{d-1}} \varrho_{A-a}(u)d\sigma(u) \le \int_{S^{d-1}} \varrho_{A-b}(u)d\sigma(u) \ \ ?$$

[1] I. Herburt, M. Moszyńska and Z. Peradzyński, *Remarks on radial centres of convex bodies*, Math. Phys. Anal. Geom. 8 (2005), 157–172.

8. Covering a larger cube (Włodzimierz Kuperberg). What is the minimum number $y(d)$ of congruent d-dimensional cubes that can cover a slightly larger cube? It can be shown that $y(d) \le d+1$ (see Fig. 1 for the case $d = 2$).

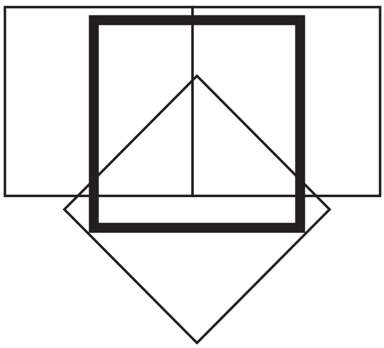

Fig. 1. $d+1$ congruent cubes can cover a larger cube

9. Self-similar Antoine's necklace (Włodzimierz Kuperberg). It is clear that a version of Antoine's necklace with many small links can have each link similar to the whole (Fig. 2). What is the minimum number of links for which this is possible, and what is the Hausdorff dimension?

10. Discs in a square (David Larman). If we place an infinite set of nonoverlapping discs in a square, how small can the dimension of the complement be (Fig. 3a)? An upper bound of 1.03 is known. This problem, posed in [1], has been open for over 40 years!

[1] D. G. Larman, *On the Besicovitch dimension of the residual set of arbitrarily packed discs in the Plane*, J. London Math. Soc. 42 (1967), 292–302.

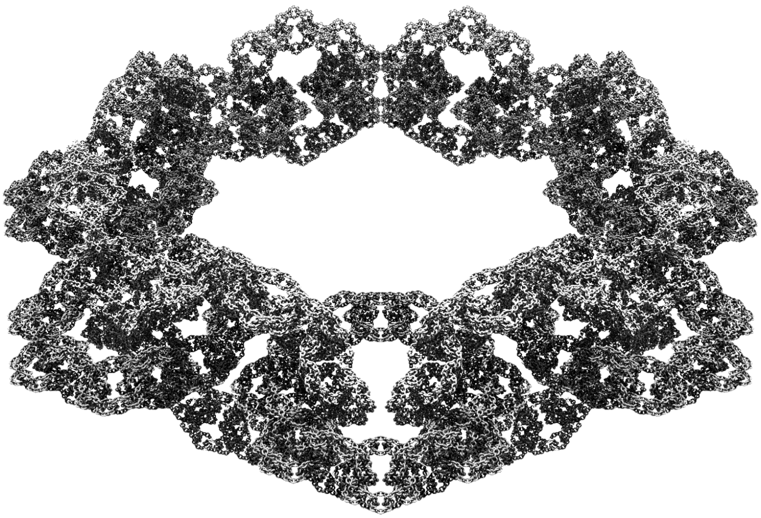

Fig. 2. A self-similar version of Antoine's necklace

11. Reconstruction from chord lengths (David Larman). Suppose that a planar body, A, contains a disc, D_r, of known radius r and centered at the origin (Fig. 3b.) Given, for each point x on the boundary of D_r, the length of the chord of A tangent to D_r at x, can A be reconstructed [1]?

[1] J. A. Barker and D.G. Larman, *Determination of convex bodies by certain sets of sectional volumes*, Discrete Mathematics 241 (2001), 79–96.

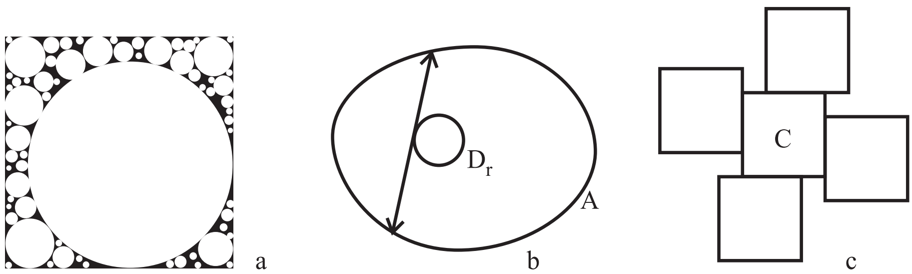

Fig. 3. Illustrations for Problems 10-12

12. Blocking numbers (David Larman). For a convex body C in $\mathbb{R}^d$, define the blocking number $b(C)$ to be the smallest number of nonoverlapping translates $C+x$ which are in contact with C at its boundary and prevent any other translate from touching it. For d-cubes the blocking number is 2^d (see Fig. 3c for the case $d = 2$); for 2-, 3- and 4-dimensional balls the blocking numbers are 4, 6, and 9 [1]. What other results can be obtained? Are there cases in which the blocking number can be reduced if we allow the blocking bodies to overlap each other (though not, of course, the blocked body)? What if contact with C is not required?

[1] L. Dalla, D. G. Larman, P. Mani-Levitska and C. Zong, *The blocking numbers of convex bodies*, Discrete and Computational Geometry 24 (2000), 267–277.

13. Selectors associated with optimal isometries (Maria Moszyńska). A selector $s : \mathcal{K}^d \to \mathbb{R}^d$, assumed equivariant under Euclidean isometries, is *associated with a metric ϱ on $\mathcal{K}^d$* (or $\mathcal{K}_0^d$) provided that whenever $f(A)$ is an isometric copy of A nearest to B, then $s(f(A)) = s(B)$.

Existence of such a selector for a metric ϱ does not imply the existence of a selector for another metric ϱ' even if ϱ' is topologically equivalent to ϱ. For instance, the Steiner point map is associated with the L_2 metric ϱ_2 (see [1]), while no selector is associated with the Hausdorff metric (see [2]).

For $p > 2$, is there a selector for $\mathcal{K}^d$ or $\mathcal{K}_0^d$, equivariant under isometries, and associated with ϱ_p?

[1] R. Arnold, *Zur L^2-Bestapproximation eines konvexen Körpers durch einen bewegten konvexen Körper*, Mh. Math. 108 (1989), 277–293.

[2] I. Herburt and M. Moszyńska, *Optimal isometries for a pair of compact convex subsets of* $\mathbb{R}^n$, this volume, 111–120.

14. Walk dimension of the Sierpiński carpet (Katarzyna Pietruska-Pałuba). For the Sierpiński gasket, assume that a resistor of resistance 1 is placed on each edge of one of the sequence of finite-network approximations (see Figure 4). Hold one corner of the triangle at a potential of 0 volts, and the other two at a potential of 1 volt. The effective restistance at stages $n = 0, 1, 2, \dots$ is $\frac{1}{2}\left(\frac{5}{3}\right)^n$, obtained by nesting.

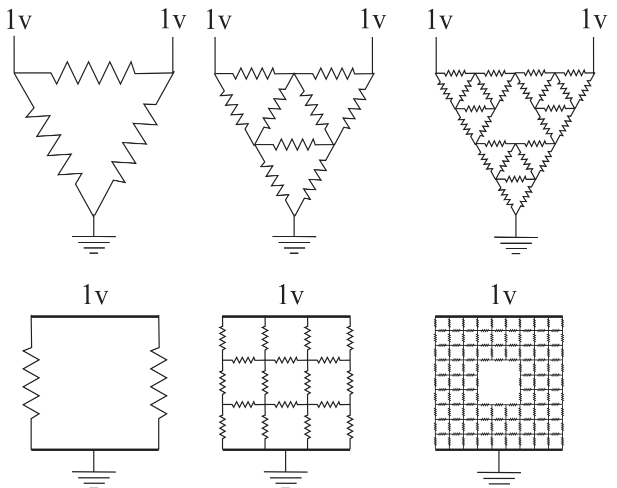

Fig. 4. Resistor networks approximating fractals

Take the n^{th} approximation of the Sierpiński carpet, put resistors along its edges except for two opposite sides which are taken to be perfect conductors, and again compute the effective resistance E_n. Is it true that $E_n \approx \rho^n E_0$ for some $\rho < 1$? Existence of such a number would give the walk dimension of the carpet.

15. Isoperimetric problems on the Sierpiński gasket (Katarzyna Pietruska-Pałuba). Among all the subsets of the Sierpiński gasket with given area (*i.e.*, Hausdorff measure at dimension $\frac{\log 3}{\log 2}$) find the one whose perimeter is smallest.

16. Are furthest points opposite? (Tudor Zamfirescu). Given a compact set C in a Hilbert space, with an intrinsic metric, and symmetric about 0. When does the fact that x and y are at maximal distance apart imply $y = -x$? In particular, what happens if C is a surface homeomorphic to the sphere or even convex? (The latter question was recently answered by the proposer in [1].)

[1] T. Zamfirescu, *Viewing and realizing diameters*, J. Geom. 88 (2008), 194–199.

17. Distribution of curvature (Tudor Zamfirescu). It is known that on most convex surfaces the curvature vanishes wherever it exists and is finite; and this occurs a.e., by Alexandrov's theorem. These are rather "flat" points (the curvature vanishes in any tangent direction; for a definition, see the first pages of [1]).

Simultaneously, a typical convex surface has a residual set of points without curvature in any tangent direction, and a dense set of points with infinite curvature in any tangent direction. This is a partition of the surface. What are the connectivity properties of these three sets of points (in the arbitrary and in the generic case)?

[1] H. Busemann, *Convex Surfaces*, Dover, 2008.