Dirac operators and Spin structures

Paul Baum

notes taken by:
Paweł Witkowski

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Chapter 1

Dirac operators and Spin structures

1.1 The Dirac operator of $\mathbb{R}^n$

First we consider n even. We shall construct matrices

$$E_1, E_2, \ldots, E_n, \quad n = 2r$$

each E_j being $2^r \times 2^r$ matrix of complex numbers. In fact each entry will be in $\{0, 1, -1, i, -i\}$.

Properties of E_j

1. $E_j^* = -E_j$,
2. each E_j is block anti-diagonal
$$E_j = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$$
and each block has size $2^{r-1} \times 2^{r-1}$,
3. $E_j^2 = I_{2^r}$,
4. $E_j E_k + E_k E_j = 0$ for $j \neq k$,
5. $$i^r E_1 E_2 \ldots E_n = \begin{bmatrix} I_{2^{r-1}} & 0 \\ 0 & -I_{2^{r-1}} \end{bmatrix}$$

We will proceed by induction on n even. For $n = 2$ we take

$$E_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Suppose we have $E_1, E_2, \ldots, E_n$ of size $2^r \times 2^r$. Then we put first n matrices of size $2^{r+1} \times 2^{r+1}$ as

$$\begin{bmatrix} 0 & E_1 \\ E_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & E_2 \\ E_2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & E_n \\ E_n & 0 \end{bmatrix}$$

and two additional matrices

$$\begin{bmatrix} 0 & -I_{2^r} \\ I_{2^r} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & iI_{2^{r-1}} & 0 \\ 0 & 0 & 0 & iI_{2^{r-1}} \\ iI_{2^{r-1}} & 0 & 0 & 0 \\ 0 & iI_{2^{r-1}} & 0 & 0 \end{bmatrix}.$$

Example 1.1. For $n = 4$ we have

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

For n odd, $n = 2r + 1$, we define matrices $E_1, E_2, \ldots, E_r$ satisfying

1. $E_j^* = -E_j$,
2. $E_j^2 = I_{2^r}$,
3. $E_jE_k + E_kE_j = 0$ for $j \neq k$,
4. $i^{r+1}E_1E_2 \ldots E_n = I_{2^r}$.

First if $n = 1$ we set

$$E_1 = [-i].$$

Then for $n = 2r + 1$ we use $2r$ matrices $E_1, E_2, \ldots, E_{n-1}$ as for the even case and as the last one we put

$$\begin{bmatrix} -iI_{2^{r-1}} & 0 \\ 0 & iI_{2^{r-1}} \end{bmatrix}.$$

From $E_1, E_2, \ldots, E_n$ we obtain:

1. The Dirac operator of $\mathbb{R}^n$ (described above)
2. The Bott generator vector bundle on S^n (n even)
3. The spin representation of $\mathrm{Spin}^c(n)$

1.1.1 Dirac operator

Now we can define **Dirac operator of** $\mathbb{R}^n$. For each n we set

$$D := \sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j}.$$

Example 1.2. For $n = 1$ we have Dirac operator of $\mathbb{R}$

$$D = -i\frac{\partial}{\partial x}.$$

For $n = 2$

$$D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \frac{\partial}{\partial x_2}$$

.

For $n = 2r$ and $n = 2r + 1$ D is an unbounded operator on the Hilbert space

$$\underbrace{L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n)}_{2^r}.$$

D is a first order elliptic differential operator on

$$\underbrace{C_c^\infty(\mathbb{R}^n) \oplus C_c^\infty(\mathbb{R}^n) \oplus \ldots \oplus C_c^\infty(\mathbb{R}^n)}_{2^r}$$

With this domain D is symmetric (that is D is formally self-adjoint) and D is essentially self-adjoint (that is D has unique self-adjoint extension). For n even

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

where D_- is the formal adjoint of D_+.

We will descirbe these notions in a general context. Let $\mathcal{H}$ be Hilbert space. An **unbounded operator** on $\mathcal{H}$ is a pair $(\mathcal{D}, T)$ such that

1. $\mathcal{D} \subset \mathcal{H}$ is a vector subspace of $\mathcal{H}$,
2. $\mathcal{D}$ is dense in $\mathcal{H}$,
3. $T\colon \mathcal{D} \to \mathcal{H}$ is a $\mathbb{C}$-linear map,
4. $(\mathcal{D}, T)$ is closeable, i.e. the closure of graph(T) in $\mathcal{H} \oplus \mathcal{H}$ is the graph of a $\mathbb{C}$-linear map

$$P(\overline{\text{graph}(T)}) \to \mathcal{H}$$

$$P(u, v) = u.$$

An unbounded operator $(\mathcal{D}, T)$ is **symmetric** if and only if

$$\langle Tu, v\rangle = \langle u, Tv\rangle \ \ \forall\, u, v \in \mathcal{D}.$$

For an unbounded operator $(\mathcal{D}, T)$ on $\mathcal{H}$ let

$$\mathcal{D}(T^*) := \big\{u \in \mathcal{H} \ \big|\ v \mapsto \langle u, Tv\rangle \text{ extends from } \mathcal{D} \text{ to } \mathcal{H} \text{ extends}$$

$$\text{to be a bounded linear functional on } \mathcal{H}\}$$

For $u \in \mathcal{D}(T^*)$ and $v \in \mathcal{H}$ there exists

$$T^*\colon \mathcal{D}(T^*) \to \mathcal{H}$$

such that

$$\langle u, Tv\rangle = \langle T^*u, v\rangle.$$

Now $(\mathcal{D}, T)$ is **self-adjoint** if and only if

$$(\mathcal{D}, T) = (\mathcal{D}(T^*), T^*).$$

Remark 1.3. Symmetric operator needs not to be self-adjoint, but a self-adjoint operator is symmetric.

Example 1.4. Take $C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ and

$$\begin{aligned}\mathcal{D} &= \left\{u \in L^2(\mathbb{R}) \;\middle|\; -i\tfrac{du}{dx} \in L^2(\mathbb{R}) \text{ in the distribution sense}\right\}\\ &= \{u \in L^2(\mathbb{R}) \mid x\hat{u} \in L^2(\mathbb{R})\},\end{aligned}$$

where $\hat{u}$ is the Fourier transform of u and

$$x\colon \mathbb{R} \to \mathbb{R},\;\; x(t) = t,\;\; \forall\, t \in \mathbb{R}.$$

Then $(C_c^\infty(\mathbb{R}), -i\frac{d}{dx})$ has unique self-adjoint extension $(\mathcal{D}, -i\frac{d}{dx})$.

Let D be Dirac operator of $\mathbb{R}^n$, $n = 2r$ or $2r+1$.

$$\begin{aligned}\Omega^1(\mathbb{R}^n) &= \{C^\infty \text{ 1-forms on } \mathbb{R}^n\}\\ &= \{f_1dx_1 + f_2dx_2 + \ldots + f_ndx_n \mid f_j\colon \mathbb{R}^n \to \mathbb{C},\; j = 1,2,\ldots,n\}\end{aligned}$$

$\Omega^1(\mathbb{R}^n)$ acts on

$$\underbrace{C_c^\infty(\mathbb{R}^n) \oplus C_c^\infty(\mathbb{R}^n) \oplus \ldots \oplus C_c^\infty(\mathbb{R}^n)}_{2^r}$$

in the following way. Let

$$\omega = f_1dx_1 + f_2dx_2 + \ldots + f_ndx_n,$$

$$s = (s_1, s_2, \ldots, s_{2^r}),\;\; s_l\colon \mathbb{R}^n \to \mathbb{C},\;\; l = 1,2,\ldots,2^r.$$

Then

$$\omega s = \sum_{j=1}^n f_jE_js.$$

There is following Leibniz rule for D

$$D(fs) = (df)s + f(Ds),$$

$$f\colon \mathbb{R}^n \to \mathbb{C},\;\; f \in C^\infty(\mathbb{R}^n),\;\; df = \sum_{j=1}^n \frac{\partial f}{\partial x_j}dx_j.$$

If M is C^∞-manifold, compact or non-compact, with or without boundary, $\dim M = M$, then the **Dirac operator of** M is an elliptic operator which is locally like the Dirac operator of $\mathbb{R}^n$.

1.1.2 Bott generator vector bundle

Let W be finite dimensional $\mathbb{C}$-vector space,

$$T \in \mathrm{Hom}_{\mathbb{C}}(W, W),\;\; T^2 = -I.$$

Then eigenvalues of T are $\pm i$ and there is decomposition

$$W = W_i \oplus W_{-i},$$

$$W_i = \{v \in W \mid Tv = iv\}$$

$$W_{-i} = \{v \in W \mid Tv = -iv\}$$

Assume that n is even, $S^n \subset \mathbb{R}^{n+1}$

$$S^n = \{(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{R}^n \bigm| a_1^2 + a_2^2 + \ldots + a_{n+1}^2 = 1\}.$$

We have a map

$$S^n \to M(2^r, \mathbb{C})$$
$$(a_1, a_2, \ldots, a_{n+1}) \mapsto a_1E_1 + a_2E_2 + \ldots + a_{n+1}E_{n+1} =: F.$$

From the properties of E_j we obtain

$$\begin{aligned} F^2 &= (a_1E_1 + a_2E_2 + \ldots + a_{n+1}E_{n+1})^2 \\ &= (-a_1^2 - a_2^2 - \ldots - a_{n+1}^2)I \\ &= -I \end{aligned}$$

so the eigenvalues of F are $\pm i$.

The Bott generator vector bundle β on S^n is given by

$$\begin{aligned} \beta_{(a_1,a_2,\ldots,a_{n+1})} &:= \text{i-eigenspace of } F \\ &= \{v \in \mathbb{C}^{2^r} \bigm| F(v) = iv\} \end{aligned}$$

For n even and $S^n \subset \mathbb{R}^{n+1}$ there is an isomorphism

$$\begin{array}{rccc} \mathrm{K}^0(S^n) = & \mathbb{Z} & \oplus & \mathbb{Z} \\ & 1 & & \beta \end{array}$$

where $1 = S^n \times \mathbb{C}$.

1.2 Spin representation and Spin^c

Let G be a topological group, Hausdorff and paracompact, X topological space Hausdorff and paracompact. A **principal G-bundle** on X is a pair (P, π) where

1. P is a Hausdorff and paracompact topological space with given continuous (right) action of G

$$P \times G \to P$$
$$(p, g) \mapsto pg$$

2. $\pi\colon P \to X$ is a continuous map, mapping P onto X

such that given any $x \in X$, there exists an open subset U of X with $x \in U$ and a homeomorphism

$$\varphi\colon U \times G \to \pi^{-1}(U)$$

with

$$\begin{aligned} &\pi\varphi(u, g) = u && \forall\ (u, g) \in U \times G \\ &\varphi(u, g_1g_2) = \varphi(u, g_1)g_2 && \forall\ (u, g_1, g_2) \in U \times G \times G \end{aligned}$$

Such $\varphi\colon U \times G \to \pi^{-1}(U)$ is referred to as a local trivialization.

Two principal G-bundles (P,π) and (Q,θ) are isomorphic if there exists a G-equivariant homeomorphism $f\colon P \to Q$ with commutativity in the diagram

$$\begin{array}{ccc} P & \xrightarrow{\quad f \quad} & Q \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \theta} & \\ & X & \end{array}$$

Let G, H be two topological groups and let (P,π), (G,θ) be a principal G-bundle and H-bundle on X. A homomorphism of principal bundles from (P,π) to (Q,θ) is a pair (η,ρ) such that

1. ρ is a homomorphism of topological groups $\rho\colon G \to H$
2. $P \to Q$ is a continuous map with commutativity in the diagrams

$$\begin{array}{ccc} P & \xrightarrow{\quad \eta \quad} & Q \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \theta} & \\ & X & \end{array} \qquad \begin{array}{ccc} P \times G & \xrightarrow{\eta \times \rho} & Q \times H \\ \downarrow & & \downarrow \\ P & \xrightarrow{\quad \eta \quad} & Q \end{array}$$

$$\pi p = \theta(\eta p) \qquad\qquad \eta(pg) = (\eta p)(\rho g)$$

A homomorphism of principal bundles on X will be denoted $\eta\colon P \to Q$ and $\rho\colon G \to H$ will be referred to as homomorphism of topological groups underlying η.

Lemma 1.5. *Let $\eta\colon P \to Q$ be a homomorphism of principal bundles on X with underlying homomorphism of topological groups $\rho\colon G \to H$. Then for any $x \in X$ there exists an open subset U of X with $x \in U$ and local trivializations*

$$\varphi\colon U \times G \to \pi^{-1}(U)$$

$$\psi\colon U \times H \to \theta^{-1}(U)$$

such that the diagram

$$\begin{array}{ccc} U \times G & \xrightarrow{\varphi} & \pi^{-1}(U) \\ {\scriptstyle \mathrm{Id}_U \times \eta}\downarrow & & \downarrow{\scriptstyle \eta} \\ U \times H & \xrightarrow{\psi} & \theta^{-1}(U) \end{array}$$

commutes.

Example 1.6. Let E be $\mathbb{R}$-vector bundle on X, $\dim_{\mathbb{R}}(E_p) = n$ for all $p \in X$. Denote

$$\Delta(E) := \{(p, v_1, v_2, \ldots, v_n) \mid p \in X,\ v_1, v_2, \ldots, v_n \text{ form a vector space basis for } E_p\}$$

$\Delta(E)$ is topologized by

$$\Delta(E) \subset \underbrace{E \oplus E \oplus \ldots \oplus E}_{n}.$$

Define an action

$$\Delta(E) \times \mathrm{GL}(n,\mathbb{R}) \to \Delta(E)$$

$$((p, v_1, v_2, \dots, v_n), [a_{ij}]) \mapsto (p, w_1, w_2, \dots, w_n),$$
$$w_j = \sum_{i=1}^{n} a_{ij} v_i, \quad [a_{ij}] \in \mathrm{GL}(n, \mathbb{R})$$

and a map

$$\theta\colon \Delta(E) \to X,$$
$$\theta(p, v_1, v_2, \dots, v_n) = p.$$

Then $(\Delta(E), \theta)$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle on X.

For $n \geqslant 3$

$$\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$$

and $\mathrm{Spin}(n)$ is the unique non-trivial 2-fold cover of $\mathrm{SO}(n)$. It is a compact connected Lie group.

$$\begin{array}{c} \mathrm{Spin}(n) \\ \downarrow \\ \mathrm{SO}(n) \subset \mathrm{GL}(n, \mathbb{R}) \end{array}$$

There is an exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

The group $\mathbb{Z}/2\mathbb{Z}$ embeds in the $\mathrm{Spin}(n)$ and S^1 as the $\{1, -1\}$. We define

$$\mathrm{Spin}^c(n) := S^1 \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(n).$$

Then there is an exact sequence

$$1 \to S^1 \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \to 1$$

$\mathrm{Spin}^c(n)$ is a compact connected Lie group

$$\begin{array}{c} \mathrm{Spin}(n) \\ \downarrow \\ \mathrm{Spin}^c(n) \\ \downarrow \\ \mathrm{SO}(n) \subset \mathrm{GL}(n, \mathbb{R}) \end{array}$$

Example 1.7. For $n = 1$

$$\mathrm{Spin}(1) = \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{SO}(1) = 1$$
$$\mathrm{Spin}^c(1) = S^1$$
$$\rho\colon S^1 \to \mathrm{pt}\,.$$

For $n = 2$

$$\mathrm{Spin}(2) = S^1 = \mathrm{SO}(2)$$
$$\mathrm{Spin}(2) \to \mathrm{SO}(2)$$
$$\zeta \mapsto \zeta^2$$

and

$$\mathrm{Spin}^c(2) = S^1 \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(2)$$
$$\rho(\lambda, \zeta) = \zeta^2.$$

Remark 1.8. Since $\mathrm{SO}(n) \subset \mathrm{GL}(n, \mathbb{R})$ we can view the standard map $\mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ as $\mathrm{Spin}^c(n) \to \mathrm{GL}(n, \mathbb{R})$.

Definition 1.9. *A* Spin^c **datum** *for an* $\mathbb{R}$*-vector bundle* $E \to X$ *is a homomorphism of principal bundles*

$$\eta\colon P \to \Delta(E),$$

where P *is a principal* $\mathrm{Spin}^c(n)$*-bundle on* X ($n = \dim_{\mathbb{R}}(E_p)$) *and the homomorphism of topological groups underlying* η *is the standard map*

$$\rho\colon \mathrm{Spin}^c(n) \to \mathrm{GL}(n, \mathbb{R}).$$

Two Spin^c data $\eta\colon P \to \Delta(E)$, $\eta'\colon P' \to \Delta(E)$ are isomorphic if there exists an isomorphism $f\colon P \to P'$ of principal $\mathrm{Spin}^c(n)$-bundles on X with commutativity in the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & P' \\ & {\scriptstyle \eta}\searrow \quad \swarrow{\scriptstyle \eta'} & \\ & \Delta(E) & \end{array} \qquad \eta = \eta' \circ f.$$

Two Spin^c data $\eta\colon P \to \Delta(E)$, $\eta'\colon P' \to \Delta(E)$ are homotopic if there exists a principal $\mathrm{Spin}^c(n)$-bundle Q on X and a continuous map

$$\Phi\colon Q \times [0, 1] \to \Delta(E)$$

such that

1. For $t \in [0, 1]$ each
$$\Phi_t = \Phi(-, t)\colon Q \to \Delta(E)$$
is a Spin^c data.

2.
$$\Phi_0\colon Q \to \Delta(E) \text{ is isomorphic to } \eta\colon P \to \Delta(E)$$
$$\Phi_1\colon Q \to \Delta(E) \text{ is isomorphic to } \eta'\colon P \to \Delta(E)$$

Definition 1.10. *A* $\mathrm{Spin}^c(n)$**-structure** *for* E *is an equivalence class of* $\mathrm{Spin}^c(n)$ *data, where the equivalence relation is homotopy.*

A Spin^c structure for an $\mathbb{R}$-bundle E determines an orientation of E. Let $w_1(E), w_2(E), \ldots$ be the Stiefel-Whitney classes of E, $w_j(E)\,\mathrm{H}^j(X; \mathbb{Z}/2\mathbb{Z})$-Cech cohomology. Then E is orientable if and only if $w_1(E) = 0$.

A **spin manifold** is a C^∞ manifold M, $\dim M = n$, for which the structure group of the tangent bundle TM has been lifted from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{Spin}(n)$. Such lifting is possible if and only if

$$\begin{array}{l} w_1(M) = 0, \quad w_1(M) \in \mathrm{H}^1(M; \mathbb{Z}/2\mathbb{Z}) \\ \text{and} \\ w_2(M) = 0, \quad w_2(M) \in \mathrm{H}^2(M; \mathbb{Z}/2\mathbb{Z}). \end{array}$$

A Spin^c **manifold** is a C^∞ manifold M, $\dim M = n$, for which the structure group of the tangent bundle TM has been lifted from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{Spin}^c(n)$. Such lifting is possible if and only if

$$\begin{array}{ll} w_1(M) = 0, & w_1(M) \in \mathrm{H}^1(M; \mathbb{Z}/2\mathbb{Z}) \\ \text{and} & \\ w_2(M) \text{ is in the image of} & \mathrm{H}^2(M; \mathbb{Z}) \to \mathrm{H}^2(M; \mathbb{Z}/2\mathbb{Z}). \end{array}$$

Various well known structures on a manifold M make M into Spin^c manifold

$$\begin{array}{ccc}
 & & \text{(complex analitic)} \\
 & & \Downarrow \\
\text{(symplectic)} & \Longrightarrow & \text{(almost complex)} \\
 & & \Downarrow \\
\text{(contact)} & \Longrightarrow & \text{(stably almost complex)} \\
 & & \Downarrow \\
\mathrm{Spin} & \Longrightarrow & \mathrm{Spin}^c \\
 & & \Downarrow \\
 & & \text{(oriented)}
\end{array}$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds. Spin^c structures behave very much like orientations. For example, an orientation on two of three $\mathbb{R}$ vector bundles in a short exact sequence determine an orientation on the third vector bundle. Analogous assertions are true for Spin^c structures.

Lemma 1.11 (Two out of three lemma). *Let*

$$0 \to E' \to E \to E'' \to 0$$

be an exact sequence of $\mathbb{R}$ *vector bundles on* X*. If* Spin^c *structures are given for any two of* E', E, E'' *then a* Spin^c *structure is determined for the third.*

Corollary 1.12. *If* M *is a* Spin^c *manifold with boundary* ∂M*, then* ∂M *is in canonocal way a* Spin^c *manifold.*

Proof. There is an exact sequence

$$0 \to T\partial M \to TM|_{\partial M} \to \partial M \times \mathbb{R} \to 0$$

□

Remark 1.13. If E is orientable ($w_1(E) = 0$), then the set of all possible orientations of E is in 1-1 correspondence with $\mathrm{H}^0(X; \mathbb{Z}/2\mathbb{Z})$. If E is Spin^c-able ($w_1(E) = 0$ and $w_2(E) \in \mathrm{im}(\mathrm{H}^2(X; \mathbb{Z}) \to \mathrm{H}^2(X; \mathbb{Z}/2\mathbb{Z}))$), then the set of all possible Spin^c-structures for E is then in 1-1 correspondence with $\mathrm{H}^0(X; \mathbb{Z}/2\mathbb{Z}) \times \mathrm{H}^2(X; \mathbb{Z})$.

1.2.1 Clifford algebras and spinor systems

Let V be a finite dimensional $\mathbb{R}$-vector space, $\langle -, - \rangle$ a positive defninite, symmetric, bilinear $\mathbb{R}$-valued inner product on V. We can form a tensor algebra

$$\mathcal{T}V := \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

with multiplication given by composing the tensors, and then define **Clifford algebra**

$$\mathrm{Cliff}(V) := \mathcal{T}V/(v \otimes v + \langle v, v \rangle \cdot 1)$$

where $(v \otimes v + \langle v, v\rangle \cdot 1)$ denotes the two-sided ideal in $\mathcal{T}V$ generated by all elements of the form

$$v \otimes v + \langle v, v\rangle \cdot 1, \;\; v \in V, \;\; 1 \in \mathbb{R}.$$

As a vector space over $\mathbb{R}$ $\mathrm{Cliff}(V)$ is canonically isomorphic to the exterior algebra

$$\Lambda^* V = \mathbb{R} \oplus V \oplus \Lambda^2 V \oplus \ldots \Lambda^n V, \;\; n = \dim_{\mathbb{R}} V.$$

Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of V. The monomials

$$e_1^{\epsilon_1} e_2^{\epsilon_2} \ldots e_n^{\epsilon_n}, \;\; \epsilon_j \in \{0, 1\}$$

form a vector space basis of $\mathrm{Cliff}(V)$. The canonical isomorphism of $\mathbb{R}$-vector spaces

$$\mathrm{Cliff}(V) \to \Lambda^* V$$

is given by

$$e_1^{\epsilon_1} e_2^{\epsilon_2} \ldots e_n^{\epsilon_n} \mapsto e_1^{\epsilon_1} \wedge e_2^{\epsilon_2} \wedge \ldots \wedge e_n^{\epsilon_n}.$$

This isomorphism does not depend on the choice of orthonormal basis of V.

$$\dim_{\mathbb{R}}(\mathrm{Cliff}(V)) = 2^n, \;\; n = \dim_{\mathbb{R}} V.$$

In $\mathrm{Cliff}(V)$ we have following identities

$$e_j^2 = -1, \;\; j = 1, 2, \ldots, n,$$

$$e_i e_j + e_j e_i = 0, \;\; i \neq j.$$

We can introduce $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Cliff}(V)$ in the following way

$$\mathrm{Cliff}(V) = (\mathrm{Cliff}(V))_0 \oplus (\mathrm{Cliff}(V))_1,$$

where $(\mathrm{Cliff}(V))_0$ is an $\mathbb{R}$-vector space spanned by $e_1^{\epsilon_1} e_2^{\epsilon_2} \ldots e_n^{\epsilon_n}$ with $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n$ even, and $(\mathrm{Cliff}(V))_1$ is an $\mathbb{R}$-vector space spanned by $e_1^{\epsilon_1} e_2^{\epsilon_2} \ldots e_n^{\epsilon_n}$ with $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n$ odd. This $\mathbb{Z}/2\mathbb{Z}$-grading does not depend on the choice of orthonormal basis of V.

Take $\mathbb{R}^n$ with the usual inner product

$$S^{n-1} \subset \mathbb{R}^n \subset \mathrm{Cliff}(\mathbb{R}^n).$$

The elements of S^{n-1} are invertible in $\mathrm{Cliff}(\mathbb{R}^n)$. Let $\mathrm{Pin}(n)$ be the subgroup of the invertible elements of $\mathrm{Cliff}(\mathbb{R}^n)$ generated by S^{n-1}. Then

$$\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap (\mathrm{Cliff}(\mathbb{R}^n))_0$$

$$\rho \colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

$$(\rho g)(x) = g x g^{-1}, \;\; g \in S^{n-1}, \;\; x \in \mathbb{R}^n.$$

For $n \geqslant 3$ this is the unique non-trivial 2-fold covering space of $\mathrm{SO}(n)$.

Consider complexification

$$\mathrm{Cliff}_{\mathbb{C}}(V) := \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cliff}(V).$$

Then $\mathrm{Cliff}_{\mathbb{C}}(V)$ is a C^*-algebra with

$$v^* = -v$$

for

$$v \in V \subset \mathrm{Cliff}(V) \subset \mathrm{Cliff}_{\mathbb{C}}(V).$$

Let

$$\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) := \mathbb{C}_{\mathbb{R}} \,\mathrm{Cliff}(\mathbb{R}^n),$$

$$\mathrm{Spin}^c(n) = S^1 \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(n) \subset \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n).$$

Then $\mathrm{Spin}^c(n)$ is a subgroup of the group of unitary elements of the C^*-algebra $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n)$.

Let us now choose an orthogonal basis $e_1, e_2, \ldots, e_n$ for even-dimensional $\mathbb{R}$-vector space V, $n = 2n = \dim_{\mathbb{R}}(V)$. Recall $2^r \times 2^r$ matrices $E_1, E_2, \ldots, E_n$ defined in the beginning of the chapter and then define a mapping

$$\mathrm{Cliff}_{\mathbb{C}}(V) \to M(2^r, \mathbb{C})$$

$$e_j \mapsto E_j, \;\; j = 1, 2, \ldots, n.$$

This gives an isomorphism of C^*-algebras $\mathrm{Cliff}_{\mathbb{C}}(V)$ and $M(2^r, \mathbb{C})$. For an odd dimension $n = 2r + 1$ recall $2^r \times 2^r$ matrices $E_1, E_2, \ldots, E_n$ and define two mappings

$$\varphi_+ \colon \mathrm{Cliff}_{\mathbb{C}}(V) \to M(2^r, \mathbb{C})$$

$$\varphi_+(e_j) = E_j, \;\; j = 1, 2, \ldots, n,$$

$$\varphi_- \colon \mathrm{Cliff}_{\mathbb{C}}(V) \to M(2^r, \mathbb{C})$$

$$\varphi_-(e_j) = -E_j, \;\; j = 1, 2, \ldots, n.$$

Then

$$\varphi_+ \oplus \varphi_- \colon \mathrm{Cliff}_{\mathbb{C}}(V) \to M(2^r, \mathbb{C}) \oplus M(2^r, \mathbb{C})$$

is an isomorphism of C^*-algebras.

Remark 1.14. This isomorphisms are non-canonical since they depend on the choice of an orthonormal basis for V.

Let E be an $\mathbb{R}$-vector bundle on X. Assume given an inner product $\langle -, - \rangle$ for E. Then define $\mathrm{Cliff}_{\mathbb{C}}(E)$ as a bundle of C^*-algebras over X whose fiber at $p \in X$ is $\mathrm{Cliff}_{\mathbb{C}}(E_p)$.

Definition 1.15. *An* **Hermitian module over** $\mathrm{Cliff}_{\mathbb{C}}(E)$ *is a complex vector bundle F on X with a $\mathbb{C}$-valued inner product $(-, -)$ and a module structure*

$$\mathrm{Cliff}_{\mathbb{C}}(E) \otimes F \to F$$

such that

1. *$(-, -)$ makes F_p into a finite dimensional Hilbert space,*
2. *for each $p \in X$, the module map*

$$\mathrm{Cliff}_{\mathbb{C}}(E_p) \to \mathcal{L}(F_p)$$

is a unital homomorphism of C^*-algebras.

Remark 1.16. Of course all structures here are assumed to be continuous. If X is a C^∞ manifold then we could take everything to be C^∞.

If E is oriented define a section ω of $\mathrm{Cliff}_{\mathbb{C}}(E)$ as follows. Given $p \in X$, choose a positively oriented orthonormal basis $e_1, e_2, \ldots, e_n$ of E_p. For n even, $n = 2r$, set

$$\omega(p) = i^r e_1 e_2 \ldots e_{2r}.$$

For $n = 2r + 1$ odd

$$\omega(p) = i^{r+1} e_1 e_2 \ldots e_{2r+1}.$$

Then $\omega(p)$ does not depend on the choice of positively oriented orthonormal basis. In $\mathrm{Cliff}_{\mathbb{C}}(E_p)$ we have

$$(\omega(p))^2 = 1.$$

If n is odd, then $\omega(p)$ is in the center of $\mathrm{Cliff}_{\mathbb{C}}(E_p)$. Note that to define ω, E must be oriented. Reversing the orientation will change ω to $-\omega$.

Definition 1.17. *Let E be an $\mathbb{R}$-vector bundle on X. A* **Spinor system** *for E is a triple $(\epsilon, \langle -, - \rangle, F)$ such that*

1. *ϵ is an orientation of E,*
2. *$\langle -, - \rangle$ is an inner product for E,*
3. *F is an Hermitian module over $\mathrm{Cliff}_{\mathbb{C}}(E)$ with each F_p an irreducible module over $\mathrm{Cliff}_{\mathbb{C}}(E_p)$,*
4. *if $n = \dim_{\mathbb{R}}(E_p)$ is odd, then $\omega(p)$ acts identically on F_p.*

Remark 1.18. The irreducibillity of F_p in (3) is equivalent to $\dim_{\mathbb{C}}(F_p) = 2^r$, where $n = 2r$ or $n = 2r+1$. In (4) note that $\omega(p)^2 = 1$ so for n odd $\omega(p)$ is in the center of $\mathrm{Cliff}_{\mathbb{C}}(E_p)$. Hence irreducibility of F_p implies that $\omega(p)$ acts either by I or $-I$ on F_p. Thus (4) normalizes the matter by requiring that $\omega(p)$ acts as I. When $n = \dim_{\mathbb{R}}(E_p)$ is even no such normalization is made.

If $(\epsilon, \langle -, - \rangle, F)$ is a Spinor system for E, then F is referred to as the **Spinor bundle**.

Suppose that $n = \dim_{\mathbb{R}}(E_p)$ is even. Let F_p^+ (F_p^+) be the $+1$ (-1) eigenspace of $\omega(p)$. We have a direct sum decomposition

$$F = F^+ \oplus F^-,$$

where F^+, F^- are $\frac{1}{2}$ $-$ Spin **bundles**. F^+ (F^-) is a vector bundle of positive (negative) spinors.

Assume we have right and left actions of the group G on topological spaces X, Y

$$X \times G \to X$$

$$G \times Y \to Y$$

Then

$$X \times_G Y := X \times Y / \sim, \;\; (xg, y) \sim (x, gy).$$

Example 1.19. Let E be an $\mathbb{R}$-vector bundle on X. Then

$$\Delta(E) \times_{\mathrm{GL}(n,\mathbb{R})} \simeq E$$

$$((p, v_1, v_2, \ldots, v_n), (a_1, a_2, \ldots, a_n)) \mapsto a_1 v_1 + a_2 v_2 + \ldots + a_n v_n.$$

Let E be an $\mathbb{R}$-vector bundle on X. A Spin^c datum

$$\eta\colon P \to \Delta(E)$$

determines a Spinor system $(\epsilon, \langle -,- \rangle, F)$ for E. For $p \in X$, given orientation ϵ, and inner product $\langle -,- \rangle$, an $\mathbb{R}$-basis $v_1, v_2, \ldots, v_n$ of E_p is positively oriented and orthonormal if and only if

$$(v_1, v_2, \ldots, v_n) \in \mathrm{im}(\eta).$$

The Spinor bundle for $n = 2r$ or $n = 2r + 1$

$$F = P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}.$$

We have to describe how $\mathrm{Spin}^c(n)$ acts on $\mathbb{C}^{2^r}$. For n odd $\mathrm{Spin}^c(n)$ has an irreducible representation known as its spin representation

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^r, \mathbb{C}), \;\; n = 2r + 1.$$

For n even $\mathrm{Spin}^c(n)$ has two irreducible representations known as its $\frac{1}{2}-$Spin representations

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C}),$$

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C}), \;\; n = 2r.$$

The direct sum

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C}) \oplus \mathrm{GL}(2^{r-1}, \mathbb{C}) \subset \mathrm{GL}(2^r, \mathbb{C}),$$

of these representations is the spin representation of $\mathrm{Spin}^c(n)$.

Consider $\mathbb{R}^n$ with its usual inner product and usual orthonormal basis $e_1, e_2, \ldots, e_n$

$$\varphi\colon \; \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) \to M(2^r, \mathbb{C})$$

$$\varphi(e_j) = E_j, \;\; j = 1, 2, \ldots, n.$$

There is a canonical inclusion

$$\mathrm{Spin}^c(n) \subset \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n)$$

and φ restricted to $\mathrm{Spin}^c(n)$ maps $\mathrm{Spin}^c(n)$ to $2^r \times 2^r$ unitary matrices

$$\mathrm{Spin}^c(n) \to \mathrm{U}(2^r) \subset \mathrm{GL}(n, \mathbb{C}).$$

This is **Spin representation** of $\mathrm{Spin}^c(n)$ and $\mathrm{Spin}^c(n)$ acts on $\mathrm{GL}(2^r, \mathbb{C})$ acts on $\mathbb{C}^{2^r}$ via this representation.

Let M be C^∞ manifold, possibly ∂M non-empty, TM the tangent bundle of M. Then

$$\begin{pmatrix} \mathrm{Spin}^c \text{ datum for } TM \\ \eta\colon P \to \Delta(TM) \end{pmatrix}$$
$$\downarrow$$
$$\begin{pmatrix} \text{Spinor system for } TM \\ (\epsilon, \langle -,- \rangle, F) \end{pmatrix}$$
$$\downarrow$$
$$\begin{pmatrix} \text{Dirac operator} \\ D\colon C_c^\infty(M, F) \to C_c^\infty(M, F) \end{pmatrix}$$

where F is the Spinor bundle on M and $C_c^\infty(M, F)$ are its C^∞ sections with compact support.

The Dirac operator

$$D\colon C_c^\infty(M, F) \to C_c^\infty(M, F)$$

is such that

1. D is $\mathbb{C}$-linear

$$D(s_1 + s_2) = Ds_1 + Ds_2,$$
$$D(\lambda s) = \lambda Ds, \quad s_1, s_2, s \in C_c^\infty(M, F), \quad \lambda \in \mathbb{C}.$$

2. If $f\colon M \to \mathbb{C}$ is a C^∞ function, then

$$D(fs) = (df)s + f(Ds).$$

3. If $s_1, s_2 \in C_c^\infty(M, F)$ then

$$\int_M (Ds_1(x), s_2(x))dx = \int_M (s_1(x), Ds_2(x))dx$$

4. If $\dim M$ is even, then D is off-diagonal

$$F = F^+ \oplus F^-$$
$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

$D\colon C_c^\infty(M, F) \to C_c^\infty(M, F)$ is an elliptic first-order differential operator. It can be viewed as an unbounded operator on the Hilbert space $L^2(M, F)$ with the scalar product

$$(s_1, s_2) := \int_M (s_1(x), s_2(x))dx.$$

Moreover it is a symmetric operator.

One proves existence of D by constructing it locally and patching together with a C^∞ partition of unity. The uniqueness of D is obtained by the fact that if D_0, D_1 satisfy conditions (1)-(4) above, then

$$D_0 - D_1\colon F \to F$$

is a vector bundle map, hence D_0, D_1 differ by lower order terms.

Example 1.20. Let n be even, $S^n \subset \mathbb{R}^{n+1}$, D-Dirac operator of S^n, F-Spinor bundle of S^n, $F = F^+ \oplus F^-$.

$$D\colon C_c^\infty(S^n, F) \to C_c^\infty(S^n, F)$$
$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$
$$D^+\colon C_c^\infty(S^n, F^+) \to C_c^\infty(S^n, F^-)$$

Then

$$\operatorname{Index}(D^+) := \dim_{\mathbb{C}}(\ker D^+) - \dim_{\mathbb{C}}(\operatorname{coker} D^+).$$

Theorem 1.21.

$$\operatorname{Index}(D^+) = 0.$$

We can tensor D^+ with the Bott generator vector bundle β from section (1.1.2)

$$D_\beta^+\colon C_c^\infty(S^n, F^+ \otimes \beta) \to C_c^\infty(S^n, F^- \otimes \beta).$$

Then we have

Theorem 1.22.

$$\operatorname{Index}(D_\beta^+) = 1.$$