‘A’, ‘an’ or ‘the’?

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Introduction

You can sometimes hear non-native speakers of English expressing opinions—not completely unfounded—that the problem of articles, and more generally of language correctness, is a side issue: “a specialist will understand”, it is the “results” that are important.

Some of those writers do not bother and choose the simplest solution: they omit the articles altogether.

Supppose however you translate a (say) Polish sentence word for word, add no articles, and obtain

Solution of equation (1) is function of form (3). $\leftarrow$ WRONG!

Although the (say, Polish) original sentence may have been perfectly grammatical, the translation is not; what is more, the reader will have difficulty guessing what the writer meant. Here are some possibilities:

- Every solution of (1) is a function of the form (3). [a statement of type $A \subset B$]
- Every function of the form (3) is a solution of (1). [$B \subset A$]
- Some solution of (1) is a function of the form (3). [$A \cap B \neq \emptyset$]

These are not all possible readings; e.g., maybe the “form” in the original sentence was misleading, and in fact the author meant a unique object:

- A function of the form (3) is the (unique) solution of (1).
- The function (3) is a solution of (1).
- The function (3) is the only solution of (1).

Each of these sentences is mathematically different. The reason for all this is that in many languages (for example in Polish) you can say grammatically something perfectly mathematically obscure; in English, the use of articles forces more precision.

‘A’ or ‘an’?

As a warm-up, we settle a simple question: when do we use ‘a’, and when ‘an’? The answer depends on *pronunciation*, and not on spelling: ‘a’ is used before a consonant, and ‘an’ before a vowel:

- a map, a unit, a y-derivative, a ξ
- an ellipse, an n, an x-section, an (ℓ, M)-tower

There is a (minor) problem with abbreviations: you can write both

- Then X is an l.c.s. [if you want it to be read ‘el si: es’]

and

- Then X is a l.c.s. [if you treat it just as shorthand and want it to be read ‘locally convex space’]

In such cases it is best to follow the common usage.

Countable and uncountable nouns

Suppose we have a noun in a sentence. As we have seen in the above examples, in most cases we have to “decorate” it, most often with an article (‘a/an’ or ‘the’). But is the article always necessary, and which one should we use?

The answer depends on the type of the noun, more precisely, on the class of nouns it belongs to.

We will not consider all classes of nouns here; for example, we will ignore proper nouns, although they do pose many article problems (Warsaw University, the University of Warsaw; London, the Hague; Denmark, the United States, the Netherlands, etc.). The relevant information can be found in many grammar books and dictionaries.

We will concentrate on two classes of nouns: countable and uncountable ones.

Countable nouns refer to separate objects, have singular and plural forms and can be counted, that is, used with numbers:

- one function, two functions
- one process, three processes
- one group, five groups

Uncountable nouns have no plural, cannot be used with numbers and do not refer to separate objects:

- compactness

- continuity
- existence
- advice
- information
- progress

The problem is that many nouns, including those in mathematical texts, are both countable and uncountable – *depending on their use*:

- Let μ and ν be two different *measures* on the measurable space (X, Ω). [countable]
- Suppose D has *measure* 1. [uncountable]
- We will give two *proofs* of this fact. [countable]
- Only (iv) needs *proof.* [uncountable]
- Consider the group of *symmetries* of F. [countable]
- The *symmetry* of G about 0 is evident. [uncountable]
- A detailed *study* of the behaviour of H_n is presented in [BX]. [countable]
- The *study* of such modules has played a major role in the development of modular representation theory. [uncountable]

Countability means that <u>in a given use</u> the noun refers to a particular object and not e.g. to a measure of something, a relation between some objects, executing some action in general (in contrast to a particular execution), a subject of investigation etc.

Note that in the last two examples, the second one can be distinguished from the first because it can be expressed with the use of the ing-form:

- Studying such modules has played a major role etc.

▷ Exercise: give examples of "countable" and "uncountable" uses of the following words: application, homeomorphism, equality.

Countable nouns

For countable nouns the rules of using articles seem simpler, starting from the following primitive "rule" (not without exceptions):

A countable singular noun cannot appear "naked".

You cannot say e.g. "for function on $[0, 1]$" – some "determiner" of the word 'function' is necessary, either before or after it. Here are some possibilities:

- for a function
- for some function
- for the function f [mentioned earlier]
- for this/that function
- for every function
- for each function
- for any function [no matter which]
- for no function
- for neither function [neither of the two mentioned earlier]
- for our function
- for Kowalski's function

A plural countable noun can appear with a determiner or without it:

- functions [functions in general; a mathematician can (*for consistency*) think that here too *there is* a determiner, namely "the indefinite article $\emptyset$"]
- the functions f_i [mentioned earlier]
- these/those functions
- some functions
- all functions
- any functions [any whatsoever]
- both functions, both the functions
- no functions

Caution: you cannot say "each functions" or "every functions": the words 'each' and 'every' require a singular noun (except for some idiomatic expressions, like 'every two hours').

$\triangleright$ The determiners that appeared above were placed before a noun (*pre-determiners*). But a determiner can sometimes follow the noun. For example, if the vertices of a graph are numbered, you can say

- We now consider vertex 2.
- This is not true of vertex i, $i = 5, \ldots, 10$.

Here is another natural example:

- The element a_{ij} appears in row i and column j of the matrix A.

Here the number is a *post-determiner*. Numbered objects appearing most often include pages, sections, steps of the proof, theorems, etc.:

- on page 3
- in Section 6
- by Theorem 5.1

▷ A mathematician would like to know whether a symbol can be a post-determiner: can one say just 'function f', or must one say 'a/the function f'? Unfortunately, I do not have a general answer. First, it seems that adding an (appropriate) article is never a mistake; secondly, in most of the cases I know (and in most areas of mathematics), articles are indeed used.

On the other hand, e.g. in elementary geometry textbooks you can see 'triangle ABC', 'angle AOB', 'line AB' (with no article)—and frankly speaking I do not know why.

In automata theory you can say e.g.

- The automaton W is in state s.

Conclusion: the best policy is to follow the habits adopted in the (good) literature in the given area of mathematics.

Which article?

In the case of countable nouns, determining which article should be used ('a/an' or 'the' in the singular, $\emptyset$ (nothing) or 'the' in the plural) requires answering a simple question:

Do they know which I mean?

'They' means the reader or listener. The article used depends on the answer:

- YES $\longrightarrow$ use 'the'
- NO & singular $\longrightarrow$ use 'a/an'
- NO & plural $\longrightarrow$ use $\emptyset$

Note that both the above question and the answer, and hence the choice of the article, *are not unconditional*: they depend on the writer's knowledge or opinion about what the readers know (or do not know). This is hardly surprising: the form of a piece of information we are transmitting depends on who we are transmitting it to and in what circumstances.

Now we shall analyze in detail various situations where the answer to the above question is 'yes' or 'no'. We start from the 'yes' answers, that is, from the cases where 'the" should be used.

'The' before countables

They know which because it's been mentioned earlier:

- Let C be a positive constant. (We do not assume that the constant is greater than 1.) [that constant]
- Let C be the constant of Lemma 3. [the constant mentioned there—even if the constant is not uniquely determined]
- We shall say that an operator is in standard form if..... Of course, the standard form is not unique.
- Let $A \subset X$. Then every element of the collection A is convex.
- Define $\exp x = \sum x^i/i!$. The series can easily be shown to converge.

They know which because it's clear:

- The set of all $n \times n$ matrices is a ring with the well-known definitions of addition and multiplication.

They know which because there is only one:

- The function $-e^{-x}$ is the derivative of e^{-x}. [Inserting 'a' here would mean mathematical ignorance: 'is a derivative' = 'is some derivative'.]
- Then x must be the centre of the ball U.
- Let $L^1(X)$ be the class of all integrable functions on X. [with 'a' we would have "some class of all integrable functions", an absurdity]

They know which because I'm specifying which:

- The function f defined by (2) is automatically continuous. [the function that has been defined there]
- Let f be the function defined by $f(x) = ...$
- Let f be the linear form $g \mapsto (g, F)$.
- Let A be the event that ξ admits k fluctuations.
- We will prove the stronger fact that there is no L^1 function such that....

Note that here the object is specified *within the same sentence*; e.g. in the last sentence you can use 'a' if you write

- We will prove a stronger fact: there is no L^1 function such that....

They know which because it is first, last, greatest, the same etc.:

- the first section

- the next example
- the last row
- the greatest common divisor
- the least such constant
- the only such map
- the same manifold

Note that 'a/an' *can* appear before an ordinal, e.g. when 'first' refers to objects that have not been numbered before:

- Here is a first relation between $K[G]$ and trivial G-modules.

Another example:

- Assume that the number x is not an nth power.

'A' or $\emptyset$ before countables

They don't know which because I mean 'some', undetermined:

- A collection τ of subsets of a set X is called a topology if.... [a collection of unspecified subsets]
- A more general theory must be sought to account for these irregularities.

They don't know which because I mean 'a certain', but I haven't specified it:

- After a change of variable we obtain.... [after some change of variables, maybe uniquely determined, but which we have not specified]
- The integral may be approximated by sums of the form... [by some sums]
- A slight strengthening of the hypotheses gives us a regular measure. [some slight strengthening]
- Then g has an additional interesting property: it is a convex functional. [has a certain additional property]
- A remarkable feature of the solution should be stressed: it is discontinuous at x.
- There exists a unique measure that represents f. [a certain uniquely determined measure]
- Combining (2) and (3) we obtain, with a new constant C,....

They don't know which because I mean 'any, no matter which':

- Let f be a solution of (1). [any solution, no matter which—assuming that we do not know at present how many solutions there are]

They don't know which because I mean 'one of many' (elements of a certain class):

- Then f is a bounded function. [belongs to the class of bounded functions]

They don't know which because I mean 'every':

- A zero of order m is counted as m zeros. [Each zero]
- A chain may be represented as a sum of paths in many ways. [Each chain]
- Closed sets are Borel sets. [All closed sets are Borel sets.]
- If H is a normal subgroup of G, left cosets and right cosets coincide.
- F_n converges to F uniformly on compact subsets of A. [on all compact subsets]
- Let A be the set of points at distance 1 from K. [of all points]

Countable nouns—more difficult cases

▷ There is no article after 'with' in certain constructions:

- There is a directed edge with label x, source a and target b.
- We first construct an automaton over A with state set S.

However, an article does appear when the noun has additional characteristics:

- We first construct an automaton over A with a larger state set S.

▷ If a plural noun refers to a class of objects as a whole, use 'the':

- The random variables of mean 0 form a hyperplane. [the family of random variables]
- Thus $\pi_n(X)$ can be interpreted as the homotopy classes of maps $S^n \to X$. [the set of all homotopy clases]
- a subalgebra of R containing the constants [containing the set of all constants]
- This group acts transitively on the vertices. [on the set of vertices]

Remember, however, that 'the' does not mean 'all'—here the earlier example is worth keeping in mind:

- Closed sets are Borel sets.

This example concerns *containment* of certain (families of) sets. When dealing with *equality* of sets, we often use 'the':

- In the plane, the open sets are those which are unions of open discs.

▷ For reasons that are not quite clear to me, 'the' is used in the plural in some constructions with 'of':

- We call the elements of V vector partitions.
- Since u is constant on the level sets of g,...
- The algebra A separates the points of X.
- A standard theorem states that it is possible to apply the operators of first-order predicate calculus to finite state automata, obtaining other finite state automata.
- Under a mild hypothesis, the trajectories of a supermartingale are almost surely of limited fluctuation.
- Let P be a set of representatives of the left cosets of H in G.
- Our next result collects some basic facts about the maximal left ideals of B.

▷ If a mathematical symbol appears alone, without the corresponding noun, it is generally understood to be singular. If you want it to be considered as plural, add 'the':

- Let Y be the set of points with coordinates $y_i = m_i/N$, where the m_i are integers with $0 \le m_i \le pN$.

▷ 'A' often appears in complementary phrases:

- We will see that automatic groups satisfy an isoperimetric inequality, a fact that can sometimes be used to prove that a group is not automatic.
- It turns out that f is injective, a fact by no means obvious, and which can be rephrased by saying that....

▷ After 'the set of' there is no article even for uniquely determined operations:

- We define S_i as the set of targets of arrows in T_i.
- the set of homotopy classes of maps $X \to Y$.

But

- the sum of the differentials of the maps f_i.

▷ 'A' does appear in certain idiomatic constructions with concrete objects:

- These two quantities differ by a factor of 2.
- We might need to cross another k arcs to get to the basepoint.

Uncountable nouns

Remembering that certain nouns are countable in some uses, and uncountable in others, let us formulate a first primitive "rule" (with exceptions):

An uncountable noun cannot appear with 'a/an'.

So you cannot thank anybody "for an advice" or consider "a continuity".

Therefore, there are two possibilities: no article or 'the'.

However, the rules one should follow here and the practical usage by native speakers are not quite clear. The examples given below should be treated as 'sufficient conditions' (i.e., sufficient for correctness): if you write like that, the text will be correct; but often it is not clear (to me) whether or not other versions are just as good.

Uncountables without article

Properties without article

When the object enjoying the property is not mentioned (even if known):

- The proof did not really use independence.
- Then $a \leq b$; equality holds if and only if...
- We see that we have strict inequality in (1.1) unless $x = y$.
- By continuity, (2) also holds for $x = 1$.
- For uniqueness, suppose that...
- We first prove sufficiency. [But you can also see 'We first prove the sufficiency']
- Openness is the trickiest part.

Similarly when talking about mathematical structures:

- Right multiplication by ρ also preserves order. [but 'preserves the order $\leq$']

When the object is mentioned, but the property is 'potential', a subject to study:

- Almost sure continuity of sample paths is an interesting probabilistic property.

If the object having the property appears in the sentence and the property is "actual", in most cases 'the' is used (see below), but probably it can often be omitted:

- Exactness of the diagram is therefore an easy consequence of (2).

Actions and processes without article

- The interplay between A and B needs further investigation.
- Consideration of asynchronous groups was first proposed by Fox.
- The property of being a small group is not invariant under change of generators. [under the changing of generators]
- But $A = B$, by direct calculation.
- Repeated application of (2) yields....
- This work continues research begun in [3].

Measures without article

After 'to have' and in certain expressions specifying the value of a measure:

- Then f has measure 1.
- Every G has norm g.
- Thus P has degree/order 2.
- an element of length 1
- We can find a representative for A in time $O(x)$.

Mathematical operations without article

- Since union and concatenation of languages are associative, many parentheses can be omitted.
- But A is also a Banach algebra under pointwise multiplication.
- This was proved in [3] with no appeal to integration.
- Using integration by parts we obtain....
- There is an action of g on X, coming from left multiplication by elements of G.
- These sublattices are isomorphic under multiplication on the right by r.
- a set closed under inversion

Areas of science or areas of mathematics

- Mathematics is fun.
- Now class field theory implies that....

Uncountables with 'the'

Properties with 'the'

When the object is mentioned, but the property is 'actual':
Probably, the use (or not) of 'the' in the examples below is facultative—but I'm not sure.

- By <u>the</u> continuity of the logarithm function,...
- We need an estimate showing <u>the</u> smallness of the tails of $\sum A_i$.
- <u>The</u> completeness of X is proved by projecting B onto....
- It is easy to prove that <u>the</u> existence of a combing is an invariant of pseudoisometry.

Measures with 'the'

- Then <u>the</u> length of B does not exceed 5.

Actions with 'the'

- <u>The</u> introduction by Gromov of this class of groups proved to be a major development in combinatorial group theory.

Mathematical operations with 'the'

(In spite of the section title, the uses below are countable; they are placed here to enable comparison with other examples involving mathematical operations.)

- <u>The</u> composition of these isotopies is therefore k-Lipschitz.
- Here A is <u>the</u> union of the disks D_i.

Uncountables with 'a' (!)

Actions with 'a'

(The uses below are countable.)

- This is customarily achieved by <u>an</u> application of the Cauchy-Schwarz inequality.
- <u>An</u> examination of the proof reveals that....
- <u>A</u> straightforward adaptation of the argument in [GY] brings in the fourth moment of G.
- We conclude this subsection by making <u>an</u> explicit calculation of....

Measures with 'a'

- At any crossing, it must have a positive slope.
- There exists a constant K such that the paths w and w' are at most a Hausdorff distance K from each other.

Mathematical operations with 'a'

- Let K be a subgroup whose conjugates have an empty intersection. [or 'have empty intersection']
- Every open set in the plane is a union of open disks. [is some union of discs]

Restricted mental activities

Here are some phrases that can appear e.g. in biographical texts:

- His parents wanted him to have a good education.
- He had a first-class knowledge of Italian.

Peculiar nouns

There exist some peculiar nouns, undoubtedly uncountable, but sometimes used with 'a':

- In this paper we wish to renew an interest in the systematic study of the relationships between....