ON RAPIDLY DECREASING DISTRIBUTIONS

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Abstract. A connection is established between the two definitions of the space of rapidly decreasing distributions on $\mathbb{R}^n$, one given by L. Schwartz and the other by J. Horváth.

Introduction

Rapidly decreasing distributions on $\mathbb{R}^n$ were defined by L. Schwartz in the form of a limit space. In the present paper it is proved that this limit space corresponds to a locally convex space with underlying set denoted by RD which is equipped with a topology denoted by $\tilde{b}$. J. Horváth's approach to rapidly decreasing distributions is different. He defines them as the members of the set H of those slowly increasing distributions on $\mathbb{R}^n$ which extend to continuous linear functionals on the inductive limit $\mathcal{O}_C = \lim_{\mu\to\infty} \mathcal{Z}_\mu$. Here $\mathcal{Z}_\mu$, $\mu \in [0,\infty[$, are some spaces of infinitely differentiable functions on $\mathbb{R}^n$ with unbounded growth as $\mu \to \infty$.

Chapter 1 of the present paper is devoted to relations between Schwartz's and Horváth's approaches from the point of view of initial topologies. In Chapter 2, by a purely analytical method, we introduce in RD the strong convolutional topology which behaves well with respect to the Fourier transformation.

1. Initial topologies in $(\mathcal{O}_C)'$ and consequences for RD

I. The J. Horváth space $\mathcal{O}_C$. For every $\mu \in \mathbb{R}$ and $p \in [1,\infty[$ consider the following Fréchet spaces of infinitely differentiable functions on $\mathbb{R}^n$:

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$$S^p_\mu = \{\phi \in C^\infty(\mathbb{R}^n) : \text{for every } \alpha \in \mathbb{N}^n_0 \text{ the function}$$
$$\mathbb{R}^n \ni x \mapsto (1+|x|)^{-\mu}\partial^\alpha\phi(x) \in \mathbb{C} \text{ belongs to } L^p(\mathbb{R}^n)\},$$
$$S_\mu = \Big\{\phi \in C^\infty(\mathbb{R}^n) : \lim_{|x|\to\infty}(1+|x|)^{-\mu}\partial^\alpha\phi(x) = 0 \text{ for every } \alpha \in \mathbb{N}^n_0\Big\},$$
$$\tilde{S}_\mu = \Big\{\phi \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}|\partial^\alpha\phi(x)| < \infty \text{ for every } \alpha \in \mathbb{N}^n_0\Big\}.$$

For $\mu \in \mathbb{R}$, $p \in [1,\infty[$ and $\lambda \in]n/p,\infty[$ one has the continuous imbeddings

$$S^p_\mu \hookrightarrow S_\mu \hookrightarrow \tilde{S}_\mu \hookrightarrow S^p_{\mu+\lambda}, \tag{1.1}$$

the proof of which (together with the definitions of the relevant seminorms) is postponed to Section III. From [B1, Remark in Sect. II.2.4] and the imbeddings (1.1) it follows that the three inductive limits $\lim_{\mu\to\infty} S^p_\mu$, $\lim_{\mu\to\infty} S_\mu$, $\lim_{\mu\to\infty} \tilde{S}_\mu$ define the same locally convex space of infinitely differentiable functions on $\mathbb{R}^n$, denoted by $\mathcal{O}_C$. This space, in the form $\mathcal{O}_C = \lim\operatorname{ind}_{\mu\to\infty} S_\mu$, was introduced by J. Horváth [H, Sect. 2.12, Example H 9].

If $\mathfrak{S}_\mu$ denotes either S^p_μ, S_μ or $\tilde{S}_\mu$, then the equality $\mathcal{O}_C = \lim_{\mu\to\infty}\mathfrak{S}_\mu$ means that $\mathcal{O}_C$ is the union $\bigcup_{\mu\in[0,\infty[}\mathfrak{S}_\mu$, equipped with the strongest locally convex topology such that $\mathfrak{S}_\mu \hookrightarrow \mathcal{O}_C$ for every $\mu \in [0,\infty[$.

Let $\mathcal{S}$ denote the set of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. Then $\mathcal{S}$ is dense in every S^p_μ and S_μ for $\mu \in [0,\infty[$, and in $\mathcal{O}_C$, so that the sets of continuous linear functionals $(S^p_\mu)'$, $(S_\mu)'$ and $(\mathcal{O}_C)'$ are sets of distributions in the sense discussed in Section IV. It is instructive to look at the inductive limit $\lim_{\mu\to\infty}\mathfrak{S}_\mu$ of the filtering family $\{\mathfrak{S}_\mu : \mu \in [0,\infty[\}$ of Fréchet spaces from the perspective of glueing the members of various spaces $\mathfrak{S}_\mu$ in accordance with the general procedure described in [B1, Sect. I.2.5.].

II. The weight functions $(1+|x|^2)^{-\mu/2}$**.** In the definitions of the spaces S^p_μ, S_μ and $\tilde{S}_\mu$ the weight functions $(1+|x|)^{-\mu}$ where $|x|^2 = x_1^2+\cdots+x_n^2$ can be replaced by $(1+|x|^2)^{-\mu/2}$. The asymptotic behaviour as $|x|\to\infty$ of $(1+|x|)^{-\mu}$ and $(1+|x|^2)^{-\mu/2}$ is the same. The advantage of using the latter will be illustrated on the example of the spaces $\tilde{S}_\mu$.

Lemma 1. *Whenever $\mu \in \mathbb{R}$, then*

$$\partial^\alpha(1+|x|^2)^{-\mu/2} = (1+|x|^2)^{-\mu/2-|\alpha|}P_\alpha(x)$$

for every multiindex $\alpha \in \mathbb{N}^n_0$ and every $x \in \mathbb{R}$ where P_α is a polynomial on $\mathbb{R}^n$ of degree no greater than $|\alpha|$.

This lemma appears in [H, Sect. 2.5, Example 8] and can be proved by induction on the length $|\alpha|$ of the multiindex α.

Lemma 2. *Whenever* $\mu \in \mathbb{R}$, *then*
$$\tilde{S}_\mu = \Big\{\phi \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}} |\partial^\alpha[(1+|x|^2)^{-\mu/2}\phi(x)]| < \infty \text{ for every } \alpha \in \mathbb{N}_0^n\Big\}.$$

Lemma 2 implies

Theorem 1. *Whenever* $\lambda, \mu \in \mathbb{R}$, *then* $(1+|x|^2)^{\lambda/2}\phi \in \tilde{S}_{\mu+\lambda}$ *for every* $\phi \in \tilde{S}_\mu$, *and the mapping*
$$\tilde{S}_\mu \ni \phi \mapsto (1+|x|^2)^{\lambda/2}\phi \in \tilde{S}_{\mu+\lambda}$$
is a linear topological isomorphism of $\tilde{S}_\mu$ *onto* $\tilde{S}_{\mu+\lambda}$. *Similar statements are true for the spaces* S^p_μ *and* S_p.

Theorem 1 in the version for the spaces S_μ was proved by J. Horváth [H, Sect. 2.5, Example 8]. Our proof is essentially the same and differs only in the organization of the argument.

Proof of Lemma 2. The topology of $\tilde{S}_\mu$ is determined by the system of seminorms $\{p_\alpha : \alpha \in \mathbb{N}_0^n\}$ where $p_\alpha(\phi) = \sup_{x\in\mathbb{R}^n}(1+|x|^2)^{-\mu/2}|\partial^\alpha\phi(x)|$ for $\phi \in \tilde{S}_\mu$. By the Leibniz formula and Lemma 1, for every $\alpha \in \mathbb{N}_0^n$, $\phi \in \tilde{S}_\mu$ and $x \in \mathbb{R}^n$,
$$\begin{aligned}
&|\partial^\alpha[(1+|x|^2)^{-\mu/2}\phi(x)]| \\
&\qquad \le \sum_{\beta\in\mathbb{N}_0^n,\,\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}|\partial^{\alpha-\beta}(1+|x|^2)^{-\mu/2}|\cdot|\partial^\beta\phi(x)| \\
&\qquad \le \sum_{\beta\in\mathbb{N}_0^n,\,\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}K_\alpha(1+|x|^2)^{-\mu/2}|\partial^\beta\phi(x)| \\
&\qquad \le \sum_{\beta\in\mathbb{N}_0^n,\,\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}K_\alpha p_\beta(\phi)
\end{aligned}$$
where
$$K_\alpha = \sup_{x\in\mathbb{R}^n,\,\beta\in\mathbb{N}_0^n,\,\beta\le\alpha}(1+|x|^2)^{-|\alpha-\beta|}|P_{\alpha-\beta}(x)| < \infty.$$
It follows that whenever $\alpha \in \mathbb{N}_0^n$, then
$$q_\alpha(\phi) = \sup_{x\in\mathbb{R}^n}|\partial^\alpha[(1+|x|^2)^{-\mu/2}\phi(x)]|, \qquad \phi \in \tilde{S}_\mu,$$
is a continuous seminorm on $\tilde{S}_\mu$, and
$$q_\alpha(\phi) \le K_\alpha \sum_{\beta\in\mathbb{N}_0^n,\,\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}p_\beta(\phi) \tag{2.1}$$
for every $\alpha \in \mathbb{N}_0^n$ and $\phi \in \tilde{S}_\mu$.

Furthermore, whenever $\mu \in \mathbb{R}$, $\phi \in \tilde{S}_\mu$ and $\alpha \in \mathbb{N}_0^n$, then

$$(2.2)\quad \begin{aligned} p_\alpha(\phi) &= \sup_{x\in\mathbb{R}^n} \Bigg| \partial^\alpha[(1+|x|^2)^{-\mu/2}\phi(x)] \\ &\qquad - \sum_{\beta\in\mathbb{N}_0^n,\, \alpha\neq\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}\partial^{\alpha-\beta}[(1+|x|^2)^{-\mu/2}\partial^\beta\phi(x)]\Bigg| \\ &\leq q_\alpha(\phi) + K_\alpha \sum_{\beta\in\mathbb{N}_0^n,\, \alpha\neq\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \sup_{x\in\mathbb{R}^n}(1+|x|^2)^{-\mu/2}|\partial^\beta\phi(x)| \\ &= q_\alpha(\phi) + K_\alpha \sum_{\beta\in\mathbb{N}_0^n,\, \alpha\neq\beta\leq\alpha} \frac{\alpha!}{\beta|(\alpha-\beta)!} p_\beta(\phi). \end{aligned}$$

Now we prove, by induction on $|\alpha|$, the following statement $\mathcal{T}(\alpha)$:

there are finite collections $\{C_{\alpha,1},\dots,C_{\alpha,k_\alpha}] \subset]0,\infty[$ *and* $\{\beta_{\alpha,1},\dots,\beta_{\alpha,k_\alpha}\} \subset \mathbb{N}_0^n$ *such that*

$$p_\alpha(\phi) \leq C_{\alpha,1} q_{\beta_{\alpha,1}}(\phi) + \cdots + C_{\alpha,k_\alpha} q_{\beta_{\alpha,k_\alpha}}(\phi)$$

for every $\phi \in \tilde{S}_\mu$.

Indeed, the statement $\mathcal{T}(0)$ is true because $p_0(\phi) \equiv q_0(\phi)$, and if we suppose that $\mathcal{T}(\alpha)$ is true for $|\alpha| \leq k$, then (2.2) implies that $\mathcal{T}(\alpha)$ is true whenever $|\alpha| \leq k+1$.

The inequalities (2.1) and (2.2) show that $\{p_\alpha : \alpha \in \mathbb{N}_0^n\}$ and $\{q_\alpha : \alpha \in \mathbb{N}_0^n\}$ are equivalent systems of seminorms on $\tilde{S}_\mu$. □

III. Proof of the imbeddings (1.1). Let $\mu \in \mathbb{R}$ and $p \in [0,\infty[$ be fixed. For every multiindex $\alpha \in \mathbb{N}_0^n$ and every function $\phi \in C^\infty(\mathbb{R}^n)$ let

$$\tilde{\rho}_{\mu,\alpha}(\phi) = \sup_{x\in\mathbb{R}^n}(1+|x|^2)^{-\mu/2}|\partial^\alpha\phi(x)|,$$

$$\pi^p_{\mu,\alpha}(\phi) = \left(\int_{\mathbb{R}^n}[(1+|x|^2)^{-\mu/2}|\partial^\alpha\phi(x)|]^p\,dx\right)^{1/p}.$$

Then $\{\rho_{\mu,\alpha} : \alpha \in \mathbb{N}_0^n\}$ is a system of seminorms in $\tilde{S}_\mu$ and S_μ defining the topology in both spaces. Moreover S_μ is a closed subspace of $\tilde{S}_\mu$ characterized by the property that $\lim_{|x|\to\infty}(1+|x|^2)^{-\mu/2}\partial^\alpha\phi(x)$ for every $\phi \in S_\mu$ and every $\alpha \in \mathbb{N}_0^n$. The system of seminorms $\{\pi^p_{\mu,\alpha} : \alpha \in \mathbb{N}_0^n\}$ defines the topology in S^p_μ.

Let $\lambda \in]n/p,\infty[$. The imbeddings $S^p_\mu \hookrightarrow \tilde{S}_\mu \hookrightarrow S^p_{\lambda+\mu}$ follow at once from the inequalities

$$\pi^p_{\lambda+\mu,\alpha}(\phi) \leq \left(\int_{\mathbb{R}^n}(1+|x|^2)^{-\lambda p}\,dx\right)^{1/p}\tilde{\rho}_{\mu,\alpha}(\phi)$$

$$\text{for every } \phi \in \tilde{S}_\mu \text{ and } \alpha \in \mathbb{N}_0^n$$

and

$$\tilde{\rho}_{\mu,\alpha}(\phi) \le c_p \pi^p_{\mu,\alpha}(\phi) \quad \text{for every } \phi \in S^p_\mu \text{ and } \alpha \in \mathbb{N}^n_0.$$

The first of these inequalities, in which $\int_{\mathbb{R}^n}(1+|x|^2)^{-\lambda p}\,dx < \infty$ because $\lambda p > n$, is easy to prove. The second, in which $c_p \in\,]0,\infty[$ is a constant independent of μ, α and $\phi \in S^p_\mu$, follows immediately by applying to $u(x) = (1+|x|^2)^{-\mu/2}\partial^\alpha\phi(x)$ the Sobolev type imbedding theorem of [A-F, Theorem 4.18, Part I, Case A].

The imbedding $S^p_\mu \subset \tilde{S}_\mu$ having been proved, the imbeddings $S^p_\mu \hookrightarrow S_\mu \hookrightarrow \tilde{S}_\mu$ follow from the inclusion $S^p_\mu \subset S_\mu$ which is a consequence of

$$\lim_{|x|\to\infty}(1+|x|^2)^{-\mu/2}\partial^\alpha\phi(x) = 0 \quad \text{for every } \phi \in S^p_\mu \text{ and } \alpha \in \mathbb{N}^n_0.$$

To prove this last equality, one applies to $u(x) = (1+|x|^2)^{-\mu/2}\partial^\alpha\phi(x)$ the following proposition.

Proposition 1. *If $u \in W^{m,p}(\mathbb{R}^n)$ where either $p = 1$ and $m = 0$, or $p \in\,]1,\infty[$, $m \in \mathbb{N}$ and $mp > n$, then u is almost everywhere on $\mathbb{R}^n$ equal to a function continuous on $\mathbb{R}^n$, denoted again by u, such that*

$$|u(x)| \le M\|u\|_{W^{m,p}(\mathbb{R}^n)}$$

for some $M \in\,]0,\infty[$ independent of u, and

$$\lim_{|x|\to\infty} u(x) = 0. \tag{3.1}$$

Proof. Apart from (3.1), the proposition is nothing but [A-F, Theorem 4.18, Part I, Case A] in a special case when the domain Ω with the cone property is the whole $\mathbb{R}^n$. The equality (3.1) will be proved by inspecting the proof of the above mentioned theorem from [A-F]. We shall consider the cases $p = 1$ and $p \in\,]1,\infty[$ separately.

Let $p = 1$ and $u \in C(\mathbb{R}^n) \cap W^{n,1}(\mathbb{R}^n)$. Then there is $r_0 \in\,]0,\infty[$ such that $C_{x,r} \subset \{y \in \mathbb{R}^n : |x-y| \le r_0\}$ for all cones $C_{x,r}$ occurring in the inequality (8) of [A-F, Lemma 4.15]. If $r \in [r_0,\infty[$ and $|x| \ge 2r$, then $C_{x,r} \subset \mathbb{R}^n \setminus B_r$ where $B_r = \{x \in \mathbb{R}^n : |x| \le r\}$. Therefore from that inequality, replacing every cone $C_{x,r}$ by $\mathbb{R}^n \setminus B_r$, one infers that there is $M' \in\,]0,\infty[$ such that

$$|u(x)| \le M' \sum_{|\alpha|\le n} \int_{\mathbb{R}^n\setminus B_r} |\partial^\alpha u(x)|\,dx \text{ whenever } r \in [r_0,\infty[\text{ and } |x| \ge 2r.$$

This inequality implies (3.1) by letting $r \to \infty$.

Let now $p \in\,]1,\infty[$, $m \in \mathbb{N}$, $mp > n$, and $u \in C(\mathbb{R}^n) \cap W^{m,p}(\mathbb{R}^n)$. Then there is $r_0 \in\,]0,\infty[$ such that $C_{x,\rho} \subset \{y \in \mathbb{R}^n : |x-y| \le r_0\}$ for all cones $C_{x,\rho}$ occurring in [A-F, Section 4.16] in estimates obtained from (8) there by means of the Hölder inequality. Again if $r \in [r_0,\infty[$ and $|x| \ge 2r$, then $C_{x,\rho} \subset \mathbb{R}^n \setminus B_r$, and one infers from the above mentioned estimates that there is $L \in \,]0,\infty[$ such that whenever $r \in [r_0,\infty[$ and $|x| \le 2r$, then

$$
(3.2)\quad |u(x)| \le K\Big[L \sum_{|\alpha|\le m-1}\Big(\int_{\mathbb{R}^n\setminus B_r} |\partial^\alpha u(x)|^p\,dx\Big)^{1/p} + \Big(\int_{C_{x,\rho}} |x-y|^{(m-n)q}\,dy\Big)^{1/p} \sum_{|\alpha|=m}\Big(\int_{\mathbb{R}^n\setminus B_r} |\partial^\alpha u(x)|^p\,dx\Big)^{1/p}\Big]
$$

where $K \in \,]0,\infty[$ is the constant from [A-F, Lemma 4.15], and $q = \frac{p}{p-1}$. Since $\int_{C_{0,\rho}} |x-y|^{(m-n)q}\,dy = \int_{C_{0,5\rho}} |y|^{(m-n)q}\,dy$ and $(m-n)q = (m-n)\frac{p}{p-1} > n(1-p)\frac{p}{p-1} = -n$, the integral $\int_{C_{x,\rho}} |x-y|^{(m-n)q}\,dy$ is independent of x and finite. Therefore it follows from (3.2) that there is $M'' \in \,]0,\infty[$ such that

$$
|u(x)| \le M'' \sum_{|\alpha|\le m}\Big(\int_{\mathbb{R}^n\setminus B_r} |\partial^\alpha u(x)|^p\,dz\Big)^{1/p} \qquad \text{whenever } r \in [r_0,\infty[\text{ and } |x| \ge 2r.
$$

Again, this inequality implies (3.1) by letting $r \to \infty$. □

IV. The strong and the weak initial topologies in $(\mathcal{O}_C)'$. Denote by $(\mathcal{O}_C)'$ the set of all continuous linear functionals on $\mathcal{O}_C$, the latter being equipped with the topology of the inductive limit $\lim_{\mu\to\infty} \mathcal{Z}_\mu$ of the Fréchet spaces $\mathcal{Z}_\mu = S^p_\mu$, S_μ or $\tilde{S}_\mu$. For every $\mu \in [0,\infty[$ denote by $\mathcal{Z}'_\mu$ the set of all continuous linear functionals on the Fréchet space $\mathcal{Z}_\mu$. Let $(\mathcal{Z}_\mu)'_b$ and $(\mathcal{Z}_\mu)'_w$ denote respectively the strong and the $*$-weak dual space of $\mathcal{Z}_\mu$.

Since $\mathcal{S}$ is dense in $\mathcal{O}_C$, it follows that *$(\mathcal{O}_C)'$ is a set of slowly increasing distributions on $\mathbb{R}^n$*. The exact meaning of the above phrase is as follows: every $f \in (\mathcal{O}_C)'$ is a continuous linear functional on the locally convex space $\mathcal{O}_C$ containing $\mathcal{S}$ as a dense subset, so that f is uniquely determined by $f|_{\mathcal{S}}$ which belongs to $\mathcal{S}'$. See [S, Sect. VI.8, pp. 199–200].

We define the strong initial topology τ_b in $(\mathcal{O}_C)'$ as the initial topology in $(\mathcal{O}_C)'$ (see [B1, Sect. I.2.3]) determined by the family of strong dual spaces $(\mathcal{Z}_\mu)'_b$, $\mu \in [0,\infty[$, and the family of projections $\mathrm{pr}_\mu : (\mathcal{O}_C)' \ni f \mapsto f|_{\mathcal{Z}_\mu} \in (\mathcal{Z}_\mu)'$, $\mu \in [0,\infty[$. Every pr_μ is dual to the continuous imbedding $\mathcal{Z}_\mu \hookrightarrow \mathcal{O}_C$. The locally convex space $((\mathcal{O}_C)',\tau_b)$ is defined by declaring that if for every $\mu \in [0,\infty[$ a family $\mathcal{V}_\mu$ of convex balanced subsets of $(\mathcal{Z}_\mu)'_b$ is a basis of neighbourhoods of zero in $(\mathcal{Z}_\mu)'_b$, then

$$(4.1)\quad \Big\{ \bigcap_{\mu\in M} \mathrm{pr}_\mu^{-1}(\mathcal{V}_\mu) : \quad M \text{ a finite subset of } [0,\infty[,\ \nu_\mu \in \mathcal{V}_\mu \text{ for every } \mu \in M \Big\}$$

is a basis of neighbourhoods of zero in $((\mathcal{O}_C)', \tau_b)$.

Two properties of τ_b will occur in the forthcoming arguments:

(4.2) If $t : E \to ((\mathcal{O}_C)', \tau_b)$ is a linear mapping of a locally convex space E into $((\mathcal{O}_C)', \tau_b)$, then t is continuous if and only if for every $\mu \in [0,\infty[$ the mapping $\mathrm{pr}_\mu \circ t$ is continuous (see [B1, Sect. I.2.3, remarks after Proposition 4], [R-R, Sect. V.4, Proposition 12], [Sf, Sect. II.5, Theorem 5]).

(4.3) The extremal property: τ_b is the coarsest among the topologies τ in $(\mathcal{O}_C)'$ such that whenever $\mu \in [0,\infty[$, then the mapping $\mathrm{pr}_\mu : ((\mathcal{O}_C)', \tau) \to (\mathfrak{Z}_\mu)'_b$ is continuous (see [B1, Sect. I.2.3, Proposition 4]).

The definition and properties of the $*$-weak initial topology τ_w in $(\mathcal{O}_C)'$ are similar.

The initial topologies τ_b and τ_w as $\mathfrak{S}$-topologies. The initial topologies τ_b and τ_w in $(\mathcal{O}_C)'$ appear to be $\mathfrak{S}$-*topologies*, i.e. topologies of uniform convergence on members of some coverings of $\mathcal{O}_C$ by its bounded subsets.

In order to exhibit the covering of $\mathcal{O}_C$ corresponding to the $\mathfrak{S}$-topology τ_b in $(\mathcal{O}_C)'$, for every $\mu \in [0,\infty[$ denote by $\mathcal{B}_\mu$ the family of all bounded subsets of $\mathfrak{Z}_\mu$. Denote by $\circ_\mu$ the forward polar in the sense of the duality $\langle \mathfrak{Z}_\mu, (\mathfrak{Z}_\mu)' \rangle$, and let $\circ$ stand for the forward polar in the sense of the duality $\langle \mathcal{O}_C, (\mathcal{O}_C)' \rangle$. Consider the continuous imbedding $t : \mathfrak{Z}_\mu \hookrightarrow \mathcal{O}_C$, and let $t' : (\mathcal{O}_C)' \to (\mathfrak{Z}_\mu)'_b$ be its adjoint mapping. Then, by [R-R, Sect. II.6, Lemma 6],

$$(4.4)\quad \mathrm{pr}_\mu^{-1}(B^{\circ_\mu}) = (t')^{-1}(B^{\circ_\mu}) = (t(B))^\circ \approx B^\circ \qquad \text{for every } B \subset \mathfrak{Z}_\mu.$$

Henceforth we follow the proof of [R-R, Sect. V.4, Proposition 15]. Since

$$\mathcal{V}_\mu = \{ B^{\circ_\mu} : B \in \mathcal{B}_\mu \}$$

is a basis of neighbourhoods of zero in $(\mathfrak{Z}_\mu)'_b$, it follows that

$$\Big\{ \bigcap_{\mu\in M} \mathrm{pr}_\mu^{-1}(B^{\circ_\mu}) : \quad M \text{ a finite subset of } [0,\infty[,\ B_\mu \in \mathcal{B}_\mu \text{ for every } \mu \in M \Big\}$$

is a basis of neighbourhoods of zero in the locally convex space $((\mathcal{O}_C)', \tau_b)$. By (4.4) this basis can be rewritten in the form

$$\Big\{\Big(\bigcap_{\mu\in M} B_\mu\Big)^\circ : M \text{ a finite subset of } [0,\infty[, B_\mu \in \mathcal{B}_\mu \text{ for every } \mu \in M\Big\}.$$

Since from a basis of neighbourhoods of zero we can remove every set larger than some other set belonging to that basis, we conclude that

$$\{C^\circ : C \in U_b\} \quad \text{where} \quad U_b = \bigcup_{\mu\in[0,\infty[} \mathcal{B}_\mu$$

is a basis of neighbourhoods of zero in $((\mathcal{O}_C)', \tau_b)$. This means that τ_b is a $\mathfrak{S}$-topology in $(\mathcal{O}_C)'$ determined by the covering U_b of $\mathcal{O}_C$. All the sets belonging to U_b are bounded subsets of $\mathcal{O}_C$ because any of them belongs to a certain $\mathcal{B}_\mu$, so that it is a bounded subset of $\mathfrak{Z}_\mu$, and since $\mathfrak{Z}_\mu \hookrightarrow \mathcal{O}_C$, it is also a bounded subset of $\mathcal{O}_C$. Therefore U_b is a covering of $\mathcal{O}_C$ by bounded subsets of $\mathcal{O}_C$. Summing up, we get the following

Theorem 2 (A. P. Robertson and W. Robertson). *The topology τ_b in the set $(\mathcal{O}_C)'$ of continuous linear functionals on $\mathcal{O}_C$ is equal to the $\mathfrak{S}$-topology corresponding to the covering $U_b = \bigcup_{\mu\in[0,\infty[} \mathcal{B}_\mu$ of $\mathcal{O}_C$.*

Similarly, τ_w is a $\mathfrak{S}$-topology in $(\mathcal{O}_C)'$ determined by the covering U_w of $\mathcal{O}_C$, where $U_w = \bigcup_{\mu\in[0,\infty[} F_\mu$, F_μ being the family of all finite subsets of $\mathfrak{Z}_\mu$.

Equivalence of τ_w and the $*$-weak topology in $(\mathcal{O}_C)'$. By [Sf, Sect. IV.4, Theorem 4.5] *the $*$-weak topology in $(\mathcal{O}_C)'$ is equivalent to the initial topology τ_w.*

This equivalence can be deduced from the extremal property of τ_w and from [R-R, Sect. V.4, Proposition 15]. Indeed, whenever $\mu \in [0,\infty[$, then $\mathfrak{Z}_\mu \hookrightarrow \mathcal{O}_C$, so that the mapping $\mathrm{pr}_\mu : (\mathcal{O}_C)' \ni \phi \mapsto \phi|_{\mathfrak{Z}_\mu} \in (\mathfrak{Z}_\mu)'$ is continuous. Comparing this with the extremal property of τ_w we infer that in $(\mathcal{O}_C)'$ the $*$-weak topology is finer than τ_w. On the other hand, by [R-R, Sect. V.4, Proposition 15], a subset A of $\mathcal{O}_C$ belongs to U_w if and only if $A\cap\mathfrak{Z}_\mu$ is finite for every $\mu \in [0,\infty[$. Thus the covering of $\mathcal{O}_C$ by its finite subsets is finer than U_w, whence, of the two $\mathfrak{S}$-topologies, the $*$-weak topology in $(\mathcal{O}_C)'$ is coarser than τ_w.

V. The set RD of rapidly decreasing distributions on $\mathbb{R}^n$ and the locally convex spaces $(RD, \tilde{b})$ and $(RD, \tilde{w})$. Whenever $\mu \in [0,\infty[$, then $\mathcal{S}$ is dense in the Fréchet space S^1_μ, and so any distribution $T \in \mathcal{S}'$ has at most one extension to a continuous linear functional on S^1_μ. If such a unique extension exists, it will be denoted by T_μ.

The set RD of rapidly decreasing distributions on $\mathbb{R}^n$ is defined as follows:

$$RD := \{T \in \mathcal{S}' : \text{for every } \mu \in [0,\infty[\text{ the distribution } T \text{ extends uniquely to } T_\mu \in (S^1_\mu)'\}.$$

The locally convex topology $\tilde{b}$ (resp. $\tilde{w}$) is induced in RD from the topological product $\prod_{\mu\in[0,\infty[}(S^1_\mu)'_b$ (resp. $\prod_{\mu\in[0,\infty[}(S^1_\mu)'_w$) via the mapping

$$RD \ni T \mapsto (T_\mu)_{\mu\in[0,\infty[} \in \prod_{\mu\in[0,\infty[}(S^1_\mu)'_b \left(\text{resp. } \prod_{\mu\in[0,\infty[}(S^1_\mu)'_w\right)$$

(see [B1, Sect. I.2.3, Example III], [R-R, Sect. V.5]). It follows that

(5.1) a net $(T_\iota)_{\iota\in J} \subset RD$ is convergent in the topology $\tilde{b}$ (resp. $\tilde{w}$) if and only if for every $\mu \in [0,\infty[$ the net of extensions $((T_\iota)_\mu)_{\iota\in J}$ is convergent in the topology of $(S^1_\mu)'_b$ (resp. $(S^1_\mu)'_w$).

Recall that J. Horváth defined the rapidly decreasing distributions as members of the set

$$H = \{T \in \mathcal{S}' : T \text{ has a unique extension to a continuous linear functional } \tilde{T} \text{ on } \mathcal{O}_C\},$$

the continuity being understood in the sense of the inductive topology in $\mathcal{O}_C = \lim_{\mu\to\infty} S^1_\mu$. J. Horváth did not discuss any topology in H.

Theorem 3. $H = RD$.

Proof of $H \subset RD$. We have to prove that if a distribution T belongs to H, then for every $\mu \in [0,\infty[$ the distribution T extends uniquely to a continuous linear functional T_μ on S^1_μ. To this end define $T_\mu := \tilde{T}|_{S^1_\mu}$. Then $T_\mu \in (S^1_\mu)'$ because $S^1_\mu \hookrightarrow \mathcal{O}_C$, and $T_\mu|_{\mathcal{S}} = T$ because $T_\mu|_{\mathcal{S}} = (\tilde{T}|_{S^1_\mu})|_{\mathcal{S}} = \tilde{T}|_{\mathcal{S}} = T$. Moreover, since $\mathcal{S}$ is dense in S^1_μ, the extension T_μ is unique. □

Proof of $RD \subset H$. Let $T \in RD$. Keep T fixed throughout the present proof. Let $(T_\mu)_{\mu\in[0,\infty[} \in \times_{\mu\in[0,\infty[}(S^1_\mu)'$ be the system of extensions of T occurring in the definition of RD. Then

(5.2) whenever $0 \le \mu < \nu < \infty$, then T_μ is a restriction of T_ν.

Indeed, if $0 \le \mu < \nu < \infty$, then $S^1_\mu \hookrightarrow S^1_\nu$, so that $T_\nu|_{S^1_\mu} \in (S^1_\mu)'$. Furthermore, since $\mathcal{S}$ is dense in S^1_μ, it follows that $T_\nu|_{S^1_\mu}$ is the unique extension of T to a continuous linear functional on S^1_μ, so that $T_\nu|_{S^1_\mu} = T_\mu$.

We are going to construct the extension of the distribution $T \in RD$ to a continuous linear functional $\tilde{T}$ on $\mathcal{O}_C$. To this end we shall use the following facts:

1° as a set, $\mathcal{O}_C$ is equal to $\bigcup_{\mu\in[0,\infty[} S^1_\mu$,
2° when $\mu \in [0,\infty[$ increases, the spaces S^1_μ increase in the sense of inclusion,
3° $\mathcal{O}_C$ is equipped with the inductive topology determined by the Fréchet spaces S^1_μ.

From 1° and (5.2) it follows that there is a unique function $\tilde{T}$ on $\mathcal{O}_C$ such that

$$\tilde{T}(\phi) = T_\mu(\phi) \quad \text{for every } \mu \in [0,\infty[\text{ and } \phi \in S^1_\mu. \tag{5.3}$$

From 1°, 2° and (5.3) it follows that for every $\mu \in [0,\infty[$ the restriction of $\tilde{T}$ to S^1_μ is equal to a continuous linear functional T_μ on S^1_μ, so that T is an algebraic linear functional on $\mathcal{O}_C$. From 3° and (5.3) it follows that $\tilde{T}$ is a continuous linear functional on $\mathcal{O}_C$ with respect to the inductive topology of $\mathcal{O}_C$.

Again by (5.3), whenever $\mu \in [0,\infty[$, then $\tilde{T}$ is an extension of T_μ. Since T_μ is an extension of T, it follows that $\tilde{T}$ is an extension of T. Finally, since $\mathcal{S}$ is dense in $\mathcal{O}_C$, it follows that $\tilde{T}$ is the unique extension of T to a continuous linear functional on $\mathcal{O}_C$, so $T \in H$. □

Remark. For every $\mu \in [0,\infty)$ denote by $\mathbf{C}_\mu$ the subset of $C(\mathbb{R}^n)$ consisting of functions g such that $\sup_{x\in\mathbb{R}^n} |x|^\mu |g(x)| < \infty$. For any $\mu \in [0,\infty[$, the set $(S^1_\mu)'$ consists of all distributions of the form $\sum_{|\alpha|\le m_\nu} \partial^\alpha g_{\alpha,\mu}$ where $m_\mu \in \mathbb{N}_0$, $g_{\alpha,\mu} \in \mathbf{C}_\mu$ for $|\alpha| \le m_\nu$, and differentiation is understood in the sense of distributions (see [G-L, Sect. 5, Exercise after Theorem 5.4], [K-R, Sect. 3, Theorem 3.4]). It follows that

$$\begin{aligned} \bigcap_{\mu\in[0,\infty[} (S^1_\mu)' &= \bigcap_{\nu\in[0,\infty[} \bigcup_{\mu\in[\nu,\infty[} (S^1_\mu)' \\ &= \Big\{ T \in S' : \underset{\nu\in[0,\infty[}{\forall} \ \underset{\mu\in[\nu,\infty[}{\exists} \Big(T = \sum_{|\alpha|\le m_\mu} \partial^\alpha g_{\alpha,\mu} \Big) \Big\}. \end{aligned} \tag{5.4}$$

The complete proofs of the above assertions are omitted. The equality (5.4) yields a new proof of [S, Sect. VII.5, Theorem IX.10].

VI. Derivation of the locally convex spaces $(RD, \tilde{b})$ and $(RD, \tilde{w})$ from the locally convex spaces $((\mathcal{O}_C)', \tau_b)$ and $((\mathcal{O}_C)', \tau_w)$. Recall that $RD = H$ and if $T \in H$, then $\tilde{T}$ denotes the unique extension of T to a continuous linear functional on $\mathcal{O}_C$.

Lemma 3. *$\varepsilon : H \ni T \mapsto \tilde{T} \in (\mathcal{O}_C)'$ is a one-to-one mapping of H onto $(\mathcal{O}_C)'$.*

Proof. Since $\mathcal{S}$ is dense in $\mathcal{O}_C$, for every $T \in H$ there is exactly one $\tilde{T} \in (\mathcal{O}_C)'$ extending T. Since $\mathcal{S} \hookrightarrow \mathcal{O}_C$, it follows that if $\tilde{T} \in (\mathcal{O}_C)'$, then $\tilde{T}|_{\mathcal{S}} \in \mathcal{S}'$, so that $\tilde{T}|_{\mathcal{S}} \in H$. □

Theorem 4. *There are isomorphisms of locally convex spaces* $(H, \varepsilon^{-1}\tau_b) \approx ((\mathcal{O}_C)', \tau_b) \approx (RD, \tilde{b})$ *and* $(H, \varepsilon^{-1}\tau_w) \approx ((\mathcal{O}_C)', \tau_w) = (\mathcal{O}_C)'_w \approx (RD, \tilde{w})$. *Here* $\varepsilon^{-1}\tau_b$ *and* $\varepsilon^{-1}\tau_w$ *denote the inverse images of the topologies* τ_b *and* τ_w *under the mapping* ε.

Proof. The proofs of the two sequences of isomorphisms being similar, we shall limit ourselves to the first. By Lemma 3 the one-to-one linear mapping ε of H onto $(\mathcal{O}_C)'$ yields a linear homeomorphism of the locally convex space $(H, \varepsilon^{-1}\tau_b)$ onto the locally convex space $((\mathcal{O}_C)', \tau_b)$. The phrase "linear homeomorphism of locally convex spaces" used by H. Jarchow [J] means "isomorphism of locally convex spaces" in common terminology.

It remains to prove that $((\mathcal{O}_C)', \tau_b)$ is isomorphic to $(RD, \tilde{b})$. To that end, consider a net $(\tilde{T}_\iota)_{\iota \in J} \subset (\mathcal{O}_C)'$ and the associated net $(T_\iota)_{\iota \in J} = (\varepsilon^{-1}\tilde{T}_\iota)_{\iota \in J} \subset H = RD$. Whenever $\mu \in [0, \infty[$, then

$$((T_\iota)_\mu)_{\iota \in J} = (\tilde{T}_\iota|_{S^1_\mu})_{\iota \in J} \tag{6.1}$$

because, by the imbedding $S^1_\mu \hookrightarrow \mathcal{O}_C$, $\tilde{T}_\iota|_{S^1_\mu}$ is an extension of T_ι to a continuous linear functional on S^1_μ, and the extension is unique, so that it must be equal to $(T_\iota)_\mu$. By (5.1) and (6.1), the net $(\tilde{T}_\iota)_{\iota \in J} \subset (\mathcal{O}_C)'$ converges in the topology τ_b if and only if $(T_\iota)_{\iota \in J} \subset RD$ converges in the topology $\tilde{b}$. This shows that the mapping $\varepsilon^{-1} : (\mathcal{O}_C)' \ni \tilde{T} \mapsto T \in H = RD$ is a linear homeomorphism of the locally convex space $((\mathcal{O}_C)', \tau_b)$ onto the locally convex space $(RD, \tilde{b})$. □

VII. Relation of the locally convex space $(RD, \tilde{b})$ and the set H to the rapidly decreasing distributions on $\mathbb{R}^n$ in the sense of L. Schwartz and in the sense of J. Horváth. L. Schwartz [S, Sect. VII.5], without using $\mathcal{O}_C$, defined the limit space $\mathcal{O}'_C$ of rapidly decreasing distributions on $\mathbb{R}^n$ by two conditions:

(a) as a set, $\mathcal{O}'_C$ is equal to $\{T \in \mathcal{S}' : (1+|\cdot|^2)^{\mu/2}T \in (\mathcal{D}_{L^1})'$ for every $\mu \in [0, \infty[\}$,
(b) a net $(T_\iota)_{\iota \in J} \subset \mathcal{O}'_C$ converges if and only if for every $\mu \in [0, \infty[$ the net $((1+|\cdot|^2)^{\mu/2}T_\iota)_{\iota \in J}$ converges in the topology of $(\mathcal{D}_{L^1})'_b$.

For a general explanation of the notion of limit space see [F] and [J, Chapter 9].

Theorem 5. *As a set, the limit space* $\mathcal{O}'_C$ *of L. Schwartz is equal to RD. A net* $(T_\iota)_{\iota \in J} \subset RD$ *converges in the sense of L. Schwartz if and only if it converges in the topology* $\tilde{b}$.

Proof. By Theorem 1, whenever $\mu \in [0, \infty[$, the mapping $S^1_0 \ni \phi \mapsto (1+|x|^2)^{\mu/2}\phi \in S^1_\mu$ is a linear homeomorphism of the Fréchet space

S_0^1 onto the Fréchet space S_μ^1. It follows that a distribution $T \in \mathcal{S}'$ satisfies condition (a) of L. Schwartz if and only if, for each $\mu \in [0,\infty[$, it extends uniquely to $T_\mu \in (S_\mu^1)'$. Thus condition (a) is satisfied if and only if $T \in RD$. Furthermore, a net $(T_\iota)_{\iota\in J} \subset RD$ converges in the sense of L. Schwartz, i.e. for every $\mu \in [0,\infty[$ the net $((1+|x|^2)^{\mu/2}T_\iota)_{\iota\in J}$ converges in the topology of $(\mathcal{D}_{L^1})'_b = (S_0^1)'_b$, if and only if for every $\mu \in [0,\infty[$ the net of extensions $((T_\iota)_\mu)_{\iota\in J} \subset (S_\mu^1)'$ converges in the topology of $(S_\mu^1)'_b$. By (5.1), the latter holds if and only if the net $(T_\iota)_{\iota\in J} \subset RD$ is convergent in the topology $\tilde{b}$. □

J. Horváth [H, Sect. 4.11, p. 420] defined the rapidly decreasing distributions on $\mathbb{R}^n$ as members of the set H, without explicitly discussing the topology. But $H = RD$ by Theorem 3, and $(RD, \tilde{b}) \approx (H, \varepsilon^{-1}\tau_b)$ by Theorem 4.

2. The strong convolutional topology in RD

VIII. Characterization of rapidly decreasing distributions by their convolutions with functions belonging to $\mathcal{S}$.

Theorem 6 (R. E. Edwards [E])**.** *For every slowly increasing distribution T on $\mathbb{R}^n$ the three conditions are equivalent:*

1º $T \in RD$,

2º *whenever* $\varphi \in \mathcal{S}$, *then* $T * \varphi \in \mathcal{S}$,

3º $[T\,*]|_{\mathcal{S}} \in L(\mathcal{S},\mathcal{S})$.

Theorem 6 is quoted in [G-L, Sect. 7•.2] as Theorem 7•.2.2 without any reference. R. E. Edwards' original proof bases on Fourier transformation, and in particular on L. Schwartz's Theorem XV from [S, Sect. VII.8] whose proof is incomplete (see L. Schwartz's own remarks in [S, pp. 269–270]). We shall obtain the equivalence 1º⇔3º as an immediate consequence of Proposition 2 below, the Fourier-theoretical proof of 3º⇔2º being postponed to Section X.

Following [Kh, Vol. 2, Sect. CC.III.30], by a *periodic partition of unity* on $\mathbb{R}^n$ we mean a partition of unity $\{\varphi(\cdot + z) : z \in \mathbb{Z}^n\} = \{\varphi_z : z \in \mathbb{Z}^n\}$ consisting of translates of a non-negative function φ in $C_c^\infty(\mathbb{R}^n)$.

Proposition 2. *For every set $\{T_\iota : \iota \in J\} \subset \mathcal{S}'$ the following conditions are equivalent:*

(a) *every T_ι has a unique extension to a continuous linear functional $\tilde{T}_\iota$ on $\mathcal{O}_C$ and $\{\tilde{T}_\iota : \iota \in J\}$ is an equicontinuous set of linear functionals on $\mathcal{O}_C$.*

(b) *$\{[T_\iota\,*]|_{\mathcal{S}} : \iota \in J\}$ is an equicontinuous set of operators belonging to $L(\mathcal{S},\mathcal{S})$,*

(c) *whenever* $\{\varphi_z : z \in \mathbb{Z}^n\}$ *is a periodic partition of unity on* $\mathbb{R}^n$*, then*

$$\sum_{z\in\mathbb{Z}^n} \sup_{\iota\in J} |T_\iota(\varphi_z\phi)| < \infty \qquad \textit{for every } \phi \in \mathcal{O}_C.$$

For the application in the proof of Theorem 6 it is not necessary to consider in Proposition 2 sets of distributions and operators, but just a single distribution and operator. However, in subsequent sections we shall use (a) and (b) in their version for sets.

We shall prove (a)⇒(b)⇒(c)⇒(a). To this end, in the domain of C^∞-functions ϕ on $\mathbb{R}^n$, we shall use the seminorms

$$\rho_{\mu,\alpha}(\phi) = \sup_{x\in\mathbb{R}^n} (1 + |x|)^{-\mu} |\partial^\alpha \phi(x)|.$$

For instance, the family of seminorms $\{\rho_{-\mu,\alpha} : \mu \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n\}$ determines the topology of $\mathcal{S}$, and for every fixed $\mu \in \mathbb{R}$ the family of seminorms $\{\rho_{\mu,\alpha} : \alpha \in \mathbb{N}_0^n\}$ determines the topology of $\tilde{S}_\mu$. The seminorms $\rho_{\mu,\alpha}$ are invariant with respect to reflection at zero, i.e. $\rho_{\mu,\alpha}(\phi^\vee) = \rho_{\mu,\alpha}(\phi)$.

Proof of (a)⇒(b).[1] Whenever $\mu \in [0, \infty[$, then $\tilde{S}_\mu \hookrightarrow \mathcal{O}_C$. Therefore (a) implies that

(a)′ whenever $\mu \in [0, \infty[$, then $\{\tilde{T}_\iota|_{\tilde{S}_\mu} : \iota \in J\}$ is an equicontinuous set of continuous linear functionals on the Fréchet space $\tilde{S}_\mu$.

From (a)′ it follows that for every $\mu \in [0, \infty[$ there are $C_\mu \in]0, \infty[$ and $\nu_\mu \in \mathbb{N}_0$ such that

$$\sup_{\iota\in J} |\tilde{T}_\iota(\phi)| \le C_\mu \sup_{|\alpha|\le\nu_\mu} \rho_{\mu,\alpha}(\phi) \qquad \text{for every } \phi \in \tilde{S}_\mu.$$

Consequently, whenever $x \in \mathbb{R}^n$, $\varphi \in \mathcal{S}$ and $\mu \in [0, \infty[$ then

$$\begin{aligned}
\sup_{\iota\in J} |(\tilde{T}_\iota * \varphi)(x)| &= \sup_{\iota\in J} |(\tilde{T}_\iota)_{(y)}(\varphi(x-y))| = \sup_{\iota\in J} |(T_\iota)_{(y)}(\varphi(x-y))| \\
&= \sup_{\iota\in J} |(\tilde{T}_\iota|_{\tilde{S}_\mu})_{(y)}(\varphi(x-y))| \\
&\le C_\mu \sup_{|\alpha|\le\nu_\mu} \sup_{y\in\mathbb{R}^n} (1+|y|)^{-\mu} |(\partial^\alpha\varphi)(x-y)| \\
&\le C_\mu \sup_{|\alpha|\le\nu_\mu} \sup_{y\in\mathbb{R}^n} (1+|y|)^{-\mu} [\rho_{-\mu,\alpha}(\varphi)(1+|x-y|)^{-\mu}] \\
&\le C_\mu \sup_{|\alpha|\le\nu_\mu} \rho_{-\mu,\alpha}(\varphi)(1+|x|)^{-\mu}
\end{aligned}$$

where the last inequality follows from $1+|x| \le (1+|y|)(1+|x-y|)$. The above estimate implies that for every $\mu \in [0, \infty[$ there are $C_\mu \in]0, \infty[$

[1] We give a slightly polished version of the proof of Proposition 2 published earlier in a preprint of Institute of Mathematics, Polish Academy of Sciences.

and $\nu_\mu \in \mathbb{N}_0$ such that

$$\sup_{\iota\in J} \rho_{-\mu,0}(T_\iota * \varphi) \le C_\mu \sup_{|\alpha|\le\nu_\mu} \rho_{-\mu,\alpha}(\varphi) \quad \text{for every } \varphi \in \mathcal{S}.$$

Applying this to $\partial^\beta\varphi$ in place of φ, we infer that

$$\sup_{\iota\in J} \rho_{-\mu,\beta}(T_\iota * \varphi) \le C_\mu \sup_{|\alpha|\le\nu_\mu} \rho_{-\mu,\alpha+\beta}(\varphi) \quad \text{for every } \varphi \in \mathcal{S} \text{ and } \beta \in \mathbb{N}_0^n.$$

It is obvious that $\{\sup_{|\beta|\le\lambda} \rho_{-\mu,\beta} : \mu \in [0,\infty[, \lambda \in \mathbb{N}_0\}$ is a filtering (see [B2, Sect. II.5.4, Remark after Proposition 4]) system of seminorms determining the topology of $\mathcal{S}$. Therefore for neighbourhoods of zero in $\mathcal{S}$ the following holds: for every $\mu \in [0,\infty[$ and $\lambda \in \mathbb{N}_0$ there are $C_\mu \in]0,\infty[$ and $\nu \in \mathbb{N}_0$ such that whenever $\varphi \in \mathcal{S}$ and $\varepsilon \in]0,\infty[$, then

$$C_\mu \sup_{|\alpha+\beta|\le\nu+\lambda} \rho_{-\mu,\alpha+\beta}(\varphi) \le \varepsilon \;\Rightarrow\; \sup_{\iota\in J} \sup_{|\beta|\le\lambda} \rho_{-\mu,\beta}(T_\iota * \varphi) \le \varepsilon$$

(see [K-A, Sect. III.2.1, Theorem 1]). The last implication means that $\{[T_\iota *]|_{\mathcal{S}} : \iota \in J\}$ is an equicontinuous subset of $L(\mathcal{S},\mathcal{S})$. □

Proof of (b)⇒(c). Let $\{\varphi_z : z \in \mathbb{Z}^n\}$ be a periodic partition of unity on $\mathbb{R}^n$, and let $(T_\iota)_{\iota\in J} \subset \mathcal{S}'$. Since $[T_\iota * \psi](x) = T_\iota((\psi_x)^\vee)$ and $(((\psi_x)^\vee)_x)^\vee = \psi$, it follows that $[T_\iota * (\psi_x)^\vee](x) = T_\iota((((\psi_x)^\vee)_x)^\vee) = T_\iota(\psi)$. If $\phi \in C^\infty(\mathbb{R}^n)$, then taking $\psi = \phi\varphi_{-z}$ and $x = z$, one obtains $T_\iota(\phi\varphi_{-z}) = [T_\iota * (\phi_z\varphi)^\vee](z)$. Hence

$$\begin{aligned}(8.1)\qquad |T_\iota(\phi\varphi_{-z})| &\le |[T_\iota * (\phi_z\varphi)^\vee](z)| \\ &\le \Big(\sup_{x\in\mathbb{R}^n} (1+|x|)^\kappa |[T_\iota * (\phi_z\varphi)^\vee](x)|\Big)\cdot(1+|z|)^{-\kappa} \\ &= \rho_{-\kappa,0}(T_\iota * (\phi_z\varphi)^\vee)\cdot(1+|z|)^{-\kappa}\end{aligned}$$

for all $\phi \in C^\infty(\mathbb{R}^n)$, $\kappa \in [0,\infty[$ and $z \in \mathbb{Z}^n$.

Assume now that (b) holds and let $\phi \in C^\infty(\mathbb{R}^n)$. Then, for every $\kappa \in [0,\infty[$,

$$p_\kappa(\psi) = \sup_{\iota\in J} \rho_{-\kappa,0}(T_\iota * \psi^\vee), \qquad \psi \in \mathcal{S},$$

is a continuous seminorm on $\mathcal{S}$. Therefore there are $C_\kappa \in]0,\infty[$, $\lambda_\kappa \in [0,\infty[$ and $\nu_\kappa \in \mathbb{N}_0$ such that

$$(8.2)\qquad p_\kappa(\psi) \le C_\kappa \sup_{|\alpha|\le\nu_\kappa} \rho_{-\lambda_\kappa,\alpha}(\psi) \quad \text{for every } \psi \in \mathcal{S}.$$

From (8.1) and (8.2) it follows that

$$\begin{aligned}
&\sup_{\iota\in J}|T_\iota(\phi\varphi_{-z})|\\
&\qquad\le \sup_{\iota\in J}\rho_{-\kappa,0}(T_\iota*(\phi_z\varphi)^\vee)\cdot(1+|z|)^{-\kappa}=p_\kappa(\phi_z\varphi)\cdot(1+|z|)^{-\kappa}\\
&\qquad\le C_\kappa \sup_{|\alpha|\le\nu_\kappa}\rho_{-\lambda_\kappa,\alpha}(\phi_z\varphi)\cdot(1+|z|)^{-\kappa}\\
&\qquad= C_\kappa \sup_{x\in\mathbb{R}^n,\,|\alpha|\le\nu_\kappa}(1+|x|)^{\lambda_\kappa}|\partial^\alpha[\phi(x+z)\varphi(x)]|\cdot(1+|z|)^{-\kappa}\\
&\qquad\le C_\kappa(1+r)^{\lambda_\kappa} \sup_{x\in\mathbb{R}^n,\,|\alpha|\le\nu_\kappa}|\partial^\alpha[\phi(x+z)\varphi(x)]|\cdot(1+|z|)^{-\kappa}
\end{aligned}$$

where

$$r=\sup\{|x|:x\in\operatorname{supp}\varphi\}.$$

From these estimates, by the Leibniz formula, it follows that if condition (b) of Proposition 2 is satisfied, then for every $\phi\in C^\infty(\mathbb{R}^n)$, $z\in\mathbb{Z}^n$ and $\kappa\in[0,\infty[$ one has

$$\sup_{\iota\in J}|T_\iota(\phi\varphi_{-z})|\le D_\kappa \sup_{x+z\in\operatorname{supp}\varphi,\,|\alpha|\le\nu_\kappa}|\partial^\alpha\phi(x+z)|\cdot(1+|z|)^{-\kappa} \tag{8.3}$$

where $D_\kappa=LC_\kappa(1+r)^{\lambda_\kappa}\sup_{x\in\operatorname{supp}\varphi,\,|\alpha|\le\nu_\kappa}|\partial^\alpha\varphi(x)|$, L being the maximum of the coefficients in the Leibniz formula. The only important thing is that D_κ is a finite non-negative constant depending only on κ.

Till now we have assumed that (b) holds and $\phi\in C^\infty(\mathbb{R}^n)$. Henceforth we shall assume that (b) holds and $\phi\in\mathcal{O}_C$. Since, as a set, $\mathcal{O}_C=\bigcup_{\mu\in[0,\infty[}\tilde S_\mu$, it follows that to every $\phi\in\mathcal{O}_C$ we can assign $\mu=\mu(\phi)\in[0,\infty[$ such that $\phi\in\tilde S_\mu$. Then, for every $z\in\mathbb{Z}^n$,

$$\begin{aligned}
\sup_{x+z\,\in\operatorname{supp}\varphi,\,|\alpha|\le\nu_\kappa}|\partial^\alpha\phi(x+z)|&\le \sup_{|x|\le r,\,|\alpha|\le\nu_\kappa}|\partial^\alpha\phi(x+z)|\\
&\le \sup_{|\alpha|\le\nu_\kappa}\rho_{\mu,\alpha}(\phi)(1+r+|z|)^\mu\\
&\le \sup_{|\alpha|\le\nu_\kappa}\rho_{\mu,\alpha}(\phi)(1+r)^\mu(1+|z|)^\mu,
\end{aligned}$$

and so from (8.3) it follows that

$$\sup_{\iota\in J}|T_\iota(\phi\varphi_{-z})|\le D_\kappa(1+r)^\mu\sup_{|\alpha|\le\nu_\kappa}\rho_{\mu,\alpha}(\phi)(1+|z|)^{\mu-\kappa}. \tag{8.4}$$

Now fix $a\in\,]n,\infty[$. Given $\phi\in\mathcal{O}_C$, choose $\kappa=a+\mu(\phi)$. From (8.4) it follows that

$$\sup_{\iota\in J}|T_\iota(\phi\varphi_{-z})|\le M(\phi)(1+|z|)^{-a}\quad\text{for every } z\in\mathbb{Z}^n, \tag{8.5}$$

where

$$M(\phi)=D_\kappa(1+r)^\mu\sup_{|\alpha|\le\nu_\kappa}\rho_{\mu,\alpha}(\phi)<\infty.$$

Condition (c) of Proposition 2 follows from (8.5) once it is shown that the series $\sum_{z\in\mathbb{Z}^n}(1+|z|)^{-a}$ is convergent. To check this, fix $\rho \in [n^{1/2}, \infty[$ and for every $z \in \mathbb{Z}^n$ define $B_z := \{x \in \mathbb{R}^n : |x - z| \le \rho\}$. Then $\{B_z : z \in \mathbb{Z}^n\}$ is a covering of $\mathbb{R}^n$. If $x \in B_z$, then $1+|x| \le 1+|z|+\rho \le (1+|z|)(1+\rho)$, so that $(1+|z|)^{-a} \le (1+\rho)^a(1+|x|)^{-a}$. Hence

$$(1+|z|)^{-a} \le V^{-1}(1+\rho)^a \int_{B_z} (1+|x|)^{-a}\,dx \quad \text{for every } z \in \mathbb{Z}^n,$$

where V is the volume of B_z, independent of z. It follows that

$$\sum_{z\in\mathbb{Z}^n}(1+|z|)^{-a} \le KV^{-1}(1+\rho)^a \int_{\mathbb{R}^n}(1+|x|)^{-a}\,dx < \infty,$$

where K denotes the order of the covering $\{B_z : z \in \mathbb{Z}^n\}$ of $\mathbb{R}^n$. □

Proof of (c)⇒(a). Suppose that (c) holds. We shall construct the extensions $\tilde{T}_\iota$ of the distributions T_ι by the series expansions

$$\tilde{T}_\iota(\phi) := \sum_{z\in\mathbb{Z}^n} T_\iota(\phi\varphi_z), \qquad \phi \in \mathcal{O}_C, \tag{8.6}$$

where $\{\varphi_z : z \in \mathbb{Z}^n\}$ is a periodic partition of unity on $\mathbb{R}^n$. For every $\iota \in J$, $k \in \mathbb{N}$, and $\phi \in \mathcal{O}_C$ let

$$\tilde{T}_{\iota,k}(\phi) := \sum_{|z|\le k} T_\iota(\phi\varphi_z).$$

Then

(8.7) each $\tilde{T}_{\iota,k}$ is a continuous linear functional on $\mathcal{O}_C$,

because the sum $\tilde{T}_{\iota,k}$ is finite and for every fixed $z \in \mathbb{Z}^n$ the mapping $\mathcal{O}_C \ni \phi \mapsto \phi\varphi_z \in C_c^\infty$ is continuous. Whenever $\iota \in J$ and $\phi \in \mathcal{O}_C$ are fixed, then, by the definition (8.6), the sequence $(\tilde{T}_{\iota,k}(\phi))_{k\in\mathbb{N}}$ of complex numbers is convergent and

$$\lim_{k\to\infty} \tilde{T}_{\iota,k}(\phi) = \tilde{T}_\iota(\phi). \tag{8.8}$$

As the inductive limit of Fréchet (and hence barrelled) spaces, $\mathcal{O}_C$ is a barrelled space. Furthermore, from (c) and (8.8) it follows that whenever $\phi \in \mathcal{O}_C$ is fixed, then $\{\tilde{T}_{\iota,k}(\phi) : \iota \in J, k \in \mathbb{N}\}$ and hence also $\{\tilde{T}_\iota(\phi) : \iota \in J\}$ are bounded subsets of $\mathbb{C}$. Since $\mathcal{O}_C$ is barrelled, from boundedness of $\{\tilde{T}_\iota(\phi) : \iota \in J\}$ for every fixed $\phi \in \mathcal{O}_C$ and from the generalized Banach–Steinhaus theorem ([B2, Sect. III.3.6, Theorem 2] or [O, Sect. 4.2, Theorem 4.16]) it follows that

$\{\tilde{T}_\iota : \iota \in J\}$ *is an equicontinuous set of linear functionals on* $\mathcal{O}_C$.

In order to complete the proof of (c)⇒(a) it remains to show that for every $\iota \in J$ the continuous linear functional $\tilde{T}_\iota$ on $\mathcal{O}_C$ defined by (8.6) is an extension of the distribution T_ι, i.e.

$$\tilde{T}_\iota(\psi) = T_\iota(\psi) \quad \text{for every } \psi \in \mathcal{S}. \tag{8.9}$$

To this end, notice that if $\psi \in C_c^\infty$ and $k \in \mathbb{N}$ is so large that $\operatorname{supp}\psi \cap \operatorname{supp}\varphi_z = \emptyset$ for $|z| > k$, then

$$\tilde{T}_\iota(\psi) = \sum_{|z|\le k} T_\iota(\psi\varphi_z) = T_\iota\Big(\psi \sum_{|z|>k} \varphi_z\Big) \\ = T_\iota\Big(\psi \sum_{z\in\mathbb{Z}^n} \varphi_z\Big) = T_\iota(\psi). \tag{8.10}$$

Now (8.10) implies (8.9) by the dense continuous imbeddings $C_c^\infty \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{O}_C$. □

Proof of 1°⇔3° *in Theorem 6.* From the equality $RD = H$ proved in Theorem 4 of Section 5 it follows that if $T \in \mathcal{S}'$, then $T \in RD$ if and only if the singleton $\{T\}$ satisfies condition (a) of Proposition 2. Thus 1°⇔3° in Theorem 6 follows from (a)⇔(b) in Proposition 2. The implication 3°⇒2° in Theorem 6 is obvious. The Fourier-theoretical proof of 2°⇒3° is postponed to Section X. □

IX. The strong convolutional topology in RD**.** This topology is induced from $L(\mathcal{S},\mathcal{S})_b$ via the mapping

$$\mathrm{pr} : RD \ni T \mapsto [T\,*]|_{\mathcal{S}} \in L(\mathcal{S},\mathcal{S}),$$

which makes sense by Theorem 6. Thus the strong convolutional topology is defined as the initial topology determined in RD by only one locally convex space $L(\mathcal{S},\mathcal{S})_b$ and only one projection operator.

The locally convex space $L(\mathcal{S},\mathcal{S})_b$ has a basis of neighbourhoods of zero of the form

$$\{U_{\mu,a,A;\varepsilon} : \mu \in [0,\infty[,\ a \in \mathbb{N}_0,\ A \text{ a bounded subset of } \mathcal{S},\ \varepsilon \in \,]0,\infty[\}$$

where

$$U_{\mu,a,A;\varepsilon} = \Big\{K \in L(\mathcal{S},\mathcal{S}) : \int_{\mathbb{R}^n} (1+|x|^2)^{\mu/2} |[\partial^\alpha K(\varphi)](x)|\,dx \le \varepsilon \\ \text{whenever } |\alpha| \le a \text{ and } \varphi \in A\Big\}.$$

Appearance of the integral in the last formula is a consequence of the integral description of neighbourhoods of zero in $\mathcal{S}$.

It follows that the strong convolutional topology in RD has a basis of neighbourhoods of zero of the form

$$V_{\mu,a,A;\varepsilon} = \Big\{T \in RD : \int_{\mathbb{R}^n} (1+|x|^2)^{\mu/2} |(T * \partial^\alpha\varphi)(x)|\,dx \le \varepsilon \\ \text{whenever } |\alpha| \le a \text{ and } \varphi \in A\Big\}.$$

Therefore the strong convolutional topology in RD is determined by the system of seminorms

$$\{p_{\mu,\alpha,A} : \mu \in [0,\infty[,\ a \in \mathbb{N}_0,\ A \text{ a bounded subset of } \mathcal{S}\}$$

where, for every $T \in RD$,

$$p_{\mu,\alpha,A}(T) = \sup_{\varphi\in A} \int_{\mathbb{R}^n} (1+|x|^2)^{\mu/2} |T_{(y)}((\partial^\alpha \varphi)(x-y))|\, dx.$$

In Section V we proved that for every $T \in RD$ and every $\mu \in [0,\infty[$ there is a unique $T_\mu \in (S^1_\mu)'$ extending T. Therefore whenever $T \in RD$, $\mu \in [0,\infty[$, $a \in \mathbb{N}_0^n$ and A is a bounded subset of $\mathcal{S}$, then

$$\begin{aligned} p_{\mu,\alpha,A}(T) &= \sup_{\varphi\in A} \int_{\mathbb{R}^n} |\partial^\alpha \varphi(x)| \cdot T_{(y)}((1+|x+y|^2)^{\mu/2})|\, dx \\ &\le \int_{\mathbb{R}^n} |T_{(y)}(\omega_{\alpha,A}(x)(1+|x+y|^2)^{\mu/2})|\, dx \\ &= \int_{\mathbb{R}^n} |(T_{\mu+n+1})_{(y)}(\omega_{\alpha,A}(x)(1+|x+y|^2)^{\mu/2})|\, dx \\ &= \left|(T_{\mu+n+1})_{(y)}\left(\int_{\mathbb{R}^n} \omega_{\alpha,A}(x)(1+|x+y|^2)^{\mu/2}\, dx\right)\right| \end{aligned}$$

where $\omega_{\alpha,A}(x) = \sup_{\varphi\in A} |\partial^\alpha \varphi(x)|$, so that $\omega_{\alpha,A}(x)$ converges rapidly to zero as $x \in \mathbb{R}^n$ and $|x| \to \infty$.

The last equality in the above estimations follows from the fact that its sides are both equal to $\lim \sum_\nu \int_{\mathbb{R}^n} |(T_{\mu+n+1})_{(y)}(\omega_{\alpha,A}(x_\nu)(1+|x_\nu+y|^2)^{\mu/2})|\,(dx)_\nu$. In the course of the proof of Theorem 7 we shall prove that the integral $\int_{\mathbb{R}^n} \omega_{\alpha,A}(x)(1+|\cdot+x|^2)^{\mu/2}\, dx$ is convergent and represents a function belonging to $S^1_{\mu+n+1}$.

Theorem 7. *The strong convolutional topology in RD is no finer than the topology $\tilde{w}$ defined in Section V.*

Proof. Similarly to the proof of Theorem 4 in Section VI, the topology $\tilde{w}$ in RD is determined by the system of seminorms $\{q_{\mu+\lambda,F} : \mu \in [0,\infty[,\ F \in \mathcal{F}^1_{\mu+\lambda}\}$ where

$$q_{\mu+\lambda,F}(T) = \sup_{\phi\in F} |T_{\mu+\lambda}(\phi)| \quad \text{for } T \in RD,$$

$\lambda \in [0,\infty[$ is any fixed constant, and $\mathcal{F}^1_{\mu+\lambda}$ denotes the family of all finite subsets of $S^1_{\mu+\lambda}$. For convenience in subsequent calculations we choose $\lambda = n+1$. Theorem 7 will follow once it is shown that

$$p_{\mu+n+1,\alpha,A}(T) \le \sup_{\phi\in F_{a,A}} |T_{\mu+n+1}(\phi)| \tag{9.1}$$

for a certain set $F_{\alpha,A} \in \mathcal{F}^1_{\mu+n+1}$. We shall see that (9.1) is true when $F_{\alpha,A}$ is the singleton

$$F_{\alpha,A} = \left\{ \int_{\mathbb{R}^n} \omega_{\alpha,A}(x)(1 + |\cdot + x|^2)^{\mu/2}\, dx \right\}.$$

First we shall prove that the integral $\int_{\mathbb{R}^n} \omega_{\alpha,A}(x)(1 + |\cdot + x|^2)^{\mu/2}\, dx$ represents a function belonging to $S^1_{\mu+n+1}$. This will follow once we check that for every $\mu \in [0, \infty[$ and $\beta \in \mathbb{N}_0^n$ the iterated integral

$$I_{\mu,\beta} = \int_{\mathbb{R}^n} (1 + |y|^2)^{-(\mu+n+1)/2} \left[\int_{\mathbb{R}^n} \omega_{\alpha,A}(x) |\partial^\beta_{(y)} (1 + |x + y|^2)^{\mu/2}|\, dx \right] dy$$

is finite. By Lemma 1 from Section II we have

$$\partial^\beta_{(y)} (1 + |x + y|^2)^{\mu/2} = (1 + |x + y|^2)^{\mu/2} (1 + |x + y|^2)^{-|\beta|} P_\beta(x + y)$$

where P_β is a polynomial on $\mathbb{R}^n$ of degree no greater than $|\beta|$. Consequently, by the inequality $1 + |x + y|^2 \le (1 + |y|^2)(1 + |x|)^2$ [Hö, Sect. II.2.1, Example 1], monotonicity of the integral, and the Fubini theorem (see [El, Sect. IV.2.4, and Sect. V.2, Theorem 2.1]), for every $\beta \in \mathbb{N}_0^n$ there is $K_\beta \in]0, \infty[$ such that

$$I_{\mu,\beta} \le K_\beta \int_{\mathbb{R}^n} \omega_{\alpha,A}(x)(1 + |x|)^\mu\, dx \int_{\mathbb{R}^n} (1 + |x + y|^2)^{-(n+1)/2}\, dy < \infty \quad \text{for every } \mu \in [0, \infty[.$$

This proves that $\int_{\mathbb{R}^n} \omega_{\alpha,A}(x)(1 + |\cdot + x|^2)^{\mu/2} \in S^1_{\mu+n+1}$, so that the singleton $F_{\alpha,A} = \{\int_{\mathbb{R}^n} \omega_{\alpha,A}(x)(1 + |\cdot + x|^2)^{\mu/2}\, dx\}$ belongs to $\mathcal{F}^1_{\mu+n+1}$, completing the proof. □

Theorem 8. *RD equipped with the strong convolutional topology is complete.*

Proof. Let $\Lambda(\mathcal{S}, \mathcal{S})_b$ be the closed subspace of $L(\mathcal{S}, \mathcal{S})_b$ consisting of all operators in $L(\mathcal{S}, \mathcal{S})$ commuting with translations. Since $\mathcal{S}$, as a Fréchet space, is bornological, applying the argument in [O, Sect. 4.3, proof of Theorem 4.20] one concludes that the locally convex space $L(\mathcal{S}, \mathcal{S})_b$ is complete, hence so is $\Lambda(\mathcal{S}, \mathcal{S})_b$. We shall prove below that the mapping $\mathrm{pr} : RD \ni T \mapsto [T\, *]|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{S})$ is a linear homeomorphism of RD onto $\Lambda(\mathcal{S}, \mathcal{S})$. Therefore RD equipped with the strong convolutional topology is complete, because $\Lambda(\mathcal{S}, \mathcal{S})_b$ is.

Since convolution with a distribution is an operator on sample functions that commutes with translations, it follows that pr indeed maps into $\Lambda(\mathcal{S}, \mathcal{S})$. It remains to prove that it is injective and maps RD onto $\Lambda(\mathcal{S}, \mathcal{S})$.

To prove injectivity it is sufficient to observe that if $[T\, *]|_{\mathcal{S}} = 0$, then $T(\varphi) = [T * \varphi^\vee](0) = 0$ for every $\varphi \in \mathcal{S}$, which means that the distribution T is equal to zero.

Surjectivity means that for every $K \in \Lambda(\mathcal{S},\mathcal{S})$ there is $T \in RD$ such that $K = [T\,*]|_{\mathcal{S}}$. To prove the latter, let $K \in \Lambda(\mathcal{S},\mathcal{S})$ and define the slowly increasing distribution T on $\mathbb{R}^n$ by $T(\varphi) = [K(\varphi^{\vee})](0)$ for every $\varphi \in \mathcal{S}$. Then $T * \varphi = T(\varphi_x^{\vee}) = [K(\varphi_x)]|_{x=0} = K(\varphi) \in \mathcal{S}$ for every $\varphi \in \mathcal{S}$, i.e. $K = [T\,*]|_{\mathcal{S}}$ and $T \in RD$, by equivalence of conditions 1° and 2° in Theorem 6. □

Another proof of Theorem 8 is by applying Theorem 9 below and the completeness of $\mathcal{O}_M$ (see [H, Sect. 2.9, Example 7]).

X. The strong convolutional topology in RD and Fourier transformation. Since the Fourier transformation $\mathfrak{F} : \mathcal{S} \to \mathcal{S}$ is a linear topological automorphism of $\mathcal{S}$, it follows from [T, Sect. II.23, Proposition 23.1] that $\mathfrak{F}'$, the transpose of $\mathfrak{F}$, is a is $*$-weakly continuous linear automorphism of $\mathcal{S}'$, so that, by [B2, Sect. IV.4.2, Proposition 6], $\mathfrak{F}'$ is also a linear topological automorphism of $\mathcal{S}'$ when $\mathcal{S}'$ is equipped with its usual strong dual topology. Moreover, since $\mathcal{S}$ is sequentially dense in $\mathcal{S}'$ (equipped with the strong dual topology), from the Parseval equality for $\mathfrak{F} : \mathcal{S} \to \mathcal{S}$ it follows that $\mathfrak{F}'$ is equal to the extension of $\mathfrak{F}$ by continuity. For this reason in what follows we shall write $\mathfrak{F}$ instead $\mathfrak{F}'$.

A function ϕ belonging to $C^{\infty}(\mathbb{R}^n)$ is called a *multiplier* of $\mathcal{S}$ if $\phi \cdot \varphi \in \mathcal{S}$ for every $\varphi \in \mathcal{S}$. The multipliers of $\mathcal{S}$ constitute a function algebra on $\mathbb{R}^n$, which will be denoted by m. The *strong multiplicational topology* in m is determined by the system of seminorms $\{s_{\mu,\alpha,B} : \mu \in [0,\infty[,\ \alpha \in \mathbb{N}_0^n,\ B$ a bounded subset of $\mathcal{S}\}$ where $s_{\mu,\alpha,B}(\phi) = \sup_{\varphi \in B} \rho_{-\mu,\alpha}(\phi \cdot \varphi)$ for every $\phi \in m$. The strong multiplicational topology in m coincides with topology defined in [S, Sect. VII.5] (see [K1, Sects. 2.1 and 2.2]).

Let $\mathcal{O}_M = \{\phi \in C^{\infty}(\mathbb{R}^n)$: for every $\alpha \in \mathbb{N}_0^n$ there is $\mu \in [0,\infty[$ such that $\rho_{\mu,\alpha}(\phi) < \infty\}$. Then $\mathcal{O}_M = m$, the inclusion $\mathcal{O}_M \subset m$ being obvious. An ingenious short proof of the inclusion $m \subset \mathcal{O}_M$ is presented in [Kh, Vol. 2, Chap. CA.III]. Roughly, every $\phi \in m$ is a slowly increasing C^{∞}-function on $\mathbb{R}^n$, so that to every ϕ belonging to m there corresponds the distribution $[\phi]$ belonging to $\mathcal{S}'$, represented by the function ϕ. The Fourier transformation is a linear bijective mapping $\mathfrak{F} : RD \to [m]$. Moreover, if $T \in RD$, and $\phi \in m$ is uniquely determined by $\mathfrak{F}(T) = [\phi]$, then $(\mathfrak{F}|_{\mathcal{S}})(T * \varphi) = \hat{\varphi} \cdot \phi$ for every $\varphi \in \mathcal{S}$.[2] Under the strong convolutional topology in RD and strong multiplicational

[2]The above statement is the last of the four theorems collected in [K2, Sects. 8 and 9] concerning the algebraic linear exchange between convolution and multiplication via Fourier transformation.

topology in m, the linear bijective mapping $\mathfrak{F} : RD \to [m]$ becomes a linear homeomorphism of locally convex spaces:

Theorem 9. *The Fourier transformation is a linear homeomorphism $\mathfrak{F} : RD \to m$ of RD equipped with the strong convolutional topology onto m equipped with the strong multiplicational topology.*

Proof. A net $(T_\iota)_{\iota\in J} \subset RD$ converges to zero in the strong convolutional topology if and only if

(10.1) $\lim_\iota T_\iota * \varphi = 0$ in the topology of $\mathcal{S}$, uniformly in φ ranging over any bounded subset of $\mathcal{S}$.

A net $(\phi_\iota)_{\iota\in J} \subset m$ converges to zero in the strong multiplicational topology if and only if

(10.2) $\lim_\iota \phi_\iota \cdot \varphi = 0$ in the topology of $\mathcal{S}$, uniformly in φ ranging over any bounded subset of $\mathcal{S}$.

If $T_\iota \in RD$ and $\mathfrak{F}(T_\iota) = [\phi_\iota]$ where $\phi_\iota \in m$, then

$$(\mathfrak{F}|_{\mathcal{S}})(T_\iota * \varphi) = \phi_\iota \cdot \hat{\varphi} \quad \text{for every } \varphi \in \mathcal{S}. \tag{10.3}$$

Theorem 9 follows once it is shown that whenever $T_\iota \in RD$ and $\phi_\iota \in m$ satisfy (10.3), then conditions (10.1) and (10.2) are equivalent. But if $T_\iota \in RD$ and $\phi_\iota \in m$ satisfy (10.3), then the equivalence of (10.1) and (10.2) is a consequence of (10.3) and the fact that $\mathfrak{F}|_{\mathcal{S}}$ is a topological linear automorphism of $\mathcal{S}$. □

Completion of the proof of Theorem 6. It remains to show that

$$\text{if } T \in \mathcal{S}' \text{ and } T * \varphi \in \mathcal{S} \text{ for every } \varphi \in \mathcal{S}, \text{ then } T \in RD. \tag{10.4}$$

By the remarks preceding Theorem 9, concerning the algebraic linear exchange between convolution and multiplication via Fourier transformation, (10.4) is equivalent to the tautological statement that if $\phi \in m$ and $\phi \cdot \varphi \in \mathcal{S}$ for every $\varphi \in \mathcal{S}$, then $\phi \in m$. □

XI. Coincidence of the strong convolutional topology and the topology $\tilde{b}$ on bounded sets.

Proposition 3. *For every set $\{T_\iota : \iota \in J\} \subset \mathcal{S}'$ the following four conditions are equivalent:*

(a) *for every $\iota \in J$ the distribution T_ι can be (uniquely) extended to a continuous linear functional $\tilde{T}_\iota$ on $\mathcal{O}_C$, and $\{T_\iota : \iota \in J\}$ is an equicontinuous set of linear functionals on $\mathcal{O}_C$,*

(a)′ *for every $\iota \in J$ the distribution T_ι can be (uniquely) extended to a continuous linear functional $\tilde{T}_\iota$ on $\mathcal{O}_C$, and $\{T_\iota : \iota \in J\}$ is a bounded subset in $(\mathcal{O}_C)'$ in any $\mathfrak{S}$-topology,*

(b) $\{[T_\iota *]|_{\mathcal{S}} : \iota \in J\}$ *is an equicontinuous subset of* $L(\mathcal{S},\mathcal{S})$,
(b)$'$ $\{[T_\iota *]|_{\mathcal{S}} : \iota \in J\}$ *is a bounded subset of* $L(\mathcal{S},\mathcal{S})_b$.

Proof. From Theorem 2 we know that (a)$\Leftrightarrow$(b). Moreover, $\mathcal{S}$ is barrelled as a Fréchet space, and $\mathcal{O}_C$ is barrelled as the inductive limit of Fréchet (and hence barrelled) spaces. Thus the equivalences (a)$\Leftrightarrow$(a)$'$ and (b)$\Leftrightarrow$(b)$'$ follow by [O, Sect. 4.2, Theorem 4.16] or by [B1, Sect. III.3.6, Proposition 7 and Theorem 2]. □

From Theorem 3 in Section V and Theorem 4 in Section VI it follows that

$$RD = \{T \in \mathcal{S}' : T \text{ has a unique extension } \tilde{T} \in (\mathcal{O}_C)'\}$$

and the mapping $RD \ni T \mapsto \tilde{T} \in (\mathcal{O}_C)'$ is a linear isomorphism.

From Theorem 4 it also follows that the topology τ_b in $(\mathcal{O}_C)'$ is determined by the system of seminorms $\{q_{\mu,B} : \mu \in [0,\infty[,\ B \in \mathcal{B}^1_\mu\}$ where $q_{\mu,B}(F) = \sup_{\phi\in B} |F(\phi)|$ for every $F \in (\mathcal{O}_C)'$, $\mu \in [0,\infty[$ and $B \in \mathcal{B}^1_\mu$, $\mathcal{B}^1_\mu$ being the family of all bounded subsets of S^1_μ. Therefore the topology $\tilde{b}$ in RD is determined by the system of seminorms $\{p_{\mu,B} : \mu \in [0,\infty[,\ B \in \mathcal{B}^1_\mu\}$ where

$$p_{\mu,B}(T) = \sup_{\phi\in B} |\tilde{T}(\phi)| \quad \text{for every } T \in RD,\ \mu \in [0,\infty[\text{ and } B \in \mathcal{B}^1_\mu.$$

If, as in Section V, T_μ denotes the (unique) extension of $T \in RD$ to a continuous linear functional on S^1_μ, then $(\tilde{T}|_{S^1_\mu})(\phi) = T_\mu(\phi)$ for every $T \in RD$, $\mu \in [0,\infty[$ and $\phi \in S^1_\mu$. Therefore

$$p_{\mu,B}(T) = \sup_{\phi\in B} |T_\mu(\phi)| \quad \text{for every } T \in RD,\ \mu \in [0,\infty[\text{ and } B \in \mathcal{B}^1_\mu.$$

Consequently, the set $\{T_\iota : \iota \in J\} \subset RD$ is bounded in the topology $\tilde{b}$ if and only if

$$\sup_{\iota\in J} p_{\mu,B}(T_\iota) < \infty \quad \text{for every } \mu \in [0,\infty[\text{ and } B \in \mathcal{B}^1_\mu.$$

The last condition is equivalent to the boundedness of $\{T_\iota : \iota \in J\}$ in the locally convex space $((\mathcal{O}_C)', \tau_b)$. By (a)$'\Leftrightarrow$(b)$'$, the latter is equivalent to the boundedness of $\{T_\iota : \iota \in J\} \subset RD$ in the strong convolutional topology. Summing up, we have proved the following

Corollary. *The boundedness of a subset of RD means the same for all the three topologies in RD: the strong convolutional topology, the topology* $\tilde{w}$ (*intermediate*) *and the topology* $\tilde{b}$.

Now we shall prove that on bounded subsets of RD, common for all the three topologies occurring in the Corollary, all these topologies coincide. This is an immediate consequence of the following theorem.

Theorem 10. *If* $(T_\iota)_{\iota\in J} \subset RD$ *is a net such that the set* $\{T_\iota : \iota \in J\}$ *of its terms is bounded (in the sense of the Corollary) and* $\lim_\iota T_\iota = 0$ *in the strong dual topology of* $\mathcal{S}'$*, then* $\lim_\iota T_\iota = 0$ *in the topology* $\tilde{b}$.

A convenient technical formulation of Theorem 9 is the following

Proposition 4. *Let* $(T_\iota)_{\iota\in J}$ *be a net in* RD*. If*

$$\lim_\iota \sup_{\varphi\in A} |T_\iota(\varphi)| = 0 \text{ for every bounded subset } A \text{ of } \mathcal{S} \tag{11.1}$$

and

$$\sup_{\iota\in J,\,\phi\in B} |\tilde{T}_\iota(\phi)| < \infty \quad \text{for every } \mu \in [0,\infty[\text{ and } B \in \mathcal{B}^1_\mu, \tag{11.2}$$

then $(T_\iota)_{\iota\in J}$ *converges to zero in the topology* $\tilde{b}$.

Proof. It is sufficient to prove that if (11.1) and (11.2) are satisfied then

$$\lim_\iota \sup_{\phi\in B} |\tilde{T}_\iota(\phi)| = 0 \quad \text{for every } \mu \in [0,\infty[\text{ and } B \in \mathcal{B}^1_\mu. \tag{11.3}$$

Indeed, (11.3) means that $(\tilde{T}_\iota)_{\iota\in J} \subset (\mathcal{O}_C)'$ converges to zero in the topology τ_b discussed in Section IV, and hence, by Theorem 4 from Section VI, $(T_\iota)_{\iota\in J} \subset RD$ converges to zero in the topology $\tilde{b}$.

In order to prove (11.3) take non-negative functions $\psi \in C^\infty_c(\mathbb{R})$ and $\eta \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \le 2\}$, $\operatorname{supp}\eta \subset \{x \in \mathbb{R}^n : |x| \ge 1\}$, and $\psi(x) + \eta(x) = 1$ for all $x \in \mathbb{R}^n$. For $r \in]0,\infty[$, let

$$\psi_r(x) = \psi(r^{-1}x) \quad \text{and} \quad \eta_r(x) = \eta(r^{-1}x).$$

Then $\operatorname{supp}\psi_r \subset \{x \in \mathbb{R}^n : |x| \le 2r\}$, $\operatorname{supp}\eta_r \subset \{x \in \mathbb{R}^n : |x| \ge r\}$ and $\psi_r(x) + \eta_r(x) = 1$ for every $x \in \mathbb{R}^n$. The equality (11.3) will follow once we prove that

$$\lim_\iota \sup_{\phi\in B} |\tilde{T}_\iota(\psi_r\phi)| = 0 \tag{11.4}$$

$$\text{for any fixed } r \in]0,\infty[,\ \mu \in [0,\infty[\text{ and } B \in \mathcal{B}^1_\mu,$$

and

$$\lim_{r\to\infty} \sup_{\iota\in J,\,\phi\in B} |\tilde{T}_\iota(\eta_r\phi)| = 0 \quad \text{whenever } \mu \in [0,\infty[\text{ and } B \in \mathcal{B}^1_\mu. \tag{11.5}$$

Proof of (11.4). Let $\mu \in [0,\infty[$ and $B \in \mathcal{B}^1_\mu$. For any fixed $r \in]0,\infty[$ the mapping $S^1_\mu \ni \phi \mapsto \psi_r\phi \in \mathcal{S}$ is continuous, so that $A =$

$\{\psi_r\phi : \phi \in B\}$ is a bounded subset of $\mathcal{S}$. Moreover $\sup_{\phi\in B} |\tilde{T}_\iota(\psi_r\phi)| = \sup_{\varphi\in A} |\tilde{T}_\iota(\varphi)|$. Hence (11.1) implies (11.4).

Proof of (11.5). For every $\phi \in \mathcal{O}_C$ let

$$s(\phi) = \sup_{\iota\in J} |\tilde{T}_\iota(\phi)|.$$

Then s is a seminorm on $\mathcal{O}_C$, and in terms of s condition (11.2) can be equivalently written as

$$\sup_{\phi\in B} s(\phi) < \infty \quad \text{whenever } \mu \in [0,\infty[\text{ and } B \in \mathcal{B}^1_\mu. \tag{11.6}$$

This means that for every $\mu \in [0,\infty[$ the seminorm s is bounded on S^1_μ. In terms of s condition (11.5) can be written as

$$\lim_{r\to\infty} \sup_{\phi\in B} s(\eta_r\phi) = 0 \quad \text{whenever } \mu \in [0,\infty[\text{ and } B \in \mathcal{B}^1_\mu. \tag{11.7}$$

Since for every $\mu \in [0,\infty[$ the space S^1_μ is bornological as a metrizable space, it follows that the restriction of the seminorm s to S^1_μ is continuous in the topology of S^1_μ (see [Y, Sect. I.7, Theorem 2]).

To prove (11.7) we shall perform some estimations. As before, for every $\mu \in [0,\infty[$ denote by $\mathcal{B}^1_\mu$ the family of all bounded subsets of the Fréchet space S^1_μ. From the definition of the functions η_r it follows that

$$\{\eta_r : r \in [r_0,\infty[\} \text{ is a bounded subset of } C^\infty_b(\mathbb{R}^n) \quad \text{for every } r_0 \in]0,\infty[. \tag{11.8}$$

From Theorem 1 in Section II, and from (11.8), it follows that whenever

$$\mu \in [0,\infty[,\ B \in \mathcal{B}^1_\mu,\ \lambda \in]0,\infty[,\ r_0 \in]0,\infty[\text{ and } r \in [r_0,\infty[,$$

then $C := \{(1+|\cdot|^2)^{\lambda/2}\phi : \phi \in B\} \in \mathcal{B}^1_{\mu+\lambda}$, so that

$$\text{if } \phi \in B \in \mathcal{B}^1_\mu, \text{ then } \psi = (1+|\cdot|^2)^{\lambda/2}\phi \in C \in \mathcal{B}^1_{\mu+\lambda}. \tag{11.9}$$

We shall prove that if

$$\zeta = \zeta_{\lambda,r_0,r} = \frac{(1+|\cdot|^2)^{\lambda/2}}{(1+r_0^2)^{-\lambda/2}}\eta_r\psi,$$

then

$$\zeta \in D \in \mathcal{B}^1_{\mu+\lambda}. \tag{11.10}$$

To this end, notice that the positive function

$$\rho(x) = \frac{(1+|x|^2)^{-\lambda/2}}{(1+r_0^2)^{-\lambda/2}}, \quad x \in \mathbb{R}^n,$$

takes values from $]0,1]$ on $\operatorname{supp}\eta_r$, and by Lemma 1 from Section II one has

$$\partial^\alpha(1+|x|^2)^{-\lambda/2} = (1+|x|^2)^{-\lambda/2}\xi_\alpha(x) \quad \text{for every } x \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{N}_0^n$$

where $\xi_\alpha(x) = (1+|x|^2)^{-|\alpha|}P_\alpha(x) \in C_b(\mathbb{R}^n)$. Therefore

$$\begin{aligned}
(11.11) \quad \pi_{\mu+\lambda,\alpha}(\zeta) &= \int_{\mathbb{R}^n}(1+|x|^2)^{-(\mu+\lambda)/2}|\partial^\alpha\zeta(x)|\,dx \\
&\leq \int_{\mathbb{R}^n}(1+|x|^2)^{-(\mu+\lambda)/2}\rho(x)\sum_{\beta+\gamma+\delta=\alpha}\frac{\alpha!}{\beta!\gamma!\delta!}|\xi_\beta(x)\partial^\gamma\eta_r(x)\partial^\delta\psi(x)|\,dx \\
&\leq \int_{\mathbb{R}^n}(1+|x|^2)^{-(\mu+\lambda)/2}\sum_{\beta+\gamma+\delta=\alpha}\frac{\alpha!}{\beta!\gamma!\delta!}|\xi_\beta(x)\partial^\gamma\eta_r(x)\partial^\delta\psi(x)|\,dx \\
&\leq L_\alpha\int_{\mathbb{R}^n}(1+|x|^2)^{-(\mu+\lambda)/2}\sum_{|\vartheta|\leq|\alpha|}|\partial^\vartheta\psi(x)|\,dx \leq K_{\alpha,C} < \infty
\end{aligned}$$

where the last two inequalities follow from the facts that $\xi_\beta \in C_b(\mathbb{R}^n)$, (11.8) holds, and $\psi \in C \in \mathcal{B}^1_{\mu+\lambda}$. The estimate (11.11) proves (11.10).

Now we are ready to complete the proof of (11.7). If $\phi \in B \in \mathcal{B}^1_\mu$, then, by (11.9) and (11.10),

$$s(\eta_r\phi) = s((1+|\cdot|^2)^{-\lambda/2}\eta_r\psi) = (1+r_0^2)^{-\lambda/2}s(\zeta) \leq (1+r_0^2)^{-\lambda/2}K_D$$

where $K_D < \infty$ because the seminorm s is bounded on $D \in \mathcal{B}^1_{\mu+\lambda}$. Thus

$$s(\eta_r\phi) \leq (1+r_0^2)^{-\lambda/2}K_D,$$

whence (11.7) follows by letting $r_0 \to \infty$. □

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