Characterization of rapidly decreasing distributions on $\mathbb{R}^n$ by convolutions with test functions

Jan Kisyński
E-mail: jan.kisynski@gmail.com

Abstract

The set of rapidly decreasing distributions on $\mathbb{R}^n$ is characterized by convolutions with test functions. In spite of logical independence, the present paper completes the results of [K].

Introduction

The set RD of rapidly decreasing distributions on $\mathbb{R}^n$ was defined in [K] as

$$RD = \Big\{ T \in \mathcal{S}'(\mathbb{R}^n) : T \text{ is continuous with respect to the topology of } \mathcal{O}_C(\mathbb{R}^n) = \liminf_{\mu\to\infty} S^1_\mu(\mathbb{R}^n) \Big\}.$$

In [K] it was proved that $RD \subset \mathbf{RD}$ where

$$\mathbf{RD} = \{T \in \mathcal{S}'(\mathbb{R}^n) : [T\,*]_{\mathcal{S}(\mathbb{R}^n)} \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))\}$$

and it was announced that the equality $RD = \mathbf{RD}$ is a consequence of

Theorem 1. *If $T \in \mathcal{S}'(\mathbb{R}^n)$ then the following three conditions are equivalent:*

(a) $T \in RD$,

(b) $T \in \mathbf{RD}$,

(c) *whenever $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\mu \in [0, \infty[$ and $\phi \in S^1_\mu(\mathbb{R}^n)$ are fixed, then the function $\mathbb{R}^n \ni z \mapsto T(\phi \circ \varphi_z) \in \mathbb{C}$ belongs to $\mathcal{S}(\mathbb{R}^n)$.*

The scheme of the proof is (a)⇒(b)⇒(c)⇒(a). Since the implication (a)⇒(b) was proved in [K], it remains to prove (b)⇒(c) and (c)⇒(a).

Proof of (b)⇒(c)

Lemma 1. *For every $\mu \in [0,\infty[$, $\phi \in S^1_\mu(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, there is $N \in]0,\infty[$ such that*

$$|\partial^\alpha \phi(x)| \le N(1+|x|)^\mu \quad \textit{for every } x \in \mathbb{R}^n.$$

Proof. Let $\mu \in [0,\infty[$, $\phi \in S^1_\mu(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Consider the function

$$u(x) = (1+|x|^2)^{-\mu/2}\partial^\alpha \phi(x), \quad x \in \mathbb{R}^n.$$

By Theorem 4.12(1) in the book [A-F] of R. A. Adams and J. Fournier there is $K \in]0,\infty[$ such that

$$\sup_{x\in\mathbb{R}^n} |u(x)| \le K \sup_{|\beta|\le n} \int_{\mathbb{R}^n} |\partial^\beta u(x)|\, dx.$$

It follows that, for every $x \in \mathbb{R}^n$,

$$|\partial^\alpha \phi(x)| \le (1+|x|^2)^{\mu/2} \\ \times K \sup_{|\beta|\le n} \int_{\mathbb{R}^n} \sum_{0\le\gamma\le\beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} |\partial^{\beta-\gamma}(1+|y|^2)^{-\mu/2}| \cdot |\partial^{\alpha+\gamma}\phi(y)|\, dy.$$

By a result of J. Horváth concerning partial derivatives of functions on $\mathbb{R}^n$ of the type $x \mapsto (1+|x|^2)^a$, $a \in \mathbb{R}$ (see [H, Sect. 2.5, Example 8, equality (3)]) there is $H \in]0,\infty[$ such that for every $x \in \mathbb{R}^n$,

$$\begin{aligned} |\partial^\alpha \phi(x)| &\le H(1+|x|^2)^{\mu/2} \sup_{|\gamma|\le n} \int_{\mathbb{R}^n} (1+|y|^2)^{-\mu/2} |\partial^{\alpha+\gamma}\phi(y)|\, dy \\ &= H(1+|x|)^{\mu/2} \sup_{|\gamma|\le n} \pi^1_{\mu,\alpha+\gamma}(\phi) \le N(1+|x|)^\mu. \end{aligned}$$

□

Lemma 2. *If $T \in$ **RD**, $\mu \in [0,\infty[$, $\phi \in S^1_\mu(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, and $\lambda \in [0,\infty[$, then there is $M \in]0,\infty[$ such that*

$$|T(\phi\cdot\varphi_z)| \le M(1+|z|)^{\mu-\lambda} \quad \textit{for every } z \in \mathbb{R}^n.$$

Proof. If $T \in \mathcal{D}'(\mathbb{R}^n)$, $\mu \in [0,\infty[$, $\phi \in S^1_\mu(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\lambda \in [0,\infty[$, then

$$\text{(d)} \quad \begin{aligned} |T(\phi\cdot\varphi_z)| &= |T((\phi_{-z}\cdot\varphi)_z)| = |[T*(\phi_{-z}\cdot\varphi)^\vee](-z)| \\ &= (1+|-z|)^\lambda |[T*(\phi_{-z}\cdot\varphi)^\vee](-z)|(1+|-z|)^{-\lambda} \\ &\le \sup_{x\in\mathbb{R}^n} (1+|x|)^\lambda |[T*(\phi_{-z}\cdot\varphi)^\vee](x)|(1+|z|)^{-\lambda} \\ &\le \rho_{\lambda,0}(T*(\phi_{-z}\cdot\varphi)^\vee)(1+|z|)^{-\lambda} \quad \text{for every } z \in \mathbb{R}^n. \end{aligned}$$

Let now $T \in \mathbf{RD}$. Then, for every $z \in \mathbb{R}^n$ and $\mu \in [0, \infty[$ the bilinear mapping

$$B_{z,\mu} : S^1_\mu(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n) \ni (\phi, \varphi) \mapsto \phi_{-z} \cdot \varphi \in \mathcal{D}(\mathbb{R}^n)$$

is continuous, and

$$\mathcal{S}(\mathbb{R}^n) \ni \psi \mapsto \rho_{\lambda,0}(T * \psi^\vee) \in [0, \infty[$$

is a continuous seminorm on $\mathcal{S}(\mathbb{R}^n)$. Comparing this seminorm with the system of seminorms $\{\rho_{\nu,\alpha} : \nu \in [0, \infty[, \alpha \in \mathbb{N}_0^n\}$ determining the topology in $\mathcal{S}(\mathbb{R}^n)$ we conclude that whenever $T \in RD$, $\phi \in S^1_\mu(\mathbb{R}^n)$ for some $\mu \in [0, \infty[$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $z \in \mathbb{R}^n$ and $\lambda \in [0, \infty[$, then there are $L \in]0, \infty[$, $\nu \in [0, \infty[$ and $\alpha \in \mathbb{N}_0^n$ such that

$$\text{(e)} \qquad \rho_{\lambda,0}(T * (\phi_{-z} \cdot \varphi)^\vee) \leq L\rho_{\nu,\alpha}(\phi_{-z} \cdot \varphi).$$

The inequalities (d) and (e) make it possible to complete the proof of Lemma 2. Indeed, from (d) and (e) it follows that whenever $T \in \mathbf{RD}$, $\mu \in [0, \infty[$, $\phi \in S^1_\mu(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\lambda \in [0, \infty[$ and $z \in \mathbb{R}^n$, then

$$\begin{aligned}
&|T(\phi_z \cdot \varphi_z)| \\
&\qquad \leq L\rho_{\nu,\alpha}(T * (\phi_{-z} \cdot \varphi)^\vee) \cdot (1 + |z|)^{-\lambda} \\
&\qquad = L \sup_{x \in \operatorname{supp} \varphi} (1 + |x|^2)^{\nu/2} |\partial^\alpha[\phi(x - z) \cdot \varphi(x)]| \cdot (1 + |z|)^{-\lambda} \\
&\qquad \leq 2^{1/2} L \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \sup_{x \in \operatorname{supp} \varphi} (1 + |x|)^\nu |\partial^\beta \phi(x - z)| \, |\partial^\gamma \varphi(x)| \cdot (1 + |x|)^{-\lambda}.
\end{aligned}$$

Let

$$r = \max\{|x| : x \in \operatorname{supp} \varphi\}.$$

Then, by Lemma 1, there is $N \in]0, \infty[$ such that $|\partial^\beta \phi(x - z)| \leq N(1 + |x - z|)^\mu \leq N(1 + r)^\mu (1 + |z|)^\mu$ for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$ and all multiindices β with length $|\beta| \leq |\alpha|$, so that

$$\begin{aligned}
&|T(\phi_z \cdot \varphi_z)| \leq \\
&LN2^{|\alpha|+1/2}(1+r)^{\nu+\mu} \cdot \Big[\sup_{|x| \leq r,\, |\gamma| \leq |\alpha|} |\partial^\gamma \varphi(x)| \Big] (1+|z|)^{\mu-\lambda} = M(1+|z|)^{\mu-\lambda}. \quad \Box
\end{aligned}$$

Lemma 3. *If* $T \in \mathbf{RD}$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\mu \in [0, \infty[$ *and* $\phi \in S^1_\mu(\mathbb{R}^n)$ *are fixed, then the function* $\mathbb{R}^n \ni z \mapsto T(\phi \cdot \varphi_z) \in \mathbb{C}$ *belongs to* $\mathcal{S}(\mathbb{R}^n)$.

Proof. Since $\partial_z^\alpha T(\phi \cdot \varphi_z) = (-1)^{|\alpha|} T(\phi \cdot (\partial^\alpha \varphi)_z)$, Lemma 3 is a consequence of Lemma 2 applied to partial derivatives of φ. $\Box$

Condition (c) and periodic partitions of unity on $\mathbb{R}^n$

Assume that a distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ has property (c), $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\{\varphi_z : z \in \mathbb{Z}^n\}$ is a periodic partition of unity on $\mathbb{R}^n$ in the sense of V.-K. Khoan [Kh, Vol. 2, Sect. CC.III.30]. We shall consider multiple series of the form $\sum_{z\in\mathbb{Z}^n} T(\phi\cdot\varphi_z)$ corresponding to the periodic partition of unity $\{\varphi_z : z \in \mathbb{Z}^n\}$.

Lemma 4. *If a distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ satisfies condition* (c)*, $\phi \in S^1_\mu(\mathbb{R}^n)$ for some $\mu \in [0,\infty[$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\{\varphi_z : z \in \mathbb{Z}^n\}$ is a periodic partition of unity on* $(\mathbb{R}^n)$*, then*

$$\sum_{z\in\mathbb{Z}^n} |T(\phi\cdot\varphi_z)| < \infty.$$

Proof. By Lemma 3, the function $f : \mathbb{R}^n \ni z \mapsto T(\phi\cdot\varphi_z) \in \mathbb{C}$ belongs to $\mathcal{S}(\mathbb{R}^n)$, so that for every $k \in \mathbb{N}$ there is $C_k \in]0,\infty[$ such that $|f(z)| \le C_k(1+|z|)^{-k}$ for every $z \in \mathbb{R}^n$. Hence

$$\sum_{z\in\mathbb{Z}^n} |T(\phi\cdot\varphi_z)| = \sum_{z\in\mathbb{Z}^n} |f(z)| \le C_{n+1}\sum_{z\in\mathbb{Z}^n}(1+|z|)^{-n-1} < \infty. \qquad \square$$

Lemma 5. *If a distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ satisfies condition* (c)*, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\{\varphi_z : z \in \mathbb{Z}^n\}$ is a periodic partition of unity on $\mathbb{R}^n$, and $\eta \in \mathcal{D}(\mathbb{R}^n)$, then*

$$\sum_{z\in\mathbb{Z}^n} T(\eta\cdot\varphi_z) = T(\eta).$$

Proof. In the multiple series above only those terms are non-zero for which $\operatorname{supp}\varphi_z \cap \operatorname{supp}\eta \ne \emptyset$. There are only finitely many such terms, so the series is in fact a finite sum. For this sum, by definition of the partition of unity, we have $\sum \eta(x)\varphi_z = \eta(x)$ for every $x \in \mathbb{R}^n$. Hence $\sum \eta\cdot\varphi_z = \eta$, and $\sum T(\eta\cdot\varphi_z) = T(\eta)$. $\square$

Lemma 6. *If a distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ satisfies condition* (c)*, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\{\varphi_z : z \in \mathbb{Z}^n\}$ is a periodic partition of unity on $\mathbb{R}^n$, and $\psi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\sum_{z\in\mathbb{Z}^n} T(\psi\cdot\varphi_z) = T(\psi).$$

Proof. The desired equality will be deduced from Lemma 5 by suitable approximation of $\psi \in \mathcal{S}(\mathbb{R}^n)$ by a sequence $(\psi_l)_{l\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$. The approximating functions $\psi_l \in \mathcal{D}(\mathbb{R}^n)$, $l \in \mathbb{N}$, will be constructed in the form $\psi_l = \psi\cdot\chi_l$ where $\chi_l \in \mathcal{D}(\mathbb{R}^n)$, $l \in \mathbb{N}$, are cut-off functions such that

$$\chi_l(x) = h_l((1+|x|^2)^{1/2}) \quad \text{for } x \in \mathbb{R}^n.$$

Here, for $l \in \mathbb{N}$, $h_l \in C^\infty[0,\infty[$ is a function such that $h_l(t) = 1$ if $t \in [0,l]$, $h_l(l+t) = h(t) \in \,]0,1[$ if $t \in \,]0,1[$ (let us stress that h does not depend on l), and $h_l(t) = 0$ if $t \in [l+1,\infty[$.

Horváth's equality (3) from [H, Sect. 2.5, Example 8] implies that for every $\alpha \in \mathbb{N}_0^n$ there is $\mathcal{C}_\alpha \in \,]0,\infty[$ such that

$$\partial^\alpha (1+|x|^2)^{1/2} \le \mathcal{C}_\alpha (1+|x|^2)^{1/2-|\alpha|}(1+|x|^2)^{|\alpha|/2} \le \mathcal{C}_\alpha (1+|x|^2)^{1/2}.$$

Hence, considering partial derivatives of all orders of the composed mappings χ_l, we infer that

$$\text{(f)} \qquad \{\chi_l, 1-\chi_l : l \in \mathbb{N}\} \subset C_b^\infty(\mathbb{R}^n).$$

By Lemma 5 for every $l \in \mathbb{N}$ we have

$$\text{(g)} \qquad \sum_{z\in\mathbb{Z}^n} T(\psi_l \cdot \varphi_z) = T(\psi_l).$$

In order to prove Lemma 6 it is sufficient to show that if $T \in \mathcal{S}'(\mathbb{R}^n)$ satisfies condition (c), $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi_l = \psi \cdot \chi_l$ for every $l \in \mathbb{N}$, then

$$\text{(h)} \qquad \lim_{l\to\infty} \sum_{z\in\mathbb{Z}^n} T(\psi_l \cdot \varphi_z) = \sum_{z\in\mathbb{Z}^n} T(\psi \cdot \varphi_z).$$

and

$$\text{(i)} \qquad \lim_{l\to\infty} T(\psi_l) = T(\psi).$$

Indeed, the equality asserted in Lemma 6 follows immediately from (g), (h) and (i).

We shall concentrate on proving (h) because (i), not involving multiple series, can be proved similarly to (h) by an argument based on (f). Recall that $\{\chi_l, 1-\chi_l : l \in \mathbb{N}\} \subset C_b^\infty(\mathbb{R}^n)$. Since $C_b^\infty(\mathbb{R}^n) \subset \mathcal{O}_M(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n)$ is the function algebra of multipliers of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, for every $l \in \mathbb{N}$ one has

$$\sum_{z\in\mathbb{Z}^n} T(\psi_l \cdot \varphi_z) = \sum_{z\in\mathbb{Z}^n} T(\psi \cdot \chi_l \cdot \varphi_z) = \sum_{z\in\mathbb{Z}^n} \chi_l T(\psi \cdot \varphi_z) = \chi_l \sum_{z\in\mathbb{Z}^n} T(\psi \cdot \varphi_z),$$

so that

$$\begin{aligned}
\sum_{z\in\mathbb{Z}^n} T(\psi_l \cdot \varphi_z) - \sum_{z\in\mathbb{Z}^n} T(\psi \cdot \varphi_z) &= -\sum_{z\in\mathbb{Z}^n} T(\psi \cdot (1-\chi_l) \cdot \varphi_z) \\
&= -\sum_{z\in\mathbb{Z}^n,\, \operatorname{supp}\varphi_z \cap \operatorname{supp}(1-\chi_l) \neq \emptyset} T(\psi \cdot (1-\chi_l) \cdot \varphi_z) \\
&= -\sum_{z\in\mathbb{Z}^n,\, \operatorname{supp}\varphi_z \cap \operatorname{supp}(1-\chi_l) \neq \emptyset} (1-\chi_l) T(\psi \cdot \varphi_z).
\end{aligned}$$

Consequently,

$$\Big|\sum_{z\in\mathbb{Z}^n} T(\psi_l\cdot\varphi_z) - \sum_{z\in\mathbb{Z}^n} T(\psi\cdot\varphi_z)\Big| \le \sum_{z\in\mathbb{Z}^n,\,\operatorname{supp}\varphi_z\subset\operatorname{supp}(1-\chi_l)\neq\emptyset} |T(\psi\cdot\varphi_z)|.$$

Let us stress that, by Lemma 4, the above calculation involves absolutely convergent multiple series. The sum on the right is a tail of the convergent series $\sum_{z\in\mathbb{Z}^n}|T(\psi\cdot\varphi_z)|$. This tail is equal to $\sum_{z\in\tau_l}|T(\psi\cdot\varphi_z)|$ where

$$\tau_l = \{z\in\mathbb{Z}^n : \operatorname{supp}\varphi_z\cap\operatorname{supp}(1-\chi_l)\neq\emptyset\}.$$

When l increases to ∞, the sets $\operatorname{supp}(1-\chi_l)$ decrease in the sense of inclusion, so that also the sets τ_l decrease. Moreover $\bigcap_{l=1}^{\infty}\tau_l=\emptyset$. Therefore

$$\lim_{l\to\infty}\sum_{z\in\mathbb{Z}^n}|T(\psi\cdot\varphi_l)|=0,$$

proving (h). □

Proof of (c)⇒(a)

Assume that $T\in\mathcal{S}'(\mathbb{R}^n)$ satisfies condition (c). If $z\in\mathbb{R}^n$, $\varphi\in\mathcal{D}(\mathbb{R}^n)$, $\{\varphi_z : z\in\mathbb{Z}^n\}$ is a periodic partition of unity on $\mathbb{R}^n$, $\mu\in[0,\infty[$ is fixed and ϕ ranges over $S^1_\mu(\mathbb{R}^n)$, then for every fixed $z\in\mathbb{Z}^n$, $S^1_\mu(\mathbb{R}^n)\ni\phi\mapsto T(\phi\cdot\varphi_z)\in\mathbb{C}$ is a continuous linear functional on $S^1_\mu(\mathbb{R}^n)$, so that

$$\text{(A)}\qquad (U_k(\phi))_{k\in\mathbb{N}} = \Big(\sum_{z\in\mathbb{Z}^n,\,|z|\le k} T(\phi\cdot\varphi_z)\Big)_{k\in\mathbb{N}}$$

is a pointwise convergent sequence of continuous linear functionals on $S^1_\mu(\mathbb{R}^n)$ with limit $U_\infty(\phi)$ equal to the sum of the multiple series $\sum_{z\in\mathbb{Z}_n} T(\phi\cdot\varphi_z)$ (see Lemmas 3 and 4).

If $\psi\in\mathcal{S}(\mathbb{R}^n)$ and $\varphi\in\mathcal{D}(\mathbb{R}^n)$, then, by Lemma 6,

$$\lim_{k\to\infty}(U_k(\psi))_{k\in\mathbb{N}} = \Big(\sum_{z\in\mathbb{Z}^n} T(\phi\cdot\varphi_z)\Big) = T(\psi).$$

Therefore *the limit* $U_\infty(\phi)$ *of the sequence* (A) *is a linear functional on* $S^1_\mu(\mathbb{R}^n)$ *extending* T. Moreover, from [B, Sect. III.3.6, Banach–Steinhaus Theorem] or [H, Sect. 3.6, Proposition 5] it follows that

$$\text{(B)}\qquad S^1_\mu(\mathbb{R}^n)\ni\phi\mapsto U_\infty(\phi)\in\mathbb{C}$$

is a continuous linear functional on $S^1_\mu(\mathbb{R}^n)$.

Consequently, $S^1_\mu(\mathbb{R}^n) \ni \phi \mapsto U_\infty(\phi) \in \mathbb{C}$ is an extension of the distribution T (satisfying condition (c)) to a continuous linear functional on $S^1_\mu(\mathbb{R}^n)$. In view of the discussion of the properties of RD in [K], this means that $S^1_\mu(\mathbb{R}^n) \ni \phi \mapsto U_\infty(\phi) \in \mathbb{C}$ is equal to the extension of T to a continuous linear functional on $S^1_\mu(\mathbb{R}^n)$. Hence (a) holds. □

Remark. The proof of continuity in (B) is difficult, related to uniform structures, equicontinuity, barrelledness and completeness. See for instance [H, Sect. 2.9], [J, Sect. I.9], [O, Sect. 4.2, Theorem 4.16(d)].

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