POISSON ALGEBRAS AND SINGULAR SYMPLECTIC FORMS ASSOCIATED TO A_k TYPE SINGULARITIES

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ABSTRACT. We show that there exists a natural Poisson algebra associated to a singular symplectic structure ω. We construct Poisson algebras for the Martinet and Roussaire types of singularities. In the special case if the singular symplectic structure is given by the pullback from the Darboux form, $\omega = F^*\omega_0$ this Poisson algebra is a basic symplectic invariant of the singularity of the smooth mapping F into symplectic space $(\mathbb{R}^{2n}, \omega_0)$. The case of A_k singularities of pullbacks were considered and Poisson algebras for $\Sigma_{2,0}, \Sigma^e_{2,2,0}, \Sigma^h_{2,2,0}$ stable singularities of 2-forms were calculated.

1. INTRODUCTION

Let ω be the germ of a closed 2-form at $0 \in \mathbb{R}^{2n}$. For a function-germ h at $0 \in \mathbb{R}^{2n}$ and nondegenerate ω, the Hamiltonian vector field of h with respect to ω is the vector field $X_{\omega,h}$ such that (see [11, 21]),

$$\omega(X_{\omega,h}, \xi) = -\xi(h) \tag{1.1}$$

for any vector field ξ on $\mathbb{R}^{2n}$.

If ω is singular, then the smooth vector field $X_{\omega,h}$, defined by the formula (1.1) may not exist (cf. [14, 19, 5]). Thus we define the space of Hamiltonians $\mathcal{H}_\omega$

$$\mathcal{H}_\omega = \{h \in \mathcal{E}_{2n} | X_{\omega,h} \text{ is smooth}\}. \tag{1.2}$$

If $h, k \in \mathcal{H}_\omega$ we show that $\{h, k\}_\omega = \omega(X_{\omega,h}, X_{\omega,k})$ belongs to $\mathcal{H}_\omega$. And under the certain generic condition we prove that $\mathcal{H}_\omega$ equipped with the bracket $\{.,.\}_\omega$ is a Poisson algebra.

Let $(\mathbb{R}^{2n}, \omega_0)$ be a symplectic space with ω_0 in Darboux form. Let θ be the Liouville 1-form on the cotangent bundle $T^*\mathbb{R}^{2n}$. Then $d\theta$ is a standard symplectic structure on $T^*\mathbb{R}^{2n}$. Let $\beta : T\mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ be the canonical bundle map defined by ω_0, $\beta : T\mathbb{R}^{2n} \ni v \mapsto \omega_0(v, \cdot) \in T^*\mathbb{R}^{2n}$. Then we can define the canonical symplectic structure $\dot\omega$ on $T\mathbb{R}^{2n}$, $\dot\omega = \beta^* d\theta = d(\beta^*\theta)$. Throughout the paper unless otherwise stated all objects are germs at 0 of smooth functions, mappings, forms etc. or their representatives on an open neighborhood of 0 in $\mathbb{R}^{2n}$.

Let $\bar F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ be a smooth map-germ. We say that $\bar F$ is isotropic if $\bar F^*\dot\omega = 0$. If we assume that $\bar F : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ is an isotropic map-germ, then the germ of a differential of a 1-form $(\beta \circ \bar F)^*\theta$ vanishes, $d(\beta \circ \bar F)^*\theta = \bar F^*\beta^* d\theta = \bar F^*\dot\omega = 0$.

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Thus $(\beta \circ \bar{F})^*\theta$ is a germ of a closed 1-form. And there exists a smooth function-germ $g : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ such that

$$(\beta \circ \bar{F})^*\theta = -dg. \tag{1.3}$$

For each smooth isotropic map-germ $\bar{F}$ the function-germ g is uniquely defined up to an additive constant.

Let $F : \mathbb{R}^{2n} \to (\mathbb{R}^{2n}, \omega_0)$ be a smooth map. $\pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $F = \pi \circ \bar{F}$. In general, $\bar{F}$ can be regarded as a vector field along F, i.e. a section of an induced fiber bundle $F^*T\mathbb{R}^{2n}$. By $\mathcal{E}_U$ ($\mathcal{E}_{\mathbb{R}^{2n}}$-respectively) we denote the $\mathbb{R}$- algebra of smooth function germs at 0 on U (and on "the target space" $\mathbb{R}^{2n}$ respectively). To each isotropic map-germ $\bar{F}$ along F there exists a unique g belonging to the maximal ideal $\mathbf{m}_U$ of $\mathcal{E}_U$, $g \in \mathbf{m}_U$ which is a generating function-germ for $\bar{F}$.

To F we associate a symplectically invariant algebra $\mathcal{R}_F$ of all generating function-germs, generating all isotropic map-germs $\bar{F}$ along F.

Let $F : \mathbb{R}^{2n} \to (\mathbb{R}^{2n}, \omega_0)$ be as above, then F induces a possibly degenerate two-form $F^*\omega_0$. For a smooth function h defined on $U \subset \mathbb{R}^{2n}$ we formally define the Hamiltonian vector field X_h (which may not be smooth) on U by the equality (1.1) replacing ω by $F^*\omega_0$. To F we associate the Poisson algebra (1.2),

$$\mathcal{H}_F = \{h \in \mathcal{E}_{2n} | X_h \text{ is smooth}\}. \tag{1.4}$$

$\mathcal{H}_F \subset \mathcal{R}_F$ is a Poisson algebra endowed with the Poisson brackets

$$\{k, h\}_{F^*\omega_0} := F^*\omega_0(X_k, X_h). \tag{1.5}$$

If $\bar{F} : (\mathbb{R}^{2n}, 0) \to T\mathbb{R}^{2n}$ is a smooth isotropic map-germ along a smooth map-germ $F : (\mathbb{R}^{2n}, 0) \to \mathbb{R}^{2n}$ such that the regular point set of F is dense, and $h : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ is a generating function-germ of $\bar{F}$. Then $\bar{F}$ is smoothly solvable (cf. [8, 9]) as an implicit differential system if and only if h belongs to the Poisson algebra $\mathcal{H}_F$. Thus elements of $\mathcal{H}_F$ are considered Hamiltonians, which fulfill the equation

$$(\beta \circ dF(X_h))^*\theta = -dh$$

In this paper we introduce the symplectic K-equivalence to classify the smooth map-germs F into symplectic space. Then we use the classified normal forms to investigate the structure of the singular pullback $F^*\omega_0$. In Section 3 we find conditions for a smooth map-germ F, such that $F^*\omega_0$ is a stable 2-form. Calculations are done for Martinet and Roussaire normal forms, but in Section 4 for the special case of A_k type singularities of mappings. Poisson-Lie algebra of singular symplectic form is introduced in Section 5 (cf. [8, 9, 10]). And the Poisson algebras for $\Sigma_{2,0}, \Sigma^e_{2,2,0}, \Sigma^h_{2,2,0}$ stable singularities of 2-forms were calculated in Sections 6 and 7.

2. Normal forms of mappings into symplectic space

Let $F : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ and $G : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ be two C^∞ map-germs, where the target space $\mathbb{R}^{2n}$ is endowed with the standard symplectic structure $\omega_0 = \sum_{i=1}^n dy_i \wedge dx_i$. We say that F and G are *symplectomorphic* if there exist a diffeomorphism-germ $\phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the source space and a symplectomorphism $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the target space such that

$$G = \Phi \circ F \circ \phi \tag{2.1}$$

In this paper, we use new (modified) pre-normal forms of A_k singularities of map germs (cf. [1, 2, 4, 12, 13]). Before that, we give an introductory pre-normal form of not necessarily stable corank 1 map-germs $F : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$.

Proposition 2.1. *(Introductory pre-normal form) Let $G : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ be corank 1 C^∞ map-germ. Then G is symplectomorphic to a map-germ of the form:*

$$\begin{aligned} F &= (f_1, \cdots f_{2n}) \\ f_i(u) &= u_i \ (i \leq 2n-1), \\ f_{2n}(u) &: \quad a\ C^\infty \text{ function.} \end{aligned} \tag{2.2}$$

Proof. Suppose that $G : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ is a corank 1 C^∞ map-germ. Then there exist a C^∞ diffeomorphism $h : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the source space and a C^∞ diffeomorphism $\varphi = (\varphi_1, \cdots, \varphi_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the target space such that

$$\begin{aligned} \varphi_i \circ G \circ h(u_1, \cdots, u_{2n}) &= u_i, \qquad (i < 2n) \\ \varphi_{2n} \circ G \circ h(u_1, \cdots, u_{2n}) &= f(u_1, \cdots, u_{2n}), \end{aligned}$$

where f is a C^∞ function with $\partial f / \partial u_{2n}(0) = 0$.

Then, there is a symplectic diffeomorphism on the target space

$$\psi = (\psi_1, \cdots, \psi_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0) \quad \text{such that} \quad \psi_{2n} = \varphi_{2n}.$$

Next, let

$$\begin{aligned} v_i &= \psi_i \circ G \circ h(u_1, \cdots, u_{2n}) \qquad (i < 2n) \\ v_{2n} &= u_{2n}. \end{aligned}$$

Then, $(v_1, \cdots, v_{2n})$ are coordinates on the source space and we have

$$\begin{aligned} \psi_i \circ G \circ h &= v_i \qquad (i < 2n) \\ \psi_{2n} \circ G \circ h & \quad \text{is a } C^\infty \text{ function of } (v_1, \cdots, v_{2n}). \end{aligned}$$

Q.E.D.

Now for A_k map-germs, we have

Proposition 2.2. *Let $G : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ be an A_k type singularity.*
1) If G is a fold map-germ, i.e. A_1, then G is symplectomorphic to a map-germ of the form:

$$F = (f_1, \cdots, f_{2n})$$

$$f_i(u) = u_i \ (i \leq 2n-1), \tag{2.3}$$

$$f_{2n}(u) = u_{2n}^2. \tag{2.4}$$

2) If G is an A_k type map-germ with $k \geq 2$, then G is symplectomorphic to a map-germ of the form:

$$f_i(u) = u_i \ (i \leq 2n-1),$$

$$f_{2n}(u) = u_{2n}^{k+1} + \sum_{i=1}^{k-1} a_i(u_1, \cdots, u_{2n-1}) u_{2n}^i + b(u_1, \cdots, u_{2n-1}), \tag{2.5}$$

where $a_1(u_1, \cdots, u_{2n-1}), \cdots, a_{k-1}(u_1, \cdots, u_{2n-1}), b(u_1, \cdots, u_{2n-1})$ *are smooth functions and* $da_1, da_2, \cdots, da_{k-1}$ *are linearly independent at the origin.*

3)(Cusp for $n = 1$*)* *If* $G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *is an* A_k *map-germ with* $k \geq 2$*, then* $k = 2$ *and it is symplectomorphic to the normal form of cusp;*

$$F = (f_1, f_2), \quad f_1(u) = u_1, \quad f_2(u) = u_2^3 + u_1 u_2. \tag{2.6}$$

Proof of 1) The proof is almost the same as the proof of Proposition 2.1. Suppose that G is a fold map-germ, i.e. A_1 map-germ. Then there exist a C^∞ diffeomorphism $h : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the source space and a C^∞ diffeomorphism $\varphi = (\varphi_1, \cdots, \varphi_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the target space such that

$$\begin{aligned} \varphi_i \circ G \circ h(u_1, \cdots, u_{2n}) &= u_i, \qquad (i < 2n) \\ \varphi_{2n} \circ G \circ h(u_1, \cdots, u_{2n}) &= u_{2n}^2. \end{aligned}$$

Then, there is a symplectic diffeomorphism on the target space

$$\psi = (\psi_1, \cdots, \psi_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0) \quad \text{such that} \quad \psi_{2n} = \varphi_{2n}.$$

Let

$$\begin{aligned} v_i &= \psi_i \circ G \circ h(u_1, \cdots, u_{2n}) \qquad (i < 2n) \\ v_{2n} &= u_{2n}. \end{aligned}$$

Then, $(v_1, \cdots, v_{2n})$ are coordinates on the source space and we have

$$\begin{aligned} \psi_i \circ G \circ h &= v_i \qquad (i < 2n) \\ \psi_{2n} \circ G \circ h &= u_{2n}^2 = v_{2n}^2. \end{aligned}$$

Q.E.D.

Proof of 2) The first half of the proof is the same as the proof of 1) of Proposition 2.2. Suppose that G is a A_k map-germ. Then, by Morin's theorem (cf. [17]), there exist a C^∞ diffeomorphism $h : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the source space and a C^∞ diffeomorphism $\varphi = (\varphi_1, \cdots, \varphi_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of the target space such that

$$\begin{aligned} \varphi_i \circ G \circ h(u_1, \cdots, u_{2n}) &= u_i, \qquad (i < 2n) \\ \varphi_{2n} \circ G \circ h(u_1, \cdots, u_{2n}) &= u_{2n}^{k+1} + \sum_{i=1}^{k-1} u_i u_{2n}^i. \end{aligned}$$

Then, there is a symplectic diffeomorphism on the target space

$$\psi = (\psi_1, \cdots, \psi_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0) \quad \text{such that} \quad \psi_{2n} = \varphi_{2n}. \tag{2.7}$$

Let

$$\begin{aligned} v_i &= \psi_i \circ G \circ h(u_1, \cdots, u_{2n}) \qquad (i < 2n) \\ v_{2n} &= u_{2n}. \end{aligned}$$

Then, $(v_1, \cdots, v_{2n})$ are coordinates on the source space and we have

$$\begin{aligned} \psi_i \circ G \circ h(v_1, \cdots, v_{2n}) &= v_i \qquad (i < 2n) \\ \psi_{2n} \circ G \circ h(v_1, \cdots, v_{2n}) &= u_{2n} = v_{2n}^{k+1} + \sum_{i=1}^{k-1} u_i(v) v_{2n}^i. \end{aligned}$$

Note that the coefficients $u_i(v)$ are functions of the variables $v_1, v_2, \cdots, v_{2n-1}, v_{2n}$. However the coefficients $u_i(v)$ are desirable to be functions of the variables $v_1, v_2, \cdots, v_{2n-1}$.

Since $u_i(v)$'s are functions of the variables $v_1, \cdots, v_{2n}$, they can be express in the forms

$$u_i(v_1, \cdots, v_{2n}) = \sum_{j=1}^{2n-1} v_j \alpha_{i,j}(v_1, \cdots, v_{2n}) + \beta_i(v_{2n}).$$

Since G is an A_k type map-germ, the order of $\beta_i(v_{2n})$ must be greater than $k-i$:

$$\text{ord } \beta_i(v_{2n}) > k - i,$$

for if ord $\beta_i(v_{2n}) \leq k-i$ then G must be an A_ℓ- singularity for some $\ell < k$.

Then with the coordinates

$$w_i = v_i \quad (i < 2n), \qquad w_{2n} = \sqrt[k+1]{u_{2n}^{k+1} + \sum_{i=1}^{k-1} \beta_i(v_{2n}) v_{2n}^i}$$

in the source space, $\psi_{2n} \circ G \circ h(w_1, \cdots, w_{2n})$ becomes an unfolding of w_{2n}^{k+1} with parameters $w_1, \cdots, w_{2n-1}$ in the sense of Unfolding Theory (see e.g. [20]);

$$\psi_{2n} \circ G \circ h(0, \cdots, 0, w_{2n}) = w_{2n}^{k+1}.$$

Then again under new coordinates of the form

$$\bar{w}_i = w_i = v_i \quad (i < 2n), \qquad \bar{w}_{2n} = \bar{w}_{2n}(v_1, \cdots, v_{2n})$$

$\psi_{2n} \circ G \circ h$ becomes of the form

$$\psi_{2n} \circ G \circ h = \bar{w}_{2n}^{k+1} + \sum_{i=1}^{k-1} \bar{a}_i(\bar{w}_1, \cdots, \bar{w}_{2n-1}) \bar{w}_{2n}^i + b(\bar{w}_1, \cdots, \bar{w}_{2n-1}). \tag{2.8}$$

Note that after (2.7) we have not changed coordinates in the target space. So the map-germ G and the map-germ $\psi \circ G \circ h$

$$\begin{aligned} \psi_i \circ G \circ h(\bar{w}) &= \bar{w}_i \qquad (i < 2n) \\ \psi_{2n} \circ G \circ h(\bar{w}) &= \bar{w}_{2n}^{k+1} + \sum_{i=1}^{k-1} \bar{a}_i(\bar{w}_1, \cdots, \bar{w}_{2n-1}) \bar{w}_{2n}^i + b(\bar{w}_1, \cdots, \bar{w}_{2n-1}). \end{aligned}$$

are symplectomorphic. This completes the proof of 2). Q.E.D.

Remark 2.3. In [7] we proved that if $F : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ is an A_k type singularity, then $F = (f_1, \cdots f_{2n})$ is symplectomorphic to a map-germ of the form:

$$f_i(u) = u_i \ (i \leq 2n-1), \qquad f_{2n}(u) = u_{2n}^{k+1} + \sum_{i=1}^{k-1} a_i(u) u_{2n}^i,$$

where $a_1(u), a_2(u), \cdots, a_{k-1}(u)$ are smooth functions such that $da_1, da_2, \cdots, da_{k-1}$ and du_{2n} are linearly independent at the origin. Let us note that the coefficients $a_i(u)$ in that version of the result are functions of the variables $(u_1, u_2, \cdots, u_{2n-1}, u_{2n})$. However in the new pre-normal form the coefficients $a_i(u)$ are functions of the variables $(u_1, u_2, \cdots, u_{2n-1})$.

Now we want to investigate the induced closed 2-forms $F^*\omega_0$. In order to avoid unnecessarily complicated calculations, we choose the following new coordinates in the target space $(\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^n dy_i \wedge dx_i)$:

$$z_1 = -x_1,\ z_2 = y_1, \cdots,\ z_{2n-1} = -x_n,\ z_{2n} = y_n.$$

Then

$$\omega_0 = dz_1 \wedge dz_2 + \cdots + dz_{2n-1} \wedge dz_{2n}.$$

Following the above change, we also use the corresponding new coordinates in the source space:

$$v_1 = -u_1,\ v_2 = u_{n+1}, \cdots,\ v_{2n-1} = -u_n,\ v_{2n} = u_{2n},$$

In this section, we formulate our results on the induced closed 2-forms $F^*\omega_0$. This is stated for the pre-normal forms of the induced mapping F.

Let $(z_1, \cdots, z_{2n})$ be the standard coordinates in the target space $\mathbb{R}^{2n}$ and let $\omega_0 = dz_1 \wedge dz_2 + \cdots + dz_{2n-1} \wedge dz_{2n}$ be the symplectic form on the target space $\mathbb{R}^{2n}$.

Let $G : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ be a corank 1 map-germ. Then, from Proposition 2.1, G is symplectomorphic to one of the following introductory pre-normal forms:

$$\begin{aligned} F &= (f_1, \cdots, f_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0), \\ f_i(v) &= v_i\ (i \leq 2n-1), \\ f_{2n}(v) &= \text{a } C^\infty \text{ function.} \end{aligned} \tag{2.9}$$

Proposition 2.4. *Let F be the above pre-normal form. Then*

$$\begin{aligned} F^*\omega_0 &= \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + \Delta dv_{2n-1} \wedge dv_{2n} \\ &\quad - \sum_{i \neq 2n-1, 2n} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_{2n-1} \end{aligned} \tag{2.10}$$

where $\Delta = \partial f_{2n}/\partial v_{2n}$ is the jacobian of F.

Proof. Expressing $F^*\omega_0$ as

$$F^*\omega_0 = \sum_{1 \leq i < j \leq 2n} \alpha_{i,j}(u) dv_i \wedge dv_j,$$

we are going to determine the coefficients $\alpha_{i,j}(v)$. Let

$$D_pF : T_p\mathbb{R}^{2n} \to T_{F(p)}\mathbb{R}^{2n}$$

denote the differential of F at a point p. Then

$$\alpha_{i,j}(p) = F^*\omega_0(p)\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right) = \omega_0\left(D_pF\left(\frac{\partial}{\partial v_i}\right), D_pF\left(\frac{\partial}{\partial v_j}\right)\right).$$

The jacobian matrix of F is

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \\ \cdots & \cdots & \partial f_{2n}/\partial v_j & \cdots & \Delta \end{pmatrix}.$$

Therefore

$$\begin{aligned} D_pF\left(\frac{\partial}{\partial v_i}\right) &= \frac{\partial}{\partial z_i} + \frac{\partial f_{2n}}{\partial v_i}\frac{\partial}{\partial z_{2n}}, \qquad (k < 2n) \\ D_pF\left(\frac{\partial}{\partial v_{2n}}\right) &= \Delta\frac{\partial}{\partial z_{2n}}. \end{aligned}$$

So, for $i < j < 2n-1$,

$$\begin{aligned} \alpha_{i,j} &= F^*\omega_0(p)\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right) = \omega_0\left(D_pF\left(\frac{\partial}{\partial v_i}\right), D_pF\left(\frac{\partial}{\partial v_j}\right)\right) \\ &= (dz_1 \wedge dz_2 + \cdots + dz_{2n-1} \wedge dz_{2n})\left(\frac{\partial}{\partial z_i} + \frac{\partial f_{2n}}{\partial v_i}\frac{\partial}{\partial z_{2n}}, \frac{\partial}{\partial z_j} + \frac{\partial f_{2n}}{\partial v_j}\frac{\partial}{\partial z_{2n}}\right) \\ &= \begin{cases} \delta_{i,j-1} & \text{for } i \text{ odd} \\ 0 & \text{for } i \text{ even} \end{cases} \end{aligned}$$

For $i < 2n-1$,

$$\begin{aligned} \alpha_{i,2n-1} &= F^*\omega_0(p)\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_{2n-1}}\right) = \omega_0\left(D_pF\left(\frac{\partial}{\partial v_i}\right), D_pF\left(\frac{\partial}{\partial v_{2n-1}}\right)\right) \\ &= (dz_1 \wedge dz_2 + \cdots + dz_{2n-1} \wedge dz_{2n})\left(\frac{\partial}{\partial z_i} + \frac{\partial f_{2n}}{\partial v_i}\frac{\partial}{\partial z_{2n}}, \frac{\partial}{\partial z_{2n-1}} + \frac{\partial f_{2n}}{\partial v_{2n-1}}\frac{\partial}{\partial z_{2n}}\right) \\ &= -\frac{\partial f_{2n}}{\partial v_i} \end{aligned}$$

For $i < 2n-1$

$$\begin{aligned} \alpha_{i,2n} &= F^*\omega_0(p)\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right) = \omega_0\left(D_pF\left(\frac{\partial}{\partial v_i}\right), D_pF\left(\frac{\partial}{\partial v_{2n}}\right)\right) \\ &= (dz_1 \wedge dz_2 + \cdots + dz_{2n-1} \wedge dz_{2n})\left(\frac{\partial}{\partial z_i} + \frac{\partial f_{2n}}{\partial v_i}\frac{\partial}{\partial z_{2n}}, \Delta\frac{\partial}{\partial z_{2n}}\right) \\ &= 0. \end{aligned}$$

$$\begin{aligned} \alpha_{2n-1,2n} &= F^*\omega_0(p)\left(\frac{\partial}{\partial v_{2n-1}}, \frac{\partial}{\partial v_{2n}}\right) = \omega_0\left(D_pF\left(\frac{\partial}{\partial v_{2n-1}}\right), D_pF\left(\frac{\partial}{\partial v_{2n}}\right)\right) \\ &= (dz_1 \wedge dz_2 + \cdots + dz_{2n-1} \wedge dz_{2n})\left(\frac{\partial}{\partial z_{2n-1}} + \frac{\partial f_{2n}}{\partial v_{2n-1}}\frac{\partial}{\partial z_{2n}}, \Delta\frac{\partial}{\partial z_{2n}}\right) \\ &= \Delta. \end{aligned}$$

Thus we have

$$\begin{aligned} F^*\omega_0 &= \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + \Delta dv_{2n-1} \wedge dv_{2n} \\ &\quad - \sum_{i \neq 2n-1,2n}\left(\sum_{\ell=1}^{k-1}\frac{\partial a_\ell}{\partial v_i}v_{2n}^\ell + \frac{\partial b}{\partial v_i}\right) dv_i \wedge dv_{2n-1} \end{aligned}$$

This completes the proof of Proposition 2.4. Q.E.D.

From now on, we assume that

$$d\Delta(0) \neq 0. \tag{2.11}$$

Let

$$\Delta(v) = \text{the jacobian of } F \text{ at } v, \quad \Delta(v) = \frac{\partial f_{2n}}{\partial v_{2n}}(v), \tag{2.12}$$

$$\Sigma_2(F^*\omega_0) = \{v \in \mathbb{R}^{2n} \mid \Delta(v) = 0\}, \tag{2.13}$$

$$A_{F^*\omega_0}(v) = \{w \in T_v\mathbb{R}^{2n} \mid i(w)F^*\omega_0(v) = 0\} \; : \text{ the kernel of } F^*\omega_0(v), \tag{2.14}$$

where $i(w)F^*\omega_0(v)$ denotes the inner product of vector w and the 2-form $F^*\omega_0(v)$.
Since $d\Delta(0) \neq 0$, $\Sigma_2(F^*\omega_0)$ is a $2n-1$ dimensional submanifold of $\mathbb{R}^{2n}$.

Proposition 2.5. *Suppose that $d\Delta(0) \neq 0$. If $v \in \Sigma_2(F^*\omega_0)$, then $\dim A_{F^*\omega_0}(v) = 2$ and it is spanned by the following two vectors:*

$$e_1 = -\sum_{i=1}^{n-1} \frac{\partial f_{2n}}{\partial v_{2i}} \frac{\partial}{\partial v_{2i-1}} + \sum_{i=1}^{n-1} \frac{\partial f_{2n}}{\partial v_{2i-1}} \frac{\partial}{\partial v_{2i}} + \frac{\partial}{\partial v_{2n-1}}, \tag{2.15}$$

$$e_2 = \frac{\partial}{\partial v_{2n}}. \tag{2.16}$$

Proof. Let $v \in \Sigma_2(F^*\omega_0)$. Since $\dim A_{F^*\omega_0(v)} = 2$ and e_1 and e_2 are linearly independent, it is enough to show that $e_1, e_2 \in A_{F^*\omega_0(v)}$.
From Proposition 2.4. we have

$$F^*\omega_0 = \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + \Delta dv_{2n-1} \wedge dv_{2n} - \sum_{i \neq 2n-1, 2n} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_{2n-1},$$

where $\Delta = \partial f_{2n}/\partial v_{2n}$ is the jacobian of F.
Since $v \in \Sigma_2(F^*\omega_0)$, $\Delta(v) = 0$. Thus

$$F^*\omega_0 = \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} - \sum_{i \neq 2n-1, 2n} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_{2n-1} \qquad \text{on} \quad \Sigma_2(F^*\omega_0).$$

Let

$$e = \sum_{i=1}^{2n} w_i \frac{\partial}{\partial v_i} \in T_v\mathbb{R}^{2n}.$$

Then

$$e \in A_{F^*\omega_0}(v) \quad \text{if and only if} \quad F^*\omega_0(v)\left(e, \frac{\partial}{\partial v_j}\right) = 0 \qquad (j = 1, \cdots, 2n).$$

We solve the following equation for the coefficients $w_1, \cdots, w_{2n}$:

$$F^*\omega_0\left(\sum_{i=1}^{2n} w_i \frac{\partial}{\partial v_i}, \; \frac{\partial}{\partial v_j}\right) = 0, \qquad (j = 1, \cdots, 2n).$$

$$
\begin{aligned}
0 &= F^*\omega_0\left(\sum_{i=1}^{2n} w_i\frac{\partial}{\partial v_i},\ \frac{\partial}{\partial v_{2j-1}}\right) \qquad (j<n)\\
&= \left(\sum_{i=1}^{n-1} dv_{2i-1}\wedge dv_{2i} - \sum_{i\neq 2n-1,2n}\frac{\partial f_{2n}}{\partial v_i}dv_i\wedge dv_{2n-1}\right)\left(\sum_{i=1}^{2n} w_i\frac{\partial}{\partial v_i},\ \frac{\partial}{\partial v_{2j-1}}\right)\\
&= -w_{2j}+\frac{\partial f_{2n}}{\partial v_{2j-1}}w_{2n-1}.\\
0 &= F^*\omega_0\left(\sum_{i=1}^{2n} w_i\frac{\partial}{\partial v_i},\ \frac{\partial}{\partial v_{2j}}\right) = w_{2j-1}+\frac{\partial f_{2n}}{\partial v_{2j}}w_{2n-1} \qquad (j<n).
\end{aligned}
$$

Thus we have

$$
w_{2j-1} = -\frac{\partial f_{2n}}{\partial v_{2j}}w_{2n-1}, \quad w_{2j} = \frac{\partial f_{2n}}{\partial v_{2j-1}}w_{2n-1}. \tag{2.17}
$$

Note that

$$
F^*\omega_0\left(\sum_{i=1}^{2n} w_i\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_{2n}}\right) = 0 \qquad \text{for arbitrary } w_1,\cdots,w_{2n-1},
$$

since $F^*\omega_0$ does not contain the term $\partial/\partial v_{2n}$.

We see also that if we let

$$
w_{2i-1} = -\frac{\partial f_{2n}}{\partial v_{2i}}w_{2n-1}, \quad w_{2j} = \frac{\partial f_{2n}}{\partial v_{2i-1}}w_{2n-1},
$$

then we have automatically

$$
F^*\omega_0\left(\sum_{i=1}^{2n} w_i\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_{2n-1}}\right) = 0.
$$

Thus we have no relations between $w_1,\cdots,w_{2n}$ other than 2.17. Therefore, as a basis of $A_{F^*\omega_0}(v)$, we can choose

$$
\begin{aligned}
e_1 &= -\sum_{i=1}^{n-1}\frac{\partial f_{2n}}{\partial v_{2i}}\frac{\partial}{\partial v_{2i-1}} + \sum_{i=1}^{n-1}\frac{\partial f_{2n}}{\partial v_{2i-1}}\frac{\partial}{\partial v_{2i}} + \frac{\partial}{\partial v_{2n-1}},\\
&\qquad\qquad \text{letting } w_{2n-1}=1,\ w_{2n}=0,\\
e_2 &= \frac{\partial}{\partial v_{2n}}, \qquad \text{letting } w_{2n-1}=0,\ w_{2n}=1.
\end{aligned}
$$

This completes the proof of Proposition (2.5). Q.E.D.

3. Classification of mappings by induced closed 2-forms

In this Section we prove the classification of singularities of corank 1 maps induced by classification of "stable" singularities of closed differential 2-forms (cf. [15, 18, 16]).

Let

$$
\omega = \sum_{1\le i<j\le 2n}\alpha_{i,j}dv_i\wedge dv_j
$$

be the germ of a closed 2-form on $\mathbb{R}^{2n}$ at 0. As a volume form on $\mathbb{R}^{2n}$, we choose

$$\Omega = dv_1 \wedge dv_2 \wedge \cdots \wedge dv_{2n}.$$

Let

$$\omega^n = f\Omega.$$

If $f(0) \neq 0$, then by Darboux's Theorem, ω is isomorphic to the Darboux form

$$dv_1 \wedge dv_2 + dv_3 \wedge dv_4 + \cdots + dv_{2n-1} \wedge dv_{2n}.$$

Now assume that $f(0) = 0$ while $df(0) \neq 0$. Let

$$\Sigma_2(\omega) = \{v \in \mathbb{R}^{2n} \mid f(v) = 0\}.$$

By the condition $df(0) \neq 0$, $\Sigma_2(\omega)$ is a dimension $2n-1$ submanifold of $\mathbb{R}^{2n}$ and at a point $v \in \Sigma_2(\omega)$, the kernel

$$A_\omega(v) = \{w \in T_v\mathbb{R}^{2n} \mid i(w)\omega(v) = 0\}$$

of $\omega(v)$ is a 2-dimensional vector subspace of $T_v\mathbb{R}^{2n}$, where $i(w)\omega(v)$ denotes the inner product of a tangent vector w and a two form ω.

Definition 3.1. *(J. Martinet) Suppose that $f(0) = 0$ while $df(0) \neq 0$. If $A_\omega(0)$ is transversal to $T_0\Sigma_2(\omega)$, we say that ω has a $\Sigma_{2,0}$ singularity at* 0.

Theorem 3.2. *(J. Martinet) If a closed 2-form ω has a $\Sigma_{2,0}$ singularity at 0, then ω is isomorphic to the following closed 2-form*

$$v_1 dv_1 \wedge dv_2 + dv_3 \wedge dv_4 + \cdots + dv_{2n-1} \wedge dv_{2n}.$$

Let us consider the set

$$\Sigma_{2,2}(\omega) = \{v \in \Sigma_2(\omega) \mid A_\omega(v) \subset T_v\Sigma_2(\omega)\}.$$

It is known that $\Sigma_{2,2}(\omega)$ is a dimension $2n-3$ submanifold of $\mathbb{R}^{2n}$.

Definition 3.3. *(J. Martinet) Suppose that $0 \in \Sigma_{2,2}(\omega)$. If $A_\omega(0)$ is transversal to $T_0\Sigma_{2,2}(\omega)$ in $T_0\Sigma_2(\omega)$, then we say that ω has a $\Sigma_{2,2,0}$ singularity at* 0.

Since $\Sigma_{2,2,0}$ singularities of closed 2-forms are classified only for $n = 2$, from now on we only consider closed 2-forms on $\mathbb{R}^4$.

Theorem 3.4. *(R. Roussaire) If a closed 2-form ω on $\mathbb{R}^4$ has a $\Sigma_{2,2,0}$ singularity at 0, then ω is isomorphic to one of the following two closed 2-forms*

$$dv_1 \wedge dv_2 + v_3 dv_2 \wedge dv_3 + d\left(v_1v_3 + v_2v_4 - \frac{v_3^3}{3}\right) \wedge dv_4,$$

$$dv_1 \wedge dv_2 + v_3 dv_2 \wedge dv_3 + d\left(v_1v_3 - v_2v_4 - \frac{v_3^3}{3}\right) \wedge dv_4,$$

Definition 3.5. *If ω is isomorphic to the first one, we say that ω has a $\Sigma^e_{2,2,0}$ (elliptic $\Sigma_{2,2,0}$) singularity at 0, and if ω is isomorphic to the second one, we say that ω has a $\Sigma^h_{2,2,0}$ (hyperbolic $\Sigma_{2,2,0}$) singularity at* 0.

These two cases are distinguished as follows: Suppose that a closed 2-form ω on $\mathbb{R}^4$ has a $\Sigma_{2,2,0}$ singularity at 0. Let Ω be a positive volume form of $\mathbb{R}^4$ with coordinates $v_1, \cdots, v_4$, say, $\Omega = dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4$. Then ω^2 has the form

$$\omega^2 = f\Omega$$

for a function f such that $f(0) = 0, \quad \text{and} \quad df(0) \neq 0.$

Let $\bar{\Omega}_{\Sigma_2(\omega)}$ be a volume form on $\Sigma_2(\omega)$ such that

$$\bar{\Omega}_{\Sigma_2(\omega)} \wedge df \text{ and } \Omega \text{ define the same orientation on } \mathbb{R}^4.$$

Let $\Sigma_2(\omega)$ be oriented in such a way that $\bar{\Omega}_{\Sigma_2(\omega)}$ is a positive volume form on $\Sigma_2(\omega)$. It is known (see [18], p. 147) that there exists a smooth vector filed X on $\Sigma_2(\omega)$ such that

$$(\omega_{|\Sigma_2(\omega)}) = i(X)(\bar{\Omega}_{\Sigma_2(\omega)})$$

where $i(X)(\bar{\Omega}_{\Sigma_2(\omega)})$ is the inner product of the vector filed X with the 3-form $\bar{\Omega}_{\Sigma_2(\omega)}$.

Let w_1, w_2, w_3 be coordinates at 0 on $\Sigma_2(\omega)$ which define a positive orientation on $\Sigma_2(\omega)$. Then the vector field X has the form

$$X = \sum_{i=1}^{3} a_i(w)\frac{\partial}{\partial w_i}.$$

By definition of $\Sigma_{2,2}(\omega)$, ω vanishes on $\Sigma_{2,2}(\omega)$. So, the jacobian matrix of X at 0

$$\left(\frac{\partial a_i}{\partial w_j}(0)\right)$$

has rank 2 and it has two non-zero eigen values $\lambda_{\omega,1}, \lambda_{\omega,2}$ which are known either both real or both imaginary ([18] p.147).

Theorem 3.6. *(R. Roussaire) Let ω has a $\Sigma_{2,2,0}$ singularity at 0.*

1) If the two eigen values $\lambda_{\omega,1}, \lambda_{\omega,2}$ are real, then ω has a $\Sigma^h_{2,2,0}$ singularity at 0.

2) If the two eigen values $\lambda_{\omega,1}, \lambda_{\omega,2}$ are imaginary, then ω has a $\Sigma^e_{2,2,0}$ singularity at 0.

Theorem 3.7. *Let F be an A_k type map-germ of the form (2.9). Then $F^*\omega_0$ is isomorphic to the following Martinet's normal form of $\Sigma_{2,0}$ singularities of closed 2-forms*

$$\sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + v_{2n-1}dv_{2n-1} \wedge dv_{2n}, \qquad (\Sigma_{2,0}) \tag{3.1}$$

if and only if

$$(e_1(\Delta)(0), e_2(\Delta)(0)) \neq (0,0). \tag{3.2}$$

Proof. By (2.9)

$$F^*\omega_0 \;=\; \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + \Delta dv_{2n-1} \wedge dv_{2n} - \sum_{i\neq 2n-1,2n} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_{2n-1}$$

we have

$$(F^*\omega_0)^n = n\Delta dv_1 \wedge dv_2 \wedge \cdots dv_{2n}.$$

Since, by the assumption that $da_1(0) \neq 0$, we have $d\Delta(0) = da_1(0) \neq 0$. So, by definition of $\Sigma_{2,0}$, it is enough to seek the condition for $A_\omega(0)$ to be transversal to $T_0\Sigma_2(\omega)$ at 0.

Since

$$\Sigma_2(\omega) = \{v \in \mathbb{R}^{2n} \mid \Delta(v) = 0\}$$

and $A_\omega(0)$ is spanned by e_1 and e_2, we know that $A_\omega(0)$ is transversal to $T_0\Sigma_2(\omega)$ at 0 if and only if $(e_1(\Delta)(0), e_2(\Delta)(0)) \neq (0,0)$. Thus, from Martinet's theorem, $F^*\omega_0$ is isomorphic to Martinet's normal form of $\Sigma_{2,0}$ if and only if $(e_1(\Delta)(0), e_2(\Delta)(0)) \neq (0,0)$. Q.E.D.

Theorem 3.8. *Suppose that $F^*\omega_0$ is not isomorphic to Martinet's normal form of $\Sigma_{2,0}$ type singularities, i.e. suppose that*

$$(e_1(\Delta)(0), e_2(\Delta)(0)) = (0,0).$$

Then $F^*\omega_0$ is isomorphic to Roussaire's $\Sigma_{2,2,0}$ normal forms if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = 2.$$

Proof. Since

$$\Sigma_{2,2}(F^*\omega_0) = \{v \in \mathbb{R}^4 \mid \Delta(v) = 0,\ e_1(\Delta)(v) = 0, \quad e_2(\Delta)(v) = 0\}$$

and $A_\omega(0)$ is spanned by e_1 and e_2, we know that $A_\omega(0)$ is transversal to $T_0\Sigma_{2,2}(\omega)$ in $T_0\Sigma_2(\omega)$ at 0 if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = 2.$$

Therefore, by definition of $\Sigma_{2,2,0}$, $F^*\omega_0$ is isomorphic to Roussaire's $\Sigma_{2,2,0}$ if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = 2,$$

which holds if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = 2,$$

for $(e_1(\Delta)(0), e_2(\Delta)(0)) = (0,0)$ by assumption. Q.E.D.

Let

$$\begin{aligned} F &= (f_1, \cdots, f_4) : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0), \\ f_i(v) &= v_i \ (i \leq 3), \\ f_4(v) &: \text{ a } C^\infty \text{ function germ.} \end{aligned}$$

be the pre-normal form of corank 1 map-germs given in Proposition 2.1 such that

$$d\Delta(0) \neq 0, \quad (e_1(\Delta)(0), e_2(\Delta)(0)) = (0,0),$$

$$\operatorname{rank}\begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = 2,$$

where

$$\begin{aligned} \Delta &= \frac{\partial f_4}{\partial v_4}, \\ e_1 &= -\frac{\partial f_4}{\partial v_2}\frac{\partial}{\partial v_1} + \frac{\partial f_4}{\partial v_1}\frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_3}, \\ e_2 &= \frac{\partial}{\partial v_4}. \end{aligned}$$

Then by Theorem 3.8, $F^*\omega_0$ is of type $\Sigma_{2,2,0}$.

Recall that

$$\begin{aligned} F^*\omega_0 &= dv_1 \wedge dv_2 + \Delta dv_3 \wedge dv_4 - \sum_{i=1,2} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_3 \\ (F^*\omega_0)^2 &= f\Omega = 2\Delta\Omega \end{aligned}$$

where $\Omega = dv_1 \wedge dv_2 \wedge \Delta dv_3 \wedge dv_4$.

Since $d\Delta(0) \neq 0$,

$$\Sigma_2(F^*\omega_0) = \{v = (v_1, \cdots, v_4) \in \mathbb{R}^4 \mid \Delta = 0\}$$

is a 3-dimensional submanifold of $\mathbb{R}^4$ and

$$(\partial\Delta/\partial v_i)(0) \neq 0 \quad \text{for some } i = 1, \cdots, 4.$$

Since $(e_1(\Delta)(0), e_2(\Delta)(0)) = (0, 0)$, we have

$$\begin{aligned} \frac{\partial \Delta}{\partial v_4}(0) &= 0. \\ \left(-\frac{\partial f_4}{\partial v_2}\frac{\partial}{\partial v_1} + \frac{\partial f_4}{\partial v_1}\frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_3}\right)\Delta(0) &= 0. \end{aligned}$$

If $\partial\Delta/\partial v_1(0) = 0$ and $\partial\Delta/\partial v_2(0) = 0$, then by the above formula we have $\partial\Delta/\partial v_3(0) = 0$, which contradicts the fact that $d\Delta(0) \neq 0$. Thus we have

Lemma 3.9.

$$\frac{\partial \Delta}{\partial v_1}(0) \neq 0 \quad \textit{or} \quad \frac{\partial \Delta}{\partial v_2}(0) \neq 0.$$

and

Lemma 3.10. *We may assume that*

$$\frac{\partial \Delta}{\partial v_1}(0) \neq 0,$$

if necessary, after the following changes of coordinates

$$\bar{z}_1 = -z_2,\ \bar{z}_2 = z_1,\ \bar{z}_3 = -z_3,\ \bar{z}_4 = z_4, \quad \textit{in the target space},$$

$$\bar{v}_1 = -v_2,\ \bar{v}_2 = v_1,\ \bar{v}_3 = -v_3,\ \bar{v}_4 = v_4, \quad \textit{in the source space.}$$

So, we assume that

$$(\partial\Delta/\partial v_1)(0) \neq 0.$$

Then, by the implicit function theorem, there is a function $\varphi(v_2, v_3, v_4)$ such that

$$\begin{aligned}\Sigma_2(F^*\omega_0) &= \{v \in \mathbb{R}^4 \mid \Delta(v) = 0\} \\ &= \{(v_1, \cdots, v_4) \in \mathbb{R}^4 \mid v_1 = \varphi(v_2, v_3, v_4)\}\end{aligned}$$

and we may choose

$$v_2, \quad v_3, \quad v_4$$

as coordinates on $\Sigma_2(F^*\omega_0)$. Let us denote

$$\begin{aligned}\alpha_2 &= -\frac{\partial f_4}{\partial v_1}\frac{\partial \varphi}{\partial v_4}, \\ \alpha_3 &= -\frac{\partial \varphi}{\partial v_4}, \\ \alpha_4 &= \frac{\partial \varphi}{\partial v_3} - \frac{\partial f_4}{\partial v_1}\frac{\partial \varphi}{\partial v_2} - \frac{\partial f_4}{\partial v_2}\end{aligned}$$

and consider the jacobian matrix of $\alpha_2, \alpha_3, \alpha_4$;

$$\left(\frac{\partial \alpha_i}{\partial v_j}(0)\right)_{2 \le i,j \le 4}.$$

Lemma 3.11.

$$\operatorname{rank}\left(\frac{\partial \alpha_i}{\partial v_j}(0)\right)_{2 \le i,j \le 4} = 2.$$

Theorem 3.12. *Let the situation be as above. Then*
1) $F^*\omega_0$ *is isomorphic to Roussaire's normal form* $\Sigma^h_{2,2,0}$ *if and only if*

the two nonzero eigen values of $\left(\frac{\partial \alpha_i}{\partial v_j}(0)\right)_{2 \le i,j \le 4}$ *are real.*

2) $F^*\omega_0$ *is isomorphic to Roussaire's normal form* $\Sigma^e_{2,2,0}$ *if and only if*

the two nonzero eigen values of $\left(\frac{\partial \alpha_i}{\partial v_j}(0)\right)_{2 \le i,j \le 4}$ *are imaginary.*

Proof. Let $\iota = (\iota_1, \cdots, \iota_4) : \Sigma_2(F^*\omega_0) \to \mathbb{R}^4$;

$$\iota(\bar{v}_2,\ \bar{v}_3,\ \bar{v}_4) = (\varphi(\bar{v}_2,\ \bar{v}_3,\ \bar{v}_4), \bar{v}_2,\ \bar{v}_3,\ \bar{v}_4)$$

be the inclusion map. Then we can easily check that

$$d\bar{v}_2 \wedge d\bar{v}_3 \wedge d\bar{v}_4 = \iota^*(dv_2 \wedge dv_3 \wedge dv_4).$$

Set

$$\bar{\Omega}_{\Sigma_2(F^*\omega_0)} = -d\bar{v}_2 \wedge d\bar{v}_3 \wedge d\bar{v}_4 = -\iota^*(dv_2 \wedge dv_3 \wedge dv_4).$$

Then,

$$\Omega = dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4 \quad \text{and} \quad \bar{\Omega}_{\Sigma_2(F^*\omega_0)} \wedge df = 2\bar{\Omega}_{\Sigma_2(F^*\omega_0)} \wedge d\Delta$$

define the same orientation on $\mathbb{R}^4$, where recall that the function f was defined by the equality

$$(F^*\omega_0)^2 = f\Omega$$

and recall also

$$\begin{aligned} F^*\omega_0 &= dv_1 \wedge dv_2 + \Delta dv_3 \wedge dv_4 - \sum_{i=1,2} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_3 \\ (F^*\omega_0)^2 &= 2\Delta\Omega. \end{aligned}$$

Now we seek the vector field X on $\Sigma_2(F^*\omega_0)$ such that

$$F^*\omega_{0|\Sigma_2(F^*\omega_0)} = i(X)(\bar{\Omega}_{\Sigma_2(F^*\omega_0)}).$$

The jacobian matrix of the inclusion map $\iota(\bar{v}_2,\ \bar{v}_3,\ \bar{v}_4) = (\varphi(\bar{v}_2,\ \bar{v}_3,\ \bar{v}_4), \bar{v}_2,\ \bar{v}_3,\ \bar{v}_4)$ is

$$\begin{pmatrix} \frac{\partial\varphi}{\partial\bar{v}_2}, & \frac{\partial\varphi}{\partial\bar{v}_3} & \frac{\partial\varphi}{\partial\bar{v}_3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, letting $D\iota$ denote the differential of ι, we have

$$\begin{aligned} D\iota\left(\frac{\partial}{\partial\bar{v}_2}\right) &= \frac{\partial\varphi}{\partial\bar{v}_2}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}, \\ D\iota\left(\frac{\partial}{\partial\bar{v}_3}\right) &= \frac{\partial\varphi}{\partial\bar{v}_3}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_3} \\ D\iota\left(\frac{\partial}{\partial\bar{v}_4}\right) &= \frac{\partial\varphi}{\partial\bar{v}_4}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_4}. \end{aligned}$$

Now, we have

$$\begin{aligned} & F^*\omega_{0|\Sigma_2(F^*\omega_0)}\left(\frac{\partial}{\partial\bar{v}_2}, \frac{\partial}{\partial\bar{v}_3}\right) = \iota^*F^*\omega_0\left(\frac{\partial\varphi}{\partial\bar{v}_2}, \frac{\partial}{\partial\bar{v}_3}\right) \\ &= F^*\omega_0\left(\frac{\partial\varphi}{\partial\bar{v}_2}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}, \frac{\partial\varphi}{\partial\bar{v}_3}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_3}\right) \\ &= \left(dv_1 \wedge dv_2 + \Delta dv_3 \wedge dv_4 - \sum_{i=1,2}\frac{\partial f_4}{\partial v_i} dv_i \wedge dv_3\right) \\ &\qquad \left(\frac{\partial\varphi}{\partial\bar{v}_2}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}, \frac{\partial\varphi}{\partial\bar{v}_3}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_3}\right) \\ &= \left(dv_1 \wedge dv_2 - \sum_{i=1,2}\frac{\partial f_4}{\partial v_i} dv_i \wedge dv_3\right) \quad (\text{since } \Delta = 0 \text{ on } \Sigma_2(F^*\omega_0)) \\ &\qquad \left(\frac{\partial\varphi}{\partial\bar{v}_2}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}, \frac{\partial\varphi}{\partial\bar{v}_3}\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_3}\right) \\ &= -\frac{\partial\varphi}{\partial\bar{v}_3} - \frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial\bar{v}_2} - \frac{\partial f_4}{\partial v_2}. \end{aligned}$$

$$
\begin{aligned}
&F^*\omega_{0|\Sigma_2(F^*\omega_0)}\left(\frac{\partial}{\partial \bar v_2},\frac{\partial}{\partial \bar v_4}\right)\\
&\quad=\left(dv_1\wedge dv_2-\sum_{i=1,2}\frac{\partial f_4}{\partial v_i}dv_i\wedge dv_3\right)\left(\frac{\partial\varphi}{\partial \bar v_2}\frac{\partial}{\partial v_1}+\frac{\partial}{\partial v_2},\frac{\partial\varphi}{\partial \bar v_4}\frac{\partial}{\partial v_1}+\frac{\partial}{\partial v_4}\right)\\
&\quad=-\frac{\partial\varphi}{\partial \bar v_4}.
\end{aligned}
$$

$$
\begin{aligned}
&F^*\omega_{0|\Sigma_2(F^*\omega_0)}\left(\frac{\partial}{\partial \bar v_3},\frac{\partial}{\partial \bar v_4}\right)\\
&\quad=\left(dv_1\wedge dv_2-\sum_{i=1,2}\frac{\partial f_4}{\partial v_i}dv_i\wedge dv_3\right)\left(\frac{\partial\varphi}{\partial \bar v_3}\frac{\partial}{\partial v_1}+\frac{\partial}{\partial v_3},\frac{\partial\varphi}{\partial \bar v_4}\frac{\partial}{\partial v_1}+\frac{\partial}{\partial v_4}\right)\\
&\quad=\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial \bar v_4}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
F^*\omega_{0|\Sigma_2(F^*\omega_0)}\left(\frac{\partial}{\partial \bar v_2},\frac{\partial}{\partial \bar v_3}\right) &= -\frac{\partial\varphi}{\partial \bar v_3}-\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial \bar v_2}-\frac{\partial f_4}{\partial v_2},\\
F^*\omega_{0|\Sigma_2(F^*\omega_0)}\left(\frac{\partial}{\partial \bar v_2},\frac{\partial}{\partial \bar v_4}\right) &= -\frac{\partial\varphi}{\partial \bar v_4},\\
F^*\omega_{0|\Sigma_2(F^*\omega_0)}\left(\frac{\partial}{\partial \bar v_3},\frac{\partial}{\partial \bar v_4}\right) &= \frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial \bar v_4}.
\end{aligned}
$$

And also

$$
\begin{aligned}
&F^*\omega_{0|\Sigma_2(F^*\omega_0)}\\
&\quad=\left(-\frac{\partial\varphi}{\partial \bar v_3}-\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial \bar v_2}-\frac{\partial f_4}{\partial v_2}\right)d\bar v_2\wedge d\bar v_3-\frac{\partial\varphi}{\partial \bar v_4}d\bar v_2\wedge d\bar v_4+\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial \bar v_4}d\bar v_3\wedge d\bar v_4.
\end{aligned}
$$

Letting

$$
X=\sum_{i=2}^{4}\alpha_i(\bar v_2,\ \bar v_3,\ \bar v_4)\frac{\partial}{\partial \bar v_i}
$$

we solve the equation

$$
F^*\omega_{0|\Sigma_2(F^*\omega_0)}=i(X)(\bar\Omega_{\Sigma_2(F^*\omega_0)}).
$$

Recall that

$$
\bar\Omega_{\Sigma_2(F^*\omega_0)}=-d\bar v_2\wedge d\bar v_3\wedge d\bar v_4.
$$

Then we have

$$
\begin{aligned}
-\frac{\partial\varphi}{\partial\bar{v}_3}-\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial\bar{v}_2}-\frac{\partial f_4}{\partial v_2} &= F^*\omega_{0|\Sigma_2(F^*\omega_0)}\left(\frac{\partial}{\partial\bar{v}_2},\frac{\partial}{\partial\bar{v}_3}\right)\\
&= i(X)(\bar{\Omega}_{\Sigma_2(F^*\omega_0)})\left(\frac{\partial}{\partial\bar{v}_2},\frac{\partial}{\partial\bar{v}_3}\right)\\
&= -d\bar{v}_2\wedge d\bar{v}_3\wedge d\bar{v}_4\left(\sum_{i=2}^{4}\alpha_i\frac{\partial}{\partial\bar{v}_i},\frac{\partial}{\partial\bar{v}_2},\frac{\partial}{\partial\bar{v}_3}\right)\\
&= -\alpha_4.\\
-\frac{\partial\varphi}{\partial\bar{v}_4} &= F^*\omega_{0|\Sigma_2(F^*\omega_0)}\left(\frac{\partial}{\partial\bar{v}_2},\frac{\partial}{\partial\bar{v}_4}\right)\\
&= -d\bar{v}_2\wedge d\bar{v}_3\wedge d\bar{v}_4\left(\sum_{i=2}^{4}\alpha_i\frac{\partial}{\partial\bar{v}_i},\frac{\partial}{\partial\bar{v}_2},\frac{\partial}{\partial\bar{v}_4}\right)\\
&= \alpha_3\\
\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial\bar{v}_4} &= F^*\omega_{0|\Sigma_2(F^*\omega_0)}\left(\frac{\partial}{\partial\bar{v}_3},\frac{\partial}{\partial\bar{v}_4}\right)\\
&= -d\bar{v}_2\wedge d\bar{v}_3\wedge d\bar{v}_4\left(\sum_{i=2}^{4}\alpha_i\frac{\partial}{\partial\bar{v}_i},\frac{\partial}{\partial\bar{v}_3},\frac{\partial}{\partial\bar{v}_4}\right)\\
&= -\alpha_2.
\end{aligned}
$$

Now we consider the jacobian matrix

$$\left(\frac{\partial\alpha_i}{\partial\bar{v}_j}(0)\right)_{2\le i,j\le 4}$$

at 0 of the coefficients

$$\left(\alpha_2=-\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial\bar{v}_4},\ \alpha_3=-\frac{\partial\varphi}{\partial\bar{v}_4},\ \alpha_4=\frac{\partial\varphi}{\partial\bar{v}_3}-\frac{\partial f_4}{\partial v_1}\frac{\partial\varphi}{\partial\bar{v}_2}-\frac{\partial f_4}{\partial v_2}\right)$$

of the vector field X. According to Roussaire's Theorem, we see that

$$\operatorname{rank}\left(\frac{\partial\alpha_i}{\partial\bar{v}_j}(0)\right)_{2\le i,j\le 4}=2,$$

which implies Lemma 3.11, and we see that

1) $F^*\omega_0$ is isomorphic to Roussaire's normal form $\Sigma^h_{2,2,0}$ if and only if

$$\text{the two nonzero eigen values of }\left(\frac{\partial\alpha_i}{\partial\bar{v}_j}(0)\right)_{2\le i,j\le 4}\text{ are real.}$$

2) $F^*\omega_0$ is isomorphic to Roussaire's normal form $\Sigma^e_{2,2,0}$ if and only if

$$\text{the two nonzero eigen values of }\left(\frac{\partial\alpha_i}{\partial\bar{v}_j}(0)\right)_{2\le i,j\le 4}\text{ are imaginary.}$$

This completes the proof of the Theorem 3.12. Q.E.D.

4. Conditions for A_k type singularities

In this section we apply the results of the previous sections to various examples containing A_k map-germs.

First we recall that for an introductory pre-normal form of corank 1 C^∞ map-germs

$$\begin{aligned} F &= (f_1, \cdots f_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0) \\ f_i(u) &= u_i \ (i \le 2n-1), \\ f_{2n}(u) &: \quad \text{a } C^\infty \text{ function} \end{aligned} \tag{4.1}$$

such that $d\Delta(0) \neq 0$, we have

$$F^*\omega_0 = \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + \Delta dv_{2n-1} \wedge dv_{2n} - \sum_{i \neq 2n-1, 2n} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_{2n-1},$$

where $\Delta = \partial f_{2n}/\partial v_{2n}$ is the jacobian of F, and that $\dim A_{F^*\omega_0}(v) = 2$ and it is spanned by the following two vectors:

$$\begin{aligned} e_1 &= -\sum_{i=1}^{n-1} \frac{\partial f_{2n}}{\partial v_{2i}} \frac{\partial}{\partial v_{2i-1}} + \sum_{i=1}^{n-1} \frac{\partial f_{2n}}{\partial v_{2i-1}} \frac{\partial}{\partial v_{2i}} + \frac{\partial}{\partial v_{2n-1}}, \\ e_2 &= \frac{\partial}{\partial v_{2n}}. \end{aligned}$$

Let F be a map-germ of the form (4.1) such that $d\Delta(0) \neq 0$. Then $F^*\omega_0$ is isomorphic to the following Martinet's normal form of $\Sigma_{2,0}$ singularities of closed 2-forms

$$\sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + v_{2n-1} dv_{2n-1} \wedge dv_{2n}, \qquad (\Sigma_{2,0}) \tag{4.2}$$

if and only if

$$(e_1(\Delta)(0), e_2(\Delta)(0)) \neq (0, 0). \tag{4.3}$$

Let F be a fold map-germ;

$$\begin{aligned} F &= (f_1, \cdots, f_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0), \\ f_i(v) &= v_i \ (i \le 2n-1), \\ f_{2n}(v) &= v_{2n}^2. \end{aligned}$$

Then

$$F^*\omega_0 = \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + 2v_{2n} dv_{2n-1} \wedge dv_{2n}.$$

The above form is obviously isomorphic to Martinet's normal form $\Sigma_{2,0}$ given in Theorem 3.7:

$$\sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + v_{2n-1} dv_{2n-1} \wedge dv_{2n}, \qquad (\Sigma_{2,0}) \tag{4.4}$$

Since

$$\Delta = 2v_{2n}, \qquad e_1(\Delta) = 0, \quad e_2(\Delta) = 2,$$

and

$$(e_1(\Delta)(0), e_2(\Delta)(0)) = (0, 2) \neq (0, 0),$$

$F^*\omega_0$ satisfies the condition given in Theorem 3.7 to be isomorphic to Martinet's normal form $\Sigma_{2,0}$.

Proposition 4.1. (A_k map-germs, $k \geq 2$) *Let* $F = (f_1, \cdots, f_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ *be an* A_k *map-germ of the form*

$$\begin{aligned} f_i(v) &= v_i \quad (i \leq 2n-1), \\ f_{2n}(v) &= v_{2n}^{k+1} + \sum_{i=1}^{k-1} a_i(v_1, \cdots, v_{2n-1}) v_{2n}^i + b(v_1, \cdots, v_{2n-1}), \quad (k \geq 2), \end{aligned}$$

in particular, when $n = 1$, *let* $F = (f_1, f_2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *be a cusp map-germ;*

$$f_1(v) = v_1, \quad f_2(v) = v_2^3 + v_1 v_2.$$

Then

1) $F^*\omega_0$ *is isomorphic to the above Martinet's normal form if and only if*

$$e_1(\Delta)(0) \neq 0, \tag{4.5}$$

equivalently if and only if

$$\frac{\partial a_1}{\partial v_{2n-1}}(0) + \sum_{i=1}^{n-1} -\frac{\partial a_1}{\partial v_{2i}}(0)\frac{\partial b}{\partial v_{2i-1}}(0) + \frac{\partial a_1}{\partial v_{2i-1}}(0)\frac{\partial b}{\partial v_{2i}}(0) \neq 0.$$

2) In particular, if $b = 0$, $F^*\omega_0$ *is isomorphic to Martinet's normal form if and only if*

$$\frac{\partial a_1}{\partial v_{2n-1}}(0) \neq 0.$$

3) If $n = 1$, *then, for the cusp map -germ*

$$F = (f_1, f_2), \quad f_1(v) = v_1, \quad f_2(v) = v_2^3 + v_1 v_2,$$

$F^*\omega_0$ *is isomorphic to Martinet's normal form.*

Proof. 1) Since $k \geq 2$, $e_2(\Delta)(0) = 0$. So,

$$(e_1(\Delta)(0), e_2(\Delta)(0)) \neq (0, 0) \quad \text{if and only if} \quad e_1(\Delta)(0) \neq 0.$$

Thus $F^*\omega_0$ is isomorphic to the above Martinet's normal form if and only if

$$e_1(\Delta)(0) \neq 0,$$

equivalently if and only if

$$\frac{\partial a_1}{\partial v_{2n-1}}(0) + \sum_{i=1}^{n-1} -\frac{\partial a_1}{\partial v_{2i}}(0)\frac{\partial b}{\partial v_{2i-1}}(0) + \frac{\partial a_1}{\partial v_{2i-1}}(0)\frac{\partial b}{\partial v_{2i}}(0) \neq 0.$$

2) and 3) easily follow from 1). Q.E.D.

Example 4.2. Consider the following two map-germs:

$$\begin{aligned} F_1 &= (f_1,\cdots,f_{2n}):(\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0),\\ f_i(v) &= v_i \ (i\le 2n-1),\\ f_{2n}(v) &= v_{2n}^3+v_{2n-1}v_{2n}, \end{aligned}$$

$$\begin{aligned} F_2 &= (f_1,\cdots,f_{2n}):G:(\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0),\\ f_i(v) &= v_i \ (i\le 2n-1),\\ f_{2n}(v) &= v_{2n}^3+v_iv_{2n}, \qquad (i<2n-1.) \end{aligned}$$

1) $F_1^*\omega_0$ is isomorphic to Martinet's normal form, since

$$\frac{\partial a_1}{\partial v_{2n-1}}(0)=\frac{\partial v_{2n-1}}{\partial v_{2n-1}}(0)=1\neq 0,$$

2) $F_2^*\omega_0$ is not isomorphic to Martinet's normal form, since

$$\frac{\partial a_1}{\partial v_{2n-1}}(0)=\frac{\partial v_i}{\partial v_{2n-1}}(0)=0.$$

Example 4.3. We revise F_2 in Example 1 adding the term b as follows:

$$\begin{aligned} F_3 &= (f_1,\cdots,f_{2n}):(\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0),\\ f_i(v) &= v_i \ (i\le 2n-1),\\ f_{2n}(v) &= v_{2n}^3+v_{2i-1}v_{2n}+v_{2i} \quad (\text{or} \quad v_{2n}^3+v_{2i}v_{2n}+v_{2i-1}), \qquad (i<n.) \end{aligned}$$

Then $F_3^*\omega_0$ is isomorphic to Martinet's normal form, since

$$\begin{aligned} e_1(\Delta)(0) &= \frac{\partial a_1}{\partial v_{2n-1}}(0)+\sum_{i=1}^{n-1}-\frac{\partial a_1}{\partial v_{2i}}(0)\frac{\partial b}{\partial v_{2i-1}}(0)+\frac{\partial a_1}{\partial v_{2i-1}}(0)\frac{\partial b}{\partial v_{2i}}(0)\\ &= -\frac{\partial a_1}{\partial v_{2i}}(0)\frac{\partial b}{\partial v_{2i-1}}(0)+\frac{\partial a_1}{\partial v_{2i-1}}(0)\frac{\partial b}{\partial v_{2i}}(0)=\pm 1\neq 0, \end{aligned}$$

Example 4.4. Let

$$\begin{aligned} F_4 &= (f_1,\cdots,f_{2n}):(\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0),\\ f_i(v) &= v_i \ (i\le 2n-1),\\ f_{2n}(v) &= v_{2n-1}v_{2n}, \qquad (i<n). \end{aligned}$$

Then, although F_4 is very degenerate as a map-germ, $F_4^*\omega_0$ is stable as a closed 2-form and isomorphic to Martinet's normal form, since

$$\Delta = v_{2n-1}$$

and

$$\begin{aligned} e_1(\Delta)(0) &= \frac{\partial v_{2n-1}}{\partial v_{2n-1}}(0)+\sum_{i=1}^{n-1}-\frac{\partial v_{2n-1}}{\partial v_{2i}}(0)\frac{\partial b}{\partial v_{2i-1}}(0)+\frac{\partial v_{2n-1}}{\partial v_{2i-1}}(0)\frac{\partial b}{\partial v_{2i}}(0)\\ &= \frac{\partial v_{2n-1}}{\partial v_{2n-1}}(0)\frac{\partial b}{\partial v_{2i}}(0)=1\neq 0. \end{aligned}$$

Since the classification of $\Sigma_{2,2,0}$ singularities of closed 2-forms is completed only for $n=2$, we consider only the case where $n=2$. In this case, we consider the introductory pre-normal form

$$\begin{aligned} F &= (f_1,\cdots,f_4):(\mathbb{R}^4,0)\to(\mathbb{R}^4,0),\\ f_i(v) &= v_i\ (i\le 3),\\ f_4(v) &: \ \text{a } C^\infty \text{ function} \end{aligned}$$

such that

$$d\Delta(0)=\frac{\partial f_4}{\partial v_4}(0)\neq 0.$$

Then,

$$\begin{aligned} e_1 &= -\frac{\partial f_4}{\partial v_2}\frac{\partial}{\partial v_1}+\frac{\partial f_4}{\partial v_1}\frac{\partial}{\partial v_2}+\frac{\partial}{\partial v_3},\\ e_2 &= \frac{\partial}{\partial v_4}. \end{aligned}$$

Let us suppose that $F^*\omega_0$ is not isomorphic to Martinet's normal form of $\Sigma_{2,0}$ type singularities, i.e. suppose that

$$e_1(\Delta)(0)=\frac{\partial a_1}{\partial v_3}(0)-\frac{\partial a_1}{\partial v_2}(0)\frac{\partial b}{\partial v_1}(0)+\frac{\partial a_1}{\partial v_1}(0)\frac{\partial b}{\partial v_2}(0)=0.$$

Then $F^*\omega_0$ is isomorphic to Roussaire's $\Sigma_{2,2,0}$ normal form (see Theorem 3.8) if and only if

$$\operatorname{rank}\begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0)\\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0)\end{pmatrix}=2.$$

Theorem 4.5. *Let $F=(f_1,\cdots,f_4):(\mathbb{R}^4,0)\to(\mathbb{R}^4,0)$ be an A_k map-germ with $b=0$ of the form*

$$\begin{aligned} f_i(v) &= v_i \quad (i\le 3),\\ f_4(v) &= v_4^{k+1}+\sum_{i=1}^{k-1}a_i(v_1,v_2,v_3)v_4^i, \quad (2\le k\le 4) \end{aligned}$$

such that $F^*\omega_0$ is not isomorphic to Martinet's normal form of $\Sigma_{2,0}$ type singularities. Then $F^*\omega_0$ is isomorphic to Roussaire's $\Sigma_{2,2,0}$ normal forms

$$dv_1\wedge dv_2+v_3dv_2\wedge dv_3+d\left(v_1v_3+v_2v_4-\frac{v_3^3}{3}\right)\wedge dv_4 \quad (\Sigma^e_{2,2,0})$$

or

$$dv_1\wedge dv_2+v_3dv_2\wedge d_3+d\left(v_1v_3-v_2v_4-\frac{v_3^3}{3}\right)\wedge dv_4 \quad (\Sigma^h_{2,2,0})$$

if and only if

$$\begin{cases} \operatorname{rank}\begin{pmatrix} \frac{\partial^2 a_1}{\partial {v_3}^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0)\\ 2\frac{\partial a_2}{\partial v_3}(0) & 6\end{pmatrix}=2 & (k=2)\\ \operatorname{rank}\begin{pmatrix} \frac{\partial^2 a_1}{\partial {v_3}^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0)\\ 2\frac{\partial a_2}{\partial v_3}(0) & 0\end{pmatrix}=2 & (k=3,4)\end{cases}$$

Proof. In this case,

$$
\begin{aligned}
\Delta &= (k+1)v_4^k + \sum_{i=1}^{k-1} i a_i(v_1,v_2,v_3)v_4^{i-1} \\
e_1 &= -\frac{\partial f_4}{\partial v_2}\frac{\partial}{\partial v_1} + \frac{\partial f_4}{\partial v_1}\frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_3} = -\left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_2}v_4^i\right)\frac{\partial}{\partial v_1} + \left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_1}v_4^i\right)\frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_3}, \\
e_2 &= \frac{\partial}{\partial v_4}, \\
e_1(\Delta) &= -\left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_2}v_4^i\right)\left(\sum_{j=1}^{k-1} j\frac{\partial a_j}{\partial v_1}v_4^{j-1}\right) + \left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_1}v_4^i\right)\left(\sum_{j=1}^{k-1} j\frac{\partial a_j}{\partial v_2}v_4^{j-1}\right) \\
&\quad + \sum_{j=1}^{k-1} j\frac{\partial a_j}{\partial v_3}v_4^{j-1}, \\
e_2(\Delta) &= (k+1)kv_4^{k-1} + \sum_{j=2}^{k-1} j(j-1)a_j v_4^{j-2},
\end{aligned}
$$

$$
\begin{aligned}
e_1(\Delta)(0) &= 0, \qquad \text{by the assumption}, \\
e_1(e_1(\Delta))(0) &= \frac{\partial}{\partial v_3}\left(\frac{\partial a_1}{\partial v_3}\right)(0) = \frac{\partial^2 a_1}{\partial {v_3}^2}(0), \\
e_2(e_1(\Delta))(0) &= \frac{\partial}{\partial v_4}\left(-\left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_2}v_4^i\right)\left(\sum_{j=1}^{k-1} j\frac{\partial a_j}{\partial v_1}v_4^{j-1}\right) + \left(\sum_{i=1}^{k-1}\frac{\partial a_i}{\partial v_1}v_4^i\right)\left(\sum_{j=1}^{k-1} j\frac{\partial a_j}{\partial v_2}v_4^{j-1}\right)\right. \\
&\qquad \left. + \sum_{j=1}^{k-1} j\frac{\partial a_j}{\partial v_3}v_4^{j-1}\right)(0) \\
&= -\frac{\partial a_1}{\partial v_2}(0)\frac{\partial a_1}{\partial v_1}(0) + \frac{\partial a_1}{\partial v_1}(0)\frac{\partial a_1}{\partial v_2}(0) + 2\frac{\partial a_2}{\partial v_3}(0) = 2\frac{\partial a_2}{\partial v_3}(0) \\
e_2(\Delta)(0) &= a_2(0) = 0, \\
e_1(e_2(\Delta))(0) &= \frac{\partial}{\partial v_3}\left((k+1)kv_4^{k-1} + \sum_{j=2}^{k-1} j(j-1)a_j v_4^{j-2}\right)(0) = 2\frac{\partial a_2}{\partial v_3}(0), \\
e_2(e_2(\Delta))(0) &= \left((k+1)k(k-1)v_4^{k-2} + \sum_{j=3}^{k-1} j(j-1)(j-2)a_j v_4^{j-3}\right)(0) \\
&= \begin{cases} 6 & (k=2) \\ 0 & (k=3,4) \end{cases}
\end{aligned}
$$

Thus

For $k=2$ $\quad \begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial^2 a_1}{\partial {v_3}^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0) \\ 0 & 2\frac{\partial a_2}{\partial v_3}(0) & 6 \end{pmatrix}$,

For $k=3,4$ $\quad \begin{pmatrix} e_1(\Delta)(0) & e_1(e_1(\Delta))(0) & e_1(e_2(\Delta))(0) \\ e_2(\Delta)(0) & e_2(e_1(\Delta))(0) & e_2(e_2(\Delta))(0) \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial^2 a_1}{\partial {v_3}^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0) \\ 0 & 2\frac{\partial a_2}{\partial v_3}(0) & 0 \end{pmatrix}$.

Therefore $F^*\omega_0$ is isomorphic to Roussaire's $\Sigma_{2,2,0}$ normal forms if and only if

$$\begin{cases} \operatorname{rank} \begin{pmatrix} \frac{\partial^2 a_1}{\partial {v_3}^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0) \\ 2\frac{\partial a_2}{\partial v_3}(0) & 6 \end{pmatrix} = 2 \qquad (k=2) \\ \operatorname{rank} \begin{pmatrix} \frac{\partial^2 a_1}{\partial {v_3}^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0) \\ 2\frac{\partial a_2}{\partial v_3}(0) & 0 \end{pmatrix} = 2 \qquad (k=3,4) \end{cases}$$

This completes the proof of Theorem 4.5. Q.E.D.

Example 4.6. Consider the following two cusp map-germs:

$$\begin{aligned} F_{5\pm} &= (f_1, \cdots, f_4) : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0), \\ f_i(v) &= v_i \ (i \leq 3), \\ f_4(v) &= v_4^3 + (v_1 \pm v_3^2)v_4, \end{aligned}$$

$$\begin{aligned} F_6 &= (f_1, \cdots, f_4) : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0), \\ f_i(v) &= v_i \ (i \leq 3), \\ f_4(v) &= v_4^3 + v_1 v_4, \end{aligned}$$

Using the Theorem 3.7 or its corollary, it can be easily checked that both of $F_{5\pm}^*\omega_0$ and $F_6^*\omega_0$ are not isomorphic to Martient's $\Sigma_{2,0}$. We see that $F_{5\pm}^*\omega_0$ is isomorphic to Roussaire's $\Sigma_{2,2,0}$ but $F_6^*\omega_0$ is not so, as follows. We apply Theorem 4.5 for $k=2$. First we consider $F_{5\pm}^*\omega_0$. In this case

$$\operatorname{rank} \begin{pmatrix} \frac{\partial^2 a_1}{\partial {v_3}^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0) \\ 2\frac{\partial a_2}{\partial v_3}(0) & 6 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \pm 2 & 0 \\ 0 & 6 \end{pmatrix} = \pm 2.$$

Therefore, by Theorem 4.5, $F_{5\pm}^*\omega_0$ is isomorphic to Roussaire's $\Sigma_{2,2,0}$.

Now we consider $F_6^*\omega_0$. In this case, since $f_4 = v_4^3 + v_1 v_4$

$$\operatorname{rank} \begin{pmatrix} \frac{\partial^2 a_1}{\partial {v_3}^2}(0) & 2\frac{\partial a_2}{\partial v_3}(0) \\ 2\frac{\partial a_2}{\partial v_3}(0) & 6 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} \neq 2.$$

Therefore, by Theorem 4.5, $F_6^*\omega_0$ is not isomorphic to Roussaire's $\Sigma_{2,2,0}$-form.

Proposition 4.7. We consider two map-germs $F_{5\pm}$ as in Example 4.6:

$$\begin{aligned} F_{5\pm} &= (f_1, \cdots, f_4) : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0), \\ f_i(v) &= v_i \ (i \leq 3), \\ f_4(v) &= v_4^3 + (v_1 \pm v_3^2)v_4. \end{aligned}$$

Then $\quad F_{5+}^*\omega_0$ is of type $\Sigma_{2,2,0}^e$ and $\quad F_{5-}^*\omega_0$ is of type $\Sigma_{2,2,0}^h$.

Proof. In the following argument, we abbreviate $F_{5\pm}$ as $F_\pm$. Then

$$\begin{aligned} \Delta &= 3v_4^2 + v_1 \pm v_3^2, \\ \Sigma_2(F_\pm^*\omega_0) &= \{v = (v_1, \cdots, v_4) \in \mathbb{R}^4 \mid \Delta = 0\}, \\ &= \{v = (v_1, \cdots, v_4) \in \mathbb{R}^4 \mid v_1 = -(3v_4^2 \pm v_3^2)\} \end{aligned}$$

In general

$$\begin{aligned} F^*\omega_0 &= \sum_{i=1}^{n-1} dv_{2i-1} \wedge dv_{2i} + \Delta dv_{2n-1} \wedge dv_{2n} \\ &\qquad - \sum_{i \neq 2n-1, 2n} \frac{\partial f_{2n}}{\partial v_i} dv_i \wedge dv_{2n-1} \end{aligned}$$

So, in this case,

$$\begin{aligned} F_\pm^*\omega_0 &= dv_1 \wedge dv_2 + \Delta dv_3 \wedge dv_4 - \sum_{i=1,2} \frac{\partial f_4}{\partial v_i} dv_i \wedge dv_3, \\ &= dv_1 \wedge dv_2 + \Delta dv_3 \wedge dv_4 - v_4 dv_1 \wedge dv_3, \\ (F_\pm^*\omega_0)^2 &= 2\Delta dv_1 \wedge \cdots \wedge dv_4 = 2\Delta\Omega, \\ f &= 2\Delta \qquad (\, f \text{ as in } (F^*\omega_0)^2 = f\Omega), \\ df &= 2d\Delta = 2dv_1 \pm 4v_3 dv_3 + 12 v_4 dv_4. \end{aligned}$$

Let $(\bar{v}_2,\ \bar{v}_3,\ \bar{v}_4)$ be coordinates on $\Sigma_2(F^*\omega_0)$ defined by

$$\bar{v}_2 = v_2,\ \bar{v}_3 = v_3,\ \bar{v}_4 = v_4,$$

and let $\iota = (\iota_1, \cdots, \iota_4) : \Sigma_2(F^*\omega_0) \to \mathbb{R}^4$;

$$\iota(\bar{v}_2,\ \bar{v}_3,\ \bar{v}_4) = (-(\bar{v}_3^2 \pm 3\bar{v}_4^2), \bar{v}_2,\ \bar{v}_3,\ \bar{v}_4)$$

be the inclusion map. Then we can easily check that

$$d\bar{v}_2 \wedge d\bar{v}_3 \wedge d\bar{v}_4 = \iota^*(dv_2 \wedge dv_3 \wedge dv_4).$$

Set

$$\bar{\Omega}_{\Sigma_2(F^*\omega_0)} = -d\bar{v}_2 \wedge d\bar{v}_3 \wedge d\bar{v}_4 = -\iota^*(dv_2 \wedge dv_3 \wedge dv_4).$$

Then,

$$\Omega = dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4 \quad \text{and} \quad \bar{\Omega}_{\Sigma_2(F^*\omega_0)} \wedge df = 2\bar{\Omega}_{\Sigma_2(F^*\omega_0)} \wedge d\Delta$$

define the same orientation on $\mathbb{R}^4$.

Now we seek the vector field

$$X_\pm = \sum_{i=2}^{4} \alpha_i(\bar{v}_2,\ \bar{v}_3,\ \bar{v}_4) \frac{\partial}{\partial \bar{v}_i} \quad \text{on} \quad \Sigma_2(F_\pm^*\omega_0)$$

such that

$$F_\pm^*\omega_{0|\Sigma_2(F_\pm^*\omega_0)} = i(X_\pm)(\bar{\Omega}_{\Sigma_2(F_\pm^*\omega_0)}).$$

The jacobian matrix of the inclusion map $\iota(\bar v_2,\ \bar v_3,\ \bar v_4) = (-(\bar v_3^2 \pm 3\bar v_4^2), \bar v_2,\ \bar v_3,\ \bar v_4)$ is

$$\begin{pmatrix} 0 & -2\bar v_3 & \mp 6\bar v_4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, letting $D\iota$ denote the differential of ι, we have

$$D\iota\left(\frac{\partial}{\partial \bar v_2}\right) = \frac{\partial}{\partial v_2}, \quad D\iota\left(\frac{\partial}{\partial \bar v_3}\right) = \frac{\partial}{\partial v_3} - 2\bar v_3\frac{\partial}{\partial v_1}, \quad D\iota\left(\frac{\partial}{\partial \bar v_4}\right) = \frac{\partial}{\partial v_4} \mp 6\bar v_4\frac{\partial}{\partial v_1}.$$

Now, we have

$$\begin{aligned} & F_\pm^*\omega_{0|\Sigma_2(F_\pm^*\omega_0)}\left(\frac{\partial}{\partial \bar v_2}, \frac{\partial}{\partial \bar v_3}\right) = \iota^* F_\pm^*\omega_0\left(\frac{\partial}{\partial \bar v_2}, \frac{\partial}{\partial \bar v_3}\right) \\ & \quad = F_\pm^*\omega_0\left(\frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3} - 2\bar v_3\frac{\partial}{\partial v_1}\right) \\ & \quad = (dv_1\wedge dv_2 + \Delta dv_3\wedge dv_4 - v_4 dv_1\wedge dv_3)\left(\frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3} - 2\bar v_3\frac{\partial}{\partial v_1}\right) = 2\bar v_3. \end{aligned}$$

Similarly, we have

$$\begin{aligned} F_\pm^*\omega_{0|\Sigma_2(F_\pm^*\omega_0)}\left(\frac{\partial}{\partial \bar v_2}, \frac{\partial}{\partial \bar v_4}\right) &= \pm 6\bar v_4, \\ F_\pm^*\omega_{0|\Sigma_2(F_\pm^*\omega_0)}\left(\frac{\partial}{\partial \bar v_3}, \frac{\partial}{\partial \bar v_4}\right) &= \mp 6\bar v_4 + \Delta = \mp 6\bar v_4, \quad (\text{ for } \Delta = 0 \text{ on } \Sigma_2(F_\pm^*\omega_0).) \end{aligned}$$

Thus

$$F_\pm^*\omega_{0|\Sigma_2(F^*\omega_0)} = 2\bar v_3 d\bar v_2\wedge d\bar v_3 \pm 6\bar v_4 d\bar v_2\wedge d\bar v_4 + 6\bar v_4^2 d\bar v_3\wedge d\bar v_4. \tag{4.6}$$

Letting

$$X_\pm = \sum_{i=2}^{4}\alpha_i(\bar v_2,\ \bar v_3,\ \bar v_4)\frac{\partial}{\partial \bar v_i}$$

we solve the equation

$$F_\pm^*\omega_{0|\Sigma_2(F_\pm^*\omega_0)} = i(X_\pm)(\bar\Omega_{\Sigma_2(F_\pm^*\omega_0)}).$$

Recall that

$$\bar\Omega_{\Sigma_2(F_\pm^*\omega_0)} = -d\bar v_2\wedge d\bar v_3\wedge d\bar v_4.$$

Then we have

$$\begin{aligned}
2\bar v_3 &= F_\pm^*\omega_{0|\Sigma_2(F_\pm^*\omega_0)}\left(\frac{\partial}{\partial\bar v_2},\frac{\partial}{\partial\bar v_3}\right)\\
&= i(X_\pm)(\bar\Omega_{\Sigma_2(F_\pm^*\omega_0)})\left(\frac{\partial}{\partial\bar v_2},\frac{\partial}{\partial\bar v_3}\right)\\
&= -d\bar v_2\wedge d\bar v_3\wedge d\bar v_4\left(\sum_{i=2}^4\alpha_i\frac{\partial}{\partial\bar v_i},\frac{\partial}{\partial\bar v_2},\frac{\partial}{\partial\bar v_3}\right)=-\alpha_4.\\
\pm 6\bar v_4 &= F_\pm^*\omega_{0|\Sigma_2(F_\pm^*\omega_0)}\left(\frac{\partial}{\partial\bar v_2},\frac{\partial}{\partial\bar v_4}\right)\\
&= -d\bar v_2\wedge d\bar v_3\wedge d\bar v_4\left(\sum_{i=2}^4\alpha_i\frac{\partial}{\partial\bar v_i},\frac{\partial}{\partial\bar v_2},\frac{\partial}{\partial\bar v_4}\right)=\alpha_3\\
\mp 6\bar v_4 &= F_\pm^*\omega_{0|\Sigma_2(F_\pm^*\omega_0)}\left(\frac{\partial}{\partial\bar v_3},\frac{\partial}{\partial\bar v_4}\right)\\
&= -d\bar v_2\wedge d\bar v_3\wedge d\bar v_4\left(\sum_{i=2}^4\alpha_i\frac{\partial}{\partial\bar v_i},\frac{\partial}{\partial\bar v_3},\frac{\partial}{\partial\bar v_4}\right)=-\alpha_2.
\end{aligned}$$

Thus

$$\alpha_2=\pm 6\bar v_4,\quad \alpha_3=\pm 6\bar v_4,\quad \alpha_4=-2\bar v_3$$

and the jacobian matrix of $(\alpha_2,\alpha_3,\alpha_4)$ at 0 is

$$\left(\frac{\partial\alpha_i}{\partial\bar v_j}(0)\right)_{2\le i,j\le 4}=\begin{pmatrix}0&0&\pm 6\\0&0&\pm 6\\0&-2&0\end{pmatrix}$$

and its non-zero eigen values are

$$\begin{cases}\pm\sqrt{-12}, & \text{for } F_+\\ \pm\sqrt{12}, & \text{for } F_-.\end{cases}$$

Thus $F_+^*\omega_0$ is of type $\Sigma^e_{2,2,0}$ and $F_-^*\omega_0$ is of type $\Sigma^h_{2,2,0}$. Q.E.D.

5. Poisson algebra of Hamiltonians associated to singular symplectic forms

In this section, we give proofs of two basic properties of the Lie-Poisson algebras of singular Hamiltonians determined by singular closed 2-forms.

Two germs ω and ω' of closed 2-forms on $\mathbb{R}^{2n}$ respectively at p and q are said to be *isomorphic* if there exists a diffeomorphism-germ $\varphi:(\mathbb{R}^{2n},q)\to(\mathbb{R}^{2n},p)$ such that $\omega'=\varphi^*\omega$.

Let ω be the germ at $0\in\mathbb{R}^{2n}$ of a closed 2-form on $\mathbb{R}^{2n}$ at 0. For a function germ h at $0\in\mathbb{R}^{2n}$, the Hamiltonian vector field of h with respect to ω is the vector field $X_{\omega,h}$, *formally* defined by the equation (cf. [11, 21]),

(5.1) $$\omega(X_{\omega,h},Y)=-Y(h)\qquad\text{for any vector field } Y \text{ on } \mathbb{R}^{2n}.$$

We often abbreviate $X_{\omega,h}$ as X_h.

The reason why we say "formally defined" in the above definition is that if ω is a degenerate closed 2-form, there are functions h for which the Hamiltonian vector fields $X_{\omega,h}$ are not defined on the singular point set of ω. (See the example at the end of this section).

For the germ ω of a closed 2-form on $\mathbb{R}^{2n}$ at $0 \in \mathbb{R}^{2n}$, we set

$$\mathcal{H}_\omega = \{h \in \mathcal{E}_{2n} |\ X_h \text{ is smooth}\}. \tag{5.2}$$

Now, for two elements $h,\ k \in \mathcal{H}_\omega$, we define formally degenerate Poisson bracket $\{h,k\}_\omega$ with respect to the degenerate 2-form ω by

$$\{h,k\}_\omega = \omega(X_h, X_k) = X_k(h) = -X_h(k). \tag{5.3}$$

In the case where ω is a degenerate 2-form, it is not trivial that $\{h,k\}_\omega \in \mathcal{H}_\omega$. However we can show $\{h,k\}_\omega \in \mathcal{H}_\omega$ under a generic condition on ω that it has a representative closed 2-form defined on an open neighborhood U of 0, which we denote also by the same symbol ω, such that

(5.4) the set $O = \{p \in U | \text{corank } \omega \text{ at } p \text{ is equal to } 0\}$ is open and dense in U.

Theorem 5.1. *Let ω be the germ of a closed 2-form satisfying the above generic condition. Then $\mathcal{H}_\omega$ is a Poisson algebra with the degenerate Poisson bracket $\{\ ,\ \}_\omega$.*

Proof. Since the restriction $\omega_{|O}$ of ω to O is a non-degenerate 2-form on O and for any smooth function h on U, the restriction $X_{h|O}$ of X_h to O is an ordinary Hamiltonian system with respect to the symplectic structure $\omega_{|O}$.

Let $h,k \in \mathcal{H}_\omega$. Then h, k, X_h, X_k are all smooth on U. Now $\{h,k\}_\omega = X_h(k)$ is smooth on O and we have

$$X_{\{h,k\}_\omega} |_O = [X_{h|O}, X_{k|O}]. \tag{5.5}$$

Since $h,k \in \mathcal{H}_\omega$, X_h and X_k are smooth on U. Therefore the right-hand side of (5.5) is extendable to the Lie bracket vector field $[X_h, X_k]$ of X_h and X_k which is smooth on U. Thus $X_{\{h,k\}_{\omega|O}}$ is also extendable to a smooth vector field on U which must be $X_{\{h,k\}_\omega}$, for O being open and dense in U. Thus $X_{\{h,k\}_\omega}$ is smooth and $\{h,k\}_\omega \in \mathcal{H}_\omega$. This completes the proof of the theorem. Q.E.D.

Theorem 5.2. *Let ω and ω' be the germs of closed 2-forms. If they are isomorphic and $\omega' = \varphi^*\omega$, then their associated Poisson algebras are isomorphic:*

$$\varphi^* : \mathcal{H}_\omega \cong \mathcal{H}_{\omega'}. \tag{5.6}$$

Let ω and ω' be the germs of closed 2-forms at $0 \in \mathbb{R}^{2n}$. Suppose that ω and ω' are isomorphic: $\omega' = \varphi^*\omega$ for the germ of a diffeomorphism $\varphi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$. To prove that $\mathcal{H}_\omega$ and $\mathcal{H}_{\omega'}$. are isomorphic, we prove that the ring isomorphism

$$\varphi^* : \mathcal{E}_{2n} \to \mathcal{E}_{2n}, \quad \varphi^*(h) = h \circ \varphi$$

induces an isomorphism

$$\varphi^* : \mathcal{H}_\omega \to \mathcal{H}_{\omega'}$$

of Lie algebras. We prove this fact by proving the following two lemmas.

Lemma 5.3. *If $h \in \mathcal{H}_\omega$, then $\varphi^*(h) \in \mathcal{H}_{\omega'}$.*

Lemma 5.4. *Let $h,k \in \mathcal{H}_\omega$. Then $\varphi^*(\{h,k\}_\omega) = \{\varphi^*(h), \varphi^*(k)\}_{\omega'}$.*

Since $\varphi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ is a diffeomorphism, from Lemma 5.3, we see that $\varphi^*(\mathcal{H}_\omega) \subset \mathcal{H}_{\omega'}$ and $(\varphi^{-1})^*(\mathcal{H}_{\omega'}) \subset \mathcal{H}_\omega$ and hence $\varphi^* : \mathcal{H}_\omega \to \mathcal{H}_{\omega'}$ is a bijection. Since $\varphi^* : \mathcal{E}_{2n} \to \mathcal{E}_{2n}$ is a ring isomorphism, we see that for $h, k \in \mathcal{H}_\omega$,

$$\varphi^*(h + k) = \varphi^*(h) + \varphi^*(k).$$

Then, with Lemma 5.4, we see that $\varphi^* : \mathcal{H}_\omega \to \mathcal{H}_{\omega'}$ is an isomorphism of Lie algebras.

5.0.1. *Proof of Lemma 5.3.* Suppose that $\omega' = \varphi^*\omega$ for a diffeomorphism-germ φ and let $h \in \mathcal{H}_\omega$. Then we are going to show that $\varphi^*(h) = h \circ \varphi \in \mathcal{H}_{\omega'}$. By definition,

$$\mathcal{H}_\omega = \{h \in \mathcal{E}_{2n} |\ X_{\omega,h} \text{ is smooth}\}$$

and $X_{\omega,h}$ is defined by the equation

$$\omega(X_{\omega,h}, Y) = -Y(h)$$

for any vector field Y on $\mathbb{R}^{2n}$.

We are going to prove that if $X_{\omega,h}$ is smooth then $X_{\omega',h\circ\varphi}$ is also smooth. We prove this using local coordinates. Let $(u_1, u_2, \cdots, u_{2n})$ be local coordinates in a neighborhood of $0 \in \mathbb{R}^{2n}$ and let $\varphi = (\varphi_1, \cdots, \varphi_{2n}) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$. Since $X_{\omega,h}$ and $X_{\omega',h\circ\varphi}$ are vector fields, they are formally of the form

$$X_{\omega,h} = \sum_{i=1}^{2n} a_i(u)\frac{\partial}{\partial u_i}, \quad X_{\omega',h\circ\varphi} = \sum_{i=1}^{2n} b_i(u)\frac{\partial}{\partial u_i}.$$

Since $\omega' = \varphi^*\omega$, we have

$$\omega'(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}) = \sum_{k=1}^{2n}\sum_{\ell=1}^{2n} \frac{\partial \varphi_k}{\partial u_i}\omega(\frac{\partial}{\partial u_k}, \frac{\partial}{\partial u_\ell})\frac{\partial \varphi_\ell}{\partial u_j}. \tag{5.7}$$

Therefore

$$\begin{aligned} \omega'(X_{\omega',h\circ\varphi}, \frac{\partial}{\partial u_j}) &= \omega'(\sum_{i=1}^{2n} b_i(u)\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}) \\ &= \sum_{i=1}^{2n} b_i(u) \sum_{k=1}^{2n}\sum_{\ell=1}^{2n} \frac{\partial \varphi_k}{\partial u_i}\omega(\frac{\partial}{\partial u_k}, \frac{\partial}{\partial u_\ell})\frac{\partial \varphi_\ell}{\partial u_j}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \omega'(X_{\omega',h\circ\varphi}, \frac{\partial}{\partial u_j}) &= -\frac{\partial}{\partial u_j}(h \circ \varphi) \quad \text{by definition} \\ &= -\sum_{m=1}^{2n} \frac{\partial h}{\partial u_m}(\varphi(u))\frac{\partial \varphi_m}{\partial u_j} \\ &= \sum_{m=1}^{2n} \omega(X_{\omega,h}(\varphi(u)), \frac{\partial}{\partial u_m})\frac{\partial \varphi_m}{\partial u_j} \quad \text{by definition of } X_{\omega,h} \\ &= \sum_{m=1}^{2n}\sum_{p=1}^{2n} a_p(\varphi(u))\omega(\frac{\partial}{\partial u_p}, \frac{\partial}{\partial u_m})\frac{\partial \varphi_m}{\partial u_j}. \end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{i=1}^{2n} b_i(u) \sum_{k=1}^{2n}\sum_{\ell=1}^{2n} \frac{\partial \varphi_k}{\partial u_i}\omega(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell})\frac{\partial \varphi_\ell}{\partial u_j} &= \omega'(X_{\omega',h\circ\varphi},\frac{\partial}{\partial u_j}) \\ &= \sum_{m=1}^{2n}\sum_{p=1}^{2n} a_p(\varphi(u))\omega(\frac{\partial}{\partial u_p},\frac{\partial}{\partial u_m})\frac{\partial \varphi_m}{\partial u_j}.\end{aligned}$$

Expressing this equality in matrices, we have

$$\begin{aligned}&(b_1(u),\cdots,b_{2n}(u))\ {}^t\left(\frac{\partial \varphi_k}{\partial u_i}(u)\right)\ \left(\omega(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell})(\varphi(u))\right)\left(\frac{\partial \varphi_\ell}{\partial u_j}(u)\right) \\ &= (a_1(\varphi(u)),\cdots,a_{2n}(\varphi(u)))\left(\omega(\frac{\partial}{\partial u_p},\frac{\partial}{\partial u_m})(\varphi(u))\right)\left(\frac{\partial \varphi_m}{\partial u_j}(u)\right)\end{aligned}$$

and hence

$$(b_1(u),\cdots,b_{2n}(u))\ {}^t\left(\frac{\partial \varphi_k}{\partial u_i}(u)\right) = (a_1(\varphi(u)),\cdots,a_{2n}(\varphi(u))). \tag{5.8}$$

Since $X_{\omega,h} = \sum_{i=1}^{2n} a_i(u)\frac{\partial}{\partial u_i}$ is smooth, $a_1,\cdots,a_{2n}$ are smooth function and $\left(\frac{\partial \varphi_k}{\partial u_i}\right)^t$ is invertible matrix smoothly depending on u, we see that $b_1,\cdots,b_{2n}$ are smooth functions. Thus $X_{\omega',h\circ\varphi} = \sum_{i=1}^{2n} b_i(u)\frac{\partial}{\partial u_i}$ is smooth and $\varphi^*(h) = h\circ\varphi \in \mathcal{H}_{\omega'}$. Q.E.D.

5.0.2. *Proof of Lemma 5.4.* By definition,

$$\{h,k\}_\omega(u) = \omega(X_{\omega,h},X_{\omega,k})(u),$$

$$\{h\circ\varphi,k\circ\varphi\}_{\omega'}(u) = \omega'(X_{\omega',h\circ\varphi},X_{\omega',k\circ\varphi})(u).$$

We express $X_{\omega,h}, X_{\omega,k}, X_{\omega',h\circ\varphi}, X_{\omega',k\circ\varphi}$ again using local coordinates $u_1,\cdots,u_{2n}$;

$$X_{\omega,h} = \sum_{i=1}^{2n} a_i(u)\frac{\partial}{\partial u_i},\quad X_{\omega',h\circ\varphi} = \sum_{i=1}^{2n} \alpha_i(u)\frac{\partial}{\partial u_i},$$

$$X_{\omega,k} = \sum_{i=1}^{2n} b_i(u)\frac{\partial}{\partial u_i},\quad X_{\omega',h\circ\varphi} = \sum_{i=1}^{2n} \beta_i(u)\frac{\partial}{\partial u_i},$$

Then from the last equality in the proof of Lemma 5.3, we have

$$(\alpha_1(u),\cdots,\alpha_{2n}(u))\ {}^t\left(\frac{\partial \varphi_k}{\partial u_i}(u)\right) = (a_1(\varphi(u)),\cdots,a_{2n}(\varphi(u))),$$

$$(\beta_1(u),\cdots,\beta_{2n}(u))\ {}^t\left(\frac{\partial \varphi_k}{\partial u_i}(u)\right) = (b_1(\varphi(u)),\cdots,b_{2n}(\varphi(u))).$$

Thus

$$\begin{aligned}\{h\circ\varphi,k\circ\varphi\}_{\omega'}(u) &= \omega'(X_{\omega',h\circ\varphi},X_{\omega',k\circ\varphi})(u)\\ &= \omega'(\sum_{i=1}^{2n}\alpha_i(u)\frac{\partial}{\partial u_i},\sum_{i=1}^{2n}\beta_i(u)\frac{\partial}{\partial u_i})\\ &= \sum_{i=1}^{2n}\sum_{j=1}^{2n}\alpha_i(u)\beta_i(u)\omega'(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j})\\ &= (\alpha_1(u),\cdots,\alpha_{2n}(u))\left(\omega'(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j})(u)\right)\ {}^t(\beta_1(u),\cdots,\beta_{2n}(u)).\end{aligned}$$

Since $\omega'=\varphi^*\omega$, from the proof of Lemma 5.3 we have

$$\omega'(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j})(u)=\sum_{k=1}^{2n}\sum_{\ell=1}^{2n}\frac{\partial\varphi_k}{\partial u_i}(u)\omega(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell})(u)\frac{\partial\varphi_\ell}{\partial u_j}(u)$$

and hence

$$\left(\omega'(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j})(u)\right)={}^t\left(\frac{\partial\varphi_k}{\partial u_i}(u)\right)\left(\omega(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell})(u)\right)\left(\frac{\partial\varphi_\ell}{\partial u_j}(u)\right).$$

Thus

$$\begin{aligned}&\{h\circ\varphi,k\circ\varphi\}_{\omega'}(u)\\ &\quad=(\alpha_1(u),\cdots,\alpha_{2n}(u))\ {}^t\left(\frac{\partial\varphi_k}{\partial u_i}(u)\right)\left(\omega(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell})(u)\right)\left(\frac{\partial\varphi_\ell}{\partial u_j}(u)\right)\\ &\qquad\times\ {}^t(\beta_1(u),\cdots,\beta_{2n}(u))\\ &\quad=(a_1(\varphi(u),\cdots,a_{2n}(\varphi(u)))\left(\omega(\frac{\partial}{\partial u_k},\frac{\partial}{\partial u_\ell})(u)\right)\\ &\qquad\times\ {}^t(b_1(\varphi(u),\cdots,b_{2n}(\varphi(u))\\ &\quad=\omega(X_{\omega,h},X_{\omega,k})(\varphi(u))=\{h,k\}_\omega(\varphi(u)).\end{aligned}$$

Thus we have

$$\{\varphi^*h,\varphi^*k\}_{\omega'}=\varphi^*\{h,k\}_\omega.$$

This completes the proof of Lemma 5.4. Q.E.D.

Example 5.5. *Consider the closed 2-form*

$$\omega=u_1du_1\wedge du_2\quad \text{on } \mathbf{R^2}$$

and a function $h=u_2$. Then X_h is defined by

$$\omega(X_h,\frac{\partial}{\partial u_i})=-\frac{\partial h}{\partial u_i}.$$

Since X_h has the form $X_h=a_1(u)\frac{\partial}{\partial u_1}+a_2(u)\frac{\partial}{\partial u_2}$, the equation becomes

$$u_1du_1\wedge du_2\left(a_1(u)\frac{\partial}{\partial u_1}+a_2(u)\frac{\partial}{\partial u_2},\frac{\partial}{\partial u_1}\right)=-\frac{\partial h}{\partial u_1},$$

$$u_1du_1\wedge du_2\left(a_1(u)\frac{\partial}{\partial u_1}+a_2(u)\frac{\partial}{\partial u_2},\frac{\partial}{\partial u_2}\right)=-\frac{\partial h}{\partial u_2}.$$

Then we have

$$-u_1a_2(u) = -\frac{\partial h}{\partial u_1} = 0, \qquad u_1a_1 = -\frac{\partial h}{\partial u_2} = -1.$$

Since $u_1(0) = 0$, there are no functions $a_1(u)$ such that $u_1a_1 = -1$. In this case, X_h is not defined on the set $\{u_1 = 0\}$ which is the singular point set of ω.

6. Poisson algebras for $\Sigma_{2,0}, \Sigma^e_{2,2,0}, \Sigma^h_{2,2,0}$ stable singularities

In this section we will characterize properties of the Poisson algebra for the singular symplectic structure of Martinet and Roussaire forms.

Proposition 6.1. *Let $\omega_{2,0}$ denote Martinet's normal form:*

$$\omega_{2,0} = v_1dv_1 \wedge dv_2 + dv_3 \wedge dv_4 + \cdots + dv_{2n-1} \wedge dv_{2n}. \tag{6.1}$$

Then

$$\mathcal{H}_{\omega_{2,0}} = \langle v_1^2\rangle_{\mathcal{E}_{v_1,\cdots,v_{2n}}} + \mathcal{E}_{v_3,\cdots,v_{2n}}. \tag{6.2}$$

Proof. In what follows, let $\partial_i = \frac{\partial}{\partial v_i}$. Then for $1 \le i \le j \le n$

$$\omega_{2,0}(\partial_i, \partial_j) = \begin{cases} v_1, & \text{for } i = 1, j = 2 \\ 1, & \text{for } i = 2k-1, j = 2k,\ 2 \le k \le n \\ 0, & \text{otherwise.} \end{cases}$$

and we have

$$(\omega_{2,0}(\partial_i, \partial_j)) = \begin{pmatrix} 0 & v_1 & 0 & 0 & & \cdots & & 0 \\ -v_1 & 0 & 0 & 0 & & \cdots & & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & -1 & 0 & & \cdots & & 0 \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & & \cdots & & & 0 & 0 & 1 \\ 0 & & \cdots & & & 0 & -1 & 0 \end{pmatrix}$$

$$(\omega_{2,0}(\partial_i, \partial_j))^{-1} = \begin{pmatrix} 0 & -1/v_1 & 0 & 0 & & \cdots & & 0 \\ 1/v_1 & 0 & 0 & 0 & & \cdots & & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & & \cdots & & 0 \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & & \cdots & & & 0 & 0 & -1 \\ 0 & & \cdots & & & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$
\begin{aligned}
& h \in \mathcal{H}_{\omega_{2,0}} \\
\longleftrightarrow \;& \begin{pmatrix} 0 & -1/v_1 & 0 & 0 & & \cdots & & 0 \\ 1/v_1 & 0 & 0 & 0 & & \cdots & & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & & \cdots & & 0 \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & & \cdots & & & 0 & 0 & -1 \\ 0 & & \cdots & & & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \partial h/\partial v_1 \\ \partial h/\partial v_2 \\ \\ \vdots \\ \\ \vdots \\ \partial h/\partial v_{2n} \end{pmatrix} \text{ is smooth} \\
\longleftrightarrow \;& \partial h/\partial v_1,\ \partial h/\partial v_2 \in \langle v_1 \rangle_{\mathcal{E}_v}.
\end{aligned}
$$

We express h in the form

$$h(v) = v_1^2\alpha(v) + v_1\beta(v_2, \cdots, v_{2n}) + \gamma(v_2, \cdots, v_{2n}).$$

Then $\partial h/\partial v_1,\ \partial h/\partial v_2 \in \langle v_1 \rangle_{\mathcal{E}_v}$ if and only if

$$\beta(v_2, \cdots, v_{2n}), \partial\gamma/\partial v_2(v_2, \cdots, v_{2n}) \in \langle v_1 \rangle_{\mathcal{E}_v}$$

which holds if and only if

$$\beta(v_2, \cdots, v_{2n}) = 0, \partial\gamma/\partial v_2(v_2, \cdots, v_{2n}) = 0$$

which holds if and only if h has the form

$$h(v) = v_1^2\alpha(v) + \gamma(v_3, \cdots, v_{2n}).$$

Therefore $h \in \mathcal{H}_{\omega_{2,0}}$ if and only if $h \in \langle v_1^2 \rangle_{\mathcal{E}_v} + \mathcal{E}_{v_3,\cdots,v_{2n}}$. Thus we have

$$\mathcal{H}_{\omega_{2,0}} = \langle v_1^2 \rangle_{\mathcal{E}_v} + \mathcal{E}_{v_3,\cdots,v_{2n}}.$$

Q.E.D. For comparison with general calculations we continue with the example.

Example 6.2. ($\Sigma_{2,2,0}$-type cusps) We consider the following two cusps $F_{5\pm}$:

$$
\begin{aligned}
F_{5\pm} &= (f_1, \cdots, f_4) : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0), \\
f_i(v) &= v_i \ (i \leq 3), \\
f_4(v) &= v_4^3 + (v_1 \pm v_3^2)v_4.
\end{aligned}
$$

Then $F_{5+}^*\omega_0$ is of type $\Sigma_{2,2,0}^e$ and $F_{5-}^*\omega_0$ is of type $\Sigma_{2,2,0}^h$ and

$$F_{5\pm}^*\omega_0 = dv_1 \wedge dv_2 - v_4 dv_1 \wedge dv_3 + (3v_4^2 + v_1 \pm v_3^2)dv_3 \wedge dv_4.$$

Let ω_e, ω_h denote the Roussaire's elliptic and hyperbolic normal forms respectively:

$$\begin{aligned}\omega_e &= dv_1\wedge dv_2 + v_3 dv_2\wedge dv_3 + d\left(v_1v_3 + v_2v_4 - \frac{v_3^3}{3}\right)\wedge dv_4 \\ &= dv_1\wedge dv_2 + v_3dv_1\wedge dv_4 + v_3dv_2\wedge dv_3 + v_4dv_2\wedge dv_4 \\ &\quad +(v_1 - v_3^2)dv_3\wedge dv_4,\end{aligned}\tag{6.3}$$

$$\begin{aligned}\omega_h &= dv_1\wedge dv_2 + v_3 dv_2\wedge dv_3 + d\left(v_1v_3 - v_2v_4 - \frac{v_3^3}{3}\right)\wedge dv_4 \\ &= dv_1\wedge dv_2 + v_3dv_1\wedge dv_4 + v_3dv_2\wedge dv_3 - v_4dv_2\wedge dv_4 \\ &\quad +(v_1 - v_3^2)dv_3\wedge dv_4.\end{aligned}\tag{6.4}$$

$$\text{(CF)}\quad F_{5\pm}^*\omega_0 = dv_1\wedge dv_2 - v_4dv_1\wedge dv_3 + (v_1 \pm v_3^2 + 3v_4^2)dv_3\wedge dv_4.$$

In what follows, let $\partial_i = \frac{\partial}{\partial v_i}$. Then from (6.3) and (6.4)

$$\begin{aligned}(\omega_e(\partial_i,\partial_j)) &= \begin{pmatrix} 0 & 1 & 0 & v_3 \\ -1 & 0 & v_3 & v_4 \\ 0 & -v_3 & 0 & v_1 - v_3^2 \\ -v_3 & -v_4 & -(v_1 - v_3^2) & 0 \end{pmatrix}, \\ (\omega_h(\partial_i,\partial_j)) &= \begin{pmatrix} 0 & 1 & 0 & v_3 \\ -1 & 0 & v_3 & -v_4 \\ 0 & -v_3 & 0 & v_1 - v_3^2 \\ -v_3 & v_4 & -(v_1 - v_3^2) & 0 \end{pmatrix}, \\ \big(F_{5\pm}^*\omega_0(\partial_i,\partial_j)\big) &= \begin{pmatrix} 0 & 1 & -v_4 & 0 \\ -1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & v_1 \pm v_3^2 + 3v_4^2 \\ 0 & 0 & -(v_1 \pm v_3^2 + 3v_4^2) & 0 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}(\omega_e(\partial_i,\partial_j))^{-1} &= \frac{1}{v_1}\begin{pmatrix} 0 & -(v_1 - v_3^2) & v_4 & -v_3 \\ (v_1 - v_3^2) & 0 & -v_3 & 0 \\ -v_4 & v_3 & 0 & -1 \\ v_3 & 0 & 1 & 0 \end{pmatrix}. \\ (\omega_h(\partial_i,\partial_j))^{-1} &= \frac{1}{v_1}\begin{pmatrix} 0 & -(v_1 - v_3^2) & -v_4 & -v_3 \\ (v_1 - v_3^2) & 0 & -v_3 & 0 \\ v_4 & v_3 & 0 & -1 \\ v_3 & 0 & 1 & 0 \end{pmatrix}, \\ \big(F_{5\pm}^*\omega_0(\partial_i,\partial_j)\big)^{-1} &= \frac{1}{v_1 \pm v_3^2 + 3v_4^2}\begin{pmatrix} 0 & -(v_1 \pm v_3^2 + 3v_4^2) & 0 & 0 \\ v_1 \pm v_3^2 + 3v_4^2 & 0 & 0 & -v_4 \\ 0 & 0 & 0 & -1 \\ 0 & v_4 & 1 & 0 \end{pmatrix}.\end{aligned}$$

Remark 6.3. *Note that*

$$\begin{aligned}\det\big(\omega_e(\partial_i,\partial_j)\big) &= \big(\omega_h(\partial_i,\partial_j)\big) = v_1^2, \\ \det\big(F_{5\pm}^*\omega_0(\partial_i,\partial_j)\big) &= \big(v_1 \pm v_3^2 + 3v_4^2\big)^2.\end{aligned}$$

Now we provide the implicit formulas for $\mathcal{H}_{\omega_e}, \mathcal{H}_{\omega_h}$. Let $\mathcal{H}_{\omega_e}, \mathcal{H}_{\omega_h}$ denote the Poisson algebras associated to Roussaire's hyperbolic and elliptic normal forms ω_e,and ω_h respectively.

Let $h \in \mathcal{E}_v$. Then $h \in \mathcal{H}_{\omega_e}$ if and only if

$$\frac{1}{v_1}\begin{pmatrix} 0 & -(v_1 - v_3^2) & v_4 & -v_3 \\ (v_1 - v_3^2) & 0 & -v_3 & 0 \\ -v_4 & v_3 & 0 & -1 \\ v_3 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \partial h/\partial v_1 \\ \partial h/\partial v_2 \\ \partial h/\partial v_3 \\ \partial h/\partial v_4 \end{pmatrix} \quad \text{is smooth,}$$

$h \in \mathcal{H}_{\omega_h}$ if and only if

$$\frac{1}{v_1}\begin{pmatrix} 0 & -(v_1 - v_3^2) & -v_4 & -v_3 \\ (v_1 - v_3^2) & 0 & -v_3 & 0 \\ v_4 & v_3 & 0 & -1 \\ v_3 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \partial h/\partial v_1 \\ \partial h/\partial v_2 \\ \partial h/\partial v_3 \\ \partial h/\partial v_4 \end{pmatrix} \quad \text{is smooth,}$$

(**CF**): $h \in \mathcal{H}_{F_{5\pm}^*\omega_0}$ if and only if

$$\frac{1}{v_1 \pm v_3^2 + 3v_4^2}\begin{pmatrix} 0 & -(v_1 \pm v_3^2 + 3v_4^2) & 0 & 0 \\ v_1 \pm v_3^2 + 3v_4^2 & 0 & 0 & -v_4 \\ 0 & 0 & 0 & -1 \\ 0 & v_4 & 1 & 0 \end{pmatrix}\begin{pmatrix} \partial h/\partial v_1 \\ \partial h/\partial v_2 \\ \partial h/\partial v_3 \\ \partial h/\partial v_4 \end{pmatrix} \quad \text{is smooth,}$$

from which we have

Proposition 6.4. (First implicit formula)

$$\mathcal{H}_{\omega_e} = \{h \in \mathcal{E}_v \mid \quad h \textit{ satisfies the following conditions}\}$$

$$-v_4\frac{\partial h}{\partial v_1} + v_3\frac{\partial h}{\partial v_2} - \frac{\partial h}{\partial v_4} \in \langle v_1\rangle_{\mathcal{E}_v}, \tag{6.5}$$

$$v_3\frac{\partial h}{\partial v_1} + \frac{\partial h}{\partial v_3} \in \langle v_1\rangle_{\mathcal{E}_v}. \tag{6.6}$$

$$\mathcal{H}_{\omega_h} = \{h \in \mathcal{E}_v \mid \quad h \textit{ satisfies the following conditions}\}.$$

$$v_4\frac{\partial h}{\partial v_1} + v_3\frac{\partial h}{\partial v_2} - \frac{\partial h}{\partial v_4} \in \langle v_1\rangle_{\mathcal{E}_v}, \tag{6.7}$$

$$v_3\frac{\partial h}{\partial v_1} + \frac{\partial h}{\partial v_3} \in \langle v_1\rangle_{\mathcal{E}_v}. \tag{6.8}$$

Proof. For a function $h \in \mathcal{E}_v$, $h \in \mathcal{H}_{\omega_e}$ if and only if

$$\frac{1}{v_1}\begin{pmatrix} 0 & -(v_1 - v_3^2) & v_4 & -v_3 \\ (v_1 - v_3^2) & 0 & -v_3 & 0 \\ -v_4 & v_3 & 0 & -1 \\ v_3 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \partial h/\partial v_1 \\ \partial h/\partial v_2 \\ \partial h/\partial v_3 \\ \partial h/\partial v_4 \end{pmatrix} \quad \text{is smooth}$$

which holds if and only if

$$
\begin{aligned}
-(v_1-v_3^2)\frac{\partial h}{\partial v_2}+v_4\frac{\partial h}{\partial v_3}-v_3\frac{\partial h}{\partial v_4} &\in \langle v_1\rangle_{\mathcal{E}_v},\\
(v_1-v_3^2)\frac{\partial h}{\partial v_1}-v_3\frac{\partial h}{\partial v_3} &\in \langle v_1\rangle_{\mathcal{E}_v},\\
-v_4\frac{\partial h}{\partial v_1}+v_3\frac{\partial h}{\partial v_2}-\frac{\partial h}{\partial v_4} &\in \langle v_1\rangle_{\mathcal{E}_v},\\
v_3\frac{\partial h}{\partial v_1}+\frac{\partial h}{\partial v_3} &\in \langle v_1\rangle_{\mathcal{E}_v},
\end{aligned}
$$

which hold if and only if

$$v_3^2\frac{\partial h}{\partial v_2}+v_4\frac{\partial h}{\partial v_3}-v_3\frac{\partial h}{\partial v_4} \in \langle v_1\rangle_{\mathcal{E}_v}, \tag{6.9}$$

$$-v_3^2\frac{\partial h}{\partial v_1}-v_3\frac{\partial h}{\partial v_3} \in \langle v_1\rangle_{\mathcal{E}_v}, \tag{6.10}$$

$$-v_4\frac{\partial h}{\partial v_1}+v_3\frac{\partial h}{\partial v_2}-\frac{\partial h}{\partial v_4} \in \langle v_1\rangle_{\mathcal{E}_v}, \tag{6.11}$$

$$v_3\frac{\partial h}{\partial v_1}+\frac{\partial h}{\partial v_3} \in \langle v_1\rangle_{\mathcal{E}_v}. \tag{6.12}$$

However, (6.10) follows from (6.12) and (6.9) follows from (6.11) and(6.12); (6.9)= $v_3\times$(6.11) $-v_4\times$(6.12). Thus $h\in\mathcal{H}_{\omega_e}$ if and only if (6.11) and (6.12) hold. But (6.11) and (6.12) are nothing but (6.5) and (6.6) in Proposition 6.4 respectively.

Similarly for a function $h\in\mathcal{E}_v$, $h\in\mathcal{H}_{\omega_h}$ if and only if

$$\frac{1}{v_1}\begin{pmatrix} 0 & -(v_1-v_3^2) & -v_4 & -v_3\\ (v_1-v_3^2) & 0 & -v_3 & 0\\ v_4 & v_3 & 0 & -1\\ v_3 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\partial h/\partial v_1\\ \partial h/\partial v_2\\ \partial h/\partial v_3\\ \partial h/\partial v_4\end{pmatrix} \quad \text{is smooth}$$

which holds if and only if

$$
\begin{aligned}
-(v_1-v_3^2)\frac{\partial h}{\partial v_2}-v_4\frac{\partial h}{\partial v_3}-v_3\frac{\partial h}{\partial v_4} &\in \langle v_1\rangle_{\mathcal{E}_v},\\
(v_1-v_3^2)\frac{\partial h}{\partial v_1}-v_3\frac{\partial h}{\partial v_3} &\in \langle v_1\rangle_{\mathcal{E}_v},\\
v_4\frac{\partial h}{\partial v_1}+v_3\frac{\partial h}{\partial v_2}-\frac{\partial h}{\partial v_4} &\in \langle v_1\rangle_{\mathcal{E}_v},\\
v_3\frac{\partial h}{\partial v_1}+\frac{\partial h}{\partial v_3} &\in \langle v_1\rangle_{\mathcal{E}_v},
\end{aligned}
$$

which hold if and only if

$$v_3^2\frac{\partial h}{\partial v_2}-v_4\frac{\partial h}{\partial v_3}-v_3\frac{\partial h}{\partial v_4} \in \langle v_1\rangle_{\mathcal{E}_v}, \tag{6.13}$$

$$-v_3^2\frac{\partial h}{\partial v_1}-v_3\frac{\partial h}{\partial v_3} \in \langle v_1\rangle_{\mathcal{E}_v}, \tag{6.14}$$

$$v_4\frac{\partial h}{\partial v_1}+v_3\frac{\partial h}{\partial v_2}-\frac{\partial h}{\partial v_4} \in \langle v_1\rangle_{\mathcal{E}_v}, \tag{6.15}$$

$$v_3\frac{\partial h}{\partial v_1}+\frac{\partial h}{\partial v_3} \in \langle v_1\rangle_{\mathcal{E}_v}. \tag{6.16}$$

However, (6.14) follows from (6.16) and (6.13) follows from (6.15), (6.16); (6.13)= $v_3\times$(6.15) $-v_4\times$(6.16). Thus $h \in \mathcal{H}_{\omega_h}$ if and only if (6.15) and (6.16) hold. But (6.15) and (6.16) are nothing but (6.7) and (6.8) in Proposition 6.4 respectively. This completes the proof of Proposition 6.4. Q.E.D.

(CF) (*See*[3]) $\mathcal{H}_{F^*_{5\pm}\omega_0} = \{h \in \mathcal{E}_v \mid h \text{ satisfies the following conditions}\}$.

$$\frac{\partial h}{\partial v_4} \in \langle v_1 \pm v_3^2+3v_4^2\rangle_{\mathcal{E}_v}$$

$$v_4\frac{\partial h}{\partial v_2}+\frac{\partial h}{\partial v_3} \in \langle v_1 \pm v_3^2+3v_4^2\rangle_{\mathcal{E}_v}$$

Next, expressing h in the form

$$h = v_1^2\alpha(v)+v_1\beta(v_2,v_3,v_4)+\gamma(v_2,v_3,v_4) \tag{6.17}$$

we have

Proposition 6.5. (Second implicit formula)

$$\mathcal{H}_{\omega_e} = \langle v_1^2\rangle_{\mathcal{E}_v}+\{v_1\beta+\gamma \mid \beta,\gamma \in \mathcal{E}_{v_2,v_3,v_4} \ \ \textit{satisfying the following equations}\}$$

$$-v_4\beta(v_2,v_3,v_4)+v_3\frac{\partial \gamma}{\partial v_2}(v_2,v_3,v_4)-\frac{\partial \gamma}{\partial v_4}(v_2,v_3,v_4) = 0, \tag{6.18}$$

$$v_3\beta(v_2,v_3,v_4)+\frac{\partial \gamma}{\partial v_3}(v_2,v_3,v_4) = 0. \tag{6.19}$$

$$\mathcal{H}_{\omega_h} = \langle v_1^2\rangle_{\mathcal{E}_v}+\{v_1\beta+\gamma \mid \beta,\gamma \in \mathcal{E}_{v_2,v_3,v_4} \ \ \textit{satisfying the following equations}\}$$

$$v_4\beta(v_2,v_3,v_4)+v_3\frac{\partial \gamma}{\partial v_2}(v_2,v_3,v_4)-\frac{\partial \gamma}{\partial v_4}(v_2,v_3,v_4) = 0, \tag{6.20}$$

$$v_3\beta(v_2,v_3,v_4)+\frac{\partial \gamma}{\partial v_3}(v_2,v_3,v_4) = 0. \tag{6.21}$$

Proof. Express $h \in \mathcal{E}_v$ in the form

$$h = v_1^2\alpha(v)+v_1\beta(v_2,v_3,v_4)+\gamma(v_2,v_3,v_4). \tag{6.22}$$

Then conditions (6.5), (6.6)

$$\begin{aligned} -v_4\frac{\partial h}{\partial v_1}+v_3\frac{\partial h}{\partial v_2}-\frac{\partial h}{\partial v_4} &\in \langle v_1\rangle_{\mathcal{E}_v}, \\ v_3\frac{\partial h}{\partial v_1}+\frac{\partial h}{\partial v_3} &\in \langle v_1\rangle_{\mathcal{E}_v} \end{aligned}$$

for $h\in\mathcal{H}_{\omega_e}$ given in Proposition 6.4 are equivalent to

$$-v_4\beta+v_3\frac{\partial \gamma}{\partial v_2}-\frac{\partial \gamma}{\partial v_4} \in \langle v_1\rangle_{\mathcal{E}_v}, \tag{6.23}$$

$$v_3\beta+\frac{\partial \gamma}{\partial v_3} \in \langle v_1\rangle_{\mathcal{E}_v}. \tag{6.24}$$

However, since β and γ are functions of the variables v_2, v_3, v_4, (6.23) and (6.24) are equivalent to

$$-v_4\beta+v_3\frac{\partial \gamma}{\partial v_2}-\frac{\partial \gamma}{\partial v_4} = 0, \tag{6.25}$$

$$v_3\beta+\frac{\partial \gamma}{\partial v_3} = 0. \tag{6.26}$$

Thus $h\in\mathcal{H}_{\omega_e}$ if and only if (6.25) and (6.26) hold. Since (6.25) and (6.26) are nothing but (6.18) and (6.19) in Proposition 6.5, this completes the proof of Proposition 6.5 for $\mathcal{H}_{\omega_e}$.

The proof that $h\in\mathcal{H}_{\omega_h}$ if and only if (6.20) and (6.21) hold is similar. This completes the proof of Proposition 6.5. Q.E.D.

Expressing $\gamma(v_2,v_3,v_4)$ in the form

$$\gamma(v_2,v_3,v_4)=v_3^2\gamma_1(v_2,v_3,v_4)+v_3\gamma_2(v_2,v_4)+\gamma_3(v_2,v_4), \tag{6.27}$$

from (6.19) (= (6.21)), after some calculations, we have

$$\gamma_2 = 0, \tag{6.28}$$

$$\gamma = v_3^2\gamma_1(v_2,v_3,v_4)+\gamma_3(v_2,v_4), \tag{6.29}$$

$$\beta = -2\gamma_1(v_2,v_3,v_4)-v_3\frac{\partial\gamma_1}{\partial v_3}(v_2,v_3,v_4). \tag{6.30}$$

Then from Proposition 6.5 we have

Proposition 6.6. (Third implicit formula)

$$\begin{aligned}\mathcal{H}_{\omega_e} = \langle v_1^2\rangle_{\mathcal{E}_v}+\Big\{-v_1\left(2\gamma_1+v_3\frac{\partial\gamma_1}{\partial v_3}\right)+v_3^2\gamma_1+\gamma_3 \,\Big|\, &\gamma_1\in\mathcal{E}_{v_2,v_3,v_4}, \gamma_3\in\mathcal{E}_{v_2,v_4}, \\ &\gamma_1(v_2,v_3,v_4) \textit{ and } \gamma_3(v_2,v_4) \textit{ satisfy the following equation}\Big\}\end{aligned}$$

$$v_4\left(2\gamma_1+v_3\frac{\partial\gamma_1}{\partial v_3}\right)+v_3\left(v_3^2\frac{\partial\gamma_1}{\partial v_2}+\frac{\partial\gamma_3}{\partial v_2}\right)-\left(v_3^2\frac{\partial\gamma_1}{\partial v_4}+\frac{\partial\gamma_3}{\partial v_4}\right)=0. \tag{6.31}$$

$$\mathcal{H}_{\omega_h} = \langle v_1^2\rangle_{\mathcal{E}_v} + \left\{ -v_1\left(2\gamma_1 + v_3\frac{\partial\gamma_1}{\partial v_3}\right) + v_3^2\gamma_1 + \gamma_3 \,\middle|\, \gamma_1 \in \mathcal{E}_{v_2,v_3,v_4}, \gamma_3 \in \mathcal{E}_{v_2,v_4}, \right.$$
$$\left. \gamma_1(v_2,v_3,v_4) \textit{ and } \gamma_3(v_2,v_4) \textit{ satisfy the following equation} \right\}$$

$$-v_4\left(2\gamma_1 + v_3\frac{\partial\gamma_1}{\partial v_3}\right) + v_3\left(v_3^2\frac{\partial\gamma_1}{\partial v_2} + \frac{\partial\gamma_3}{\partial v_2}\right) - \left(v_3^2\frac{\partial\gamma_1}{\partial v_4} + \frac{\partial\gamma_3}{\partial v_4}\right) = 0. \tag{6.32}$$

Proof. We express $\gamma(v_2, v_3, v_4)$ in the form

$$\gamma(v_2,v_3,v_4) = v_3^2\gamma_1(v_2,v_3,v_4) + v_3\gamma_2(v_2,v_4) + \gamma_3(v_2,v_4) \tag{6.33}$$

Then (6.24) and (6.26) $v_3\beta + \frac{\partial\gamma}{\partial v_3} = 0$ are equivalent to

$$\gamma_2(v_2,v_4) = -v_3\left(\beta + 2\gamma_1 + v_3\frac{\partial\gamma_1}{\partial v_3}\right). \tag{6.34}$$

Since the left hand side of (6.34) does not contain the variable v_3 and the right hand side does, (6.34) is equivalent to

$$\gamma_2 = 0, \tag{6.35}$$

$$\beta = -2\gamma_1(v_2,v_3,v_4) - v_3\frac{\partial\gamma_1}{\partial v_3}(v_2,v_3,v_4). \tag{6.36}$$

Note that (6.35) and (6.36) are the same as (6.28) and (6.30).

Substituting (6.35) and (6.36) into

$$-v_4\beta + v_3\frac{\partial\gamma}{\partial v_2} - \frac{\partial\gamma}{\partial v_4} = 0$$

we see that $h = v_1^2\alpha + v_1\beta + \gamma \in \mathcal{H}_{\omega_h}$ if and only if

$$v_4\left(2\gamma_1 + v_3\frac{\partial\gamma_1}{\partial v_3}\right) + v_3\left(v_3^2\frac{\partial\gamma_1}{\partial v_2} + \frac{\partial\gamma_3}{\partial v_2}\right) - \left(v_3^2\frac{\partial\gamma_1}{\partial v_4} + \frac{\partial\gamma_3}{\partial v_4}\right) = 0, \tag{6.37}$$

here (6.37) is equals to (6.31) in Proposition 6.6.

The proof that $h \in \mathcal{H}_{\omega_h}$ if and only if (6.32) holds is similar. This completes the proof of Proposition 6.6. Q.E.D.

7. Examples

In this section we give implicit formulas for Poisson Algebras $\mathcal{H}_{\bar{F}_{5\pm}}$associated to the closed 2-forms induced from the Darboux form by the maps $\bar{F}_{5\pm}$ given in the following example.

We consider the following two map-germs $F_{5\pm}$:

$$\begin{aligned} \bar{F}_{5\pm} &= (f_1, \cdots, f_4) : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, \omega_0), \\ &\quad \omega_0 = dz_1 \wedge dz_2 + dz_3 \wedge dz_4 \\ f_i(v) &= v_i \ (i \le 3), \\ f_4(v) &= v_4^3 + (v_1 \pm v_3^2)v_4, \end{aligned}$$

Then $\bar{F}^*_{5+}\omega_0$ is of type $\Sigma^e_{2,2,0}$ and $\bar{F}^*_{5-}\omega_0$ is of type $\Sigma^h_{2,2,0}$.

In what follows sometimes we abbreviate $\bar{F}_{5\pm}$ as $\bar{F}_{\pm}$. We have

$$\begin{aligned} F_{5\pm}^*\omega_0 &= dv_1 \wedge dv_2 + \Delta_\pm dv_3 \wedge dv_4 - v_4 dv_1 \wedge dv_3 \\ &= dv_1 \wedge dv_2 + (3v_4^2 + v_1 \pm v_3^2) dv_3 \wedge dv_4 - v_4 dv_1 \wedge dv_3, \end{aligned} \tag{7.1}$$

where

$$\Delta_\pm = \det J\bar{F}_{5\pm} = 3v_4^2 + v_1 \pm v_3^2. \tag{7.2}$$

(**CF**) Recall Roussaire's normal forms:

$$dv_1 \wedge dv_2 + v_3 dv_2 \wedge dv_3 + d\left(v_1 v_3 + v_2 v_4 - \frac{v_3^3}{3}\right) \wedge dv_4, \quad (\Sigma^e_{2,2,0})$$
$$dv_1 \wedge dv_2 + v_3 dv_2 \wedge dv_3 + d\left(v_1 v_3 - v_2 v_4 - \frac{v_3^3}{3}\right) \wedge dv_4. \quad (\Sigma^h_{2,2,0})$$

$F_\pm^*\omega_0$ and Roussaire's normal forms have similar forms but they are different. However they must be equivalent according to our theorem.

For a smooth function $h(v_1, v_2, v_3, v_4)$, $h \in \mathcal{H}_{\bar{F}_\pm}$ if and only if

$$J\bar{F}_\pm^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} {}^t J\bar{F}_\pm^{-1} \begin{pmatrix} \partial h/\partial v_1 \\ \partial h/\partial v_2 \\ \partial h/\partial v_3 \\ \partial h/\partial v_4 \end{pmatrix} \text{ is smooth.}$$

From that property we have

Proposition 7.1. (First implicit formula)

$$\mathcal{H}_{\bar{F}_\pm} = \{h \in \mathcal{E}_v \mid \quad h \textit{ satisfies the following conditions}\}$$

$$\frac{\partial h}{\partial v_4} \in \langle \Delta_\pm \rangle_{\mathcal{E}_v}, \tag{7.3}$$

$$v_4 \frac{\partial h}{\partial v_2} + \frac{\partial h}{\partial v_3} \in \langle \Delta_\pm \rangle_{\mathcal{E}_v}. \tag{7.4}$$

Proof. Recall

$$\begin{aligned} \bar{F}_\pm &= (f_1, \cdots, f_4) : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0), \\ f_i(v) &= v_i \ (i \leq 3), \\ f_4(v) &= v_4^3 + (v_1 \pm v_3^2) v_4, \end{aligned}$$

Thus

$$J\bar{F}_\pm = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v_4 & 0 & \pm 2v_3 v_4 & \Delta_\pm \end{pmatrix}, \qquad \text{where} \quad \Delta_\pm = v_1 \pm v_3^2 + 3v_4^2.$$

Then

$$h \in \mathcal{H}_{\bar{F}_\pm} \Longleftrightarrow J\bar{F}_\pm^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} {}^t J\bar{F}_\pm^{-1} \begin{pmatrix} \partial h/\partial v_1 \\ \partial h/\partial v_2 \\ \partial h/\partial v_3 \\ \partial h/\partial v_4 \end{pmatrix} \text{ is smooth.} \tag{7.5}$$

We have

$$J\bar{F}_{\pm}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v_4/\Delta_\pm & 0 & \mp 2v_3v_4/\Delta_\pm & 1/\Delta_\pm \end{pmatrix},$$

$${}^tJ\bar{F}_{\pm}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -v_4/\Delta_\pm \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mp 2v_3v_4/\Delta_\pm \\ 0 & 0 & 0 & 1/\Delta_\pm \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} {}^tJ\bar{F}_{\pm}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & v_4/\Delta_\pm \\ 0 & 0 & 0 & 1/\Delta \\ 0 & 0 & -1 & \pm 2v_3v_4/\Delta_\pm \end{pmatrix},$$

$$\not{J}\bar{F}_{\pm}^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} {}^tJ\bar{F}_{\pm}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v_4/\Delta_\pm & 0 & \mp 2v_3v_4/\Delta_\pm & 1/\Delta_\pm \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & v_4/\Delta_\pm \\ 0 & 0 & 0 & 1/\Delta_\pm \\ 0 & 0 & -1 & \pm 2v_3v_4/\Delta_\pm \end{pmatrix}$$

$$(7.6) \qquad = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & v_4/\Delta_\pm \\ 0 & 0 & 0 & 1/\Delta_\pm \\ 0 & -v_4/\Delta_\pm & -1/\Delta_\pm & \end{pmatrix} = -\big(\bar{F}_{\pm}^{*}\omega_0(e_i, e_j)\big)^{-1}.$$

Where $(\bar{F}_{\pm}^{*}\omega_0(e_i, e_j))$ is the matrix representation of $\bar{F}_{\pm}^{*}\omega_0$ and $e_i = \partial/\partial v_i$.

Let $h \in \mathcal{E}_v$. Then $h \in \mathcal{H}_{\bar{F}_{5\pm}}$ if and only if

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & v_4/\Delta_\pm \\ 0 & 0 & 0 & 1/\Delta_\pm \\ 0 & -v_4/\Delta_\pm & -1/\Delta_\pm & \end{pmatrix} \begin{pmatrix} \partial h/\partial v_1 \\ \partial h/\partial v_2 \\ \partial h/\partial v_3 \\ \partial h/\partial v_4 \end{pmatrix} \quad \text{is smooth,}$$

which holds if and only if

$$(7.7) \qquad \frac{\partial h}{\partial v_4} \in \langle \Delta_\pm \rangle_{\mathcal{E}_v},$$

$$(7.8) \qquad v_4\frac{\partial h}{\partial v_2} + \frac{\partial h}{\partial v_3} \in \langle \Delta_\pm \rangle_{\mathcal{E}_v}.$$

The conditions (7.7), (7.8) are nothing but (7.3), (7.4) in Proposition 7.1. This completes the proof of Proposition 7.1. Q.E.D. Now we express h in the form

$$(7.9) \qquad h = \Delta_\pm^2\alpha(v) + \Delta_\pm\beta(v_2, v_3, v_4) + \gamma(v_2, v_3, v_4).$$

Proposition 7.2. (Second implicit formula)

$$\mathcal{H}_{\bar{F}_\pm} = \langle \Delta_\pm^2 \rangle_{\mathcal{E}_v} + \Big\{ \Delta_\pm \beta + \gamma \ \mid \beta, \gamma \in \mathcal{E}_{v_2,v_3,v_4} \textit{ and satisfy the following conditions} \}$$

$$6v_4\beta(v_2, v_3, v_4) + \frac{\partial \gamma}{\partial v_4}(v_2, v_3, v_4) = 0,$$

$$v_4 \frac{\partial \gamma}{\partial v_2} \pm 2v_3\beta + \frac{\partial \gamma}{\partial v_3} = 0.$$

Proof. Since

$$\Delta_\pm = 3v_4^2 + v_1 \pm v_3^2,$$

we may regard $(\Delta_\pm, v_2, v_3, v_4)$ as local coordinates in a small neighborhood of the origin. We express h in the form

$$h = \Delta_\pm^2 \alpha(v) + \Delta_\pm \beta(v_2, v_3, v_4) + \gamma(v_2, v_3, v_4).$$

Then the condition (7.3)

$$\frac{\partial h}{\partial v_4} \in \langle \Delta_\pm \rangle_{\mathcal{E}_v}$$

in the first formula is equivalent to

$$\frac{\partial \Delta_\pm}{\partial v_4}\beta + \frac{\partial \gamma}{\partial v_4} = 6v_4\beta(v_2, v_3, v_4) + \frac{\partial \gamma}{\partial v_4}(v_2, v_3, v_4) \in \langle \Delta_\pm \rangle_{\mathcal{E}_v}.$$

Thus (7.3) is equivalent to

$$6v_4\beta(v_2, v_3, v_4) + \frac{\partial \gamma}{\partial v_4}(v_2, v_3, v_4) = 0,$$

since $\Delta_\pm$ contains v_1 as a term while β and γ are functions of v_2, v_3, v_4.

Now consider the condition (7.4) in the first formula:

$$v_4 \frac{\partial h}{\partial v_2} + \frac{\partial h}{\partial v_3} \in \langle \Delta_\pm \rangle_{\mathcal{E}_v}.$$

The condition (7.4) is equivalent to

$$v_4 \frac{\partial}{\partial v_2}\left(\Delta_\pm^2 \alpha + \Delta_\pm \beta + \gamma\right) + \frac{\partial}{\partial v_3}\left(\Delta_\pm^2 \alpha + \Delta_\pm \beta + \gamma\right) \in \langle \Delta_\pm \rangle_{\mathcal{E}_v},$$

recalling that $\Delta_\pm = v_1 \pm v_3^2 + 3v_4^2$, which is equivalent to

$$v_4 \frac{\partial \gamma}{\partial v_2} \pm 2v_3\beta + \frac{\partial \gamma}{\partial v_3} \in \langle \Delta_\pm \rangle_{\mathcal{E}_v},$$

which is equivalent to

$$v_4 \frac{\partial \gamma}{\partial v_2} \pm 2v_3\beta + \frac{\partial \gamma}{\partial v_3} = 0, \tag{7.10}$$

since β and γ are functions of the variables v_2, v_3, v_4 and $\Delta_\pm$ has v_1 as a term.

Thus we see that $h \in \mathcal{H}_{\bar{F}_{5\pm}}$ if and only if

$$6v_4\beta(v_2, v_3, v_4) + \frac{\partial \gamma}{\partial v_4}(v_2, v_3, v_4) = 0, \tag{7.11}$$

$$v_4 \frac{\partial \gamma}{\partial v_2} \pm 2v_3\beta + \frac{\partial \gamma}{\partial v_3} = 0. \tag{7.12}$$

This completes the proof of the second formula. Q.E.D.

Expressing γ in the form

$$\gamma(v_2, v_3, v_4) = v_4^2\gamma_1(v_2, v_3, v_4) + v_4\gamma_2(v_2, v_3) + \gamma_3(v_2, v_3) \tag{7.13}$$

we will see that (7.10) is equivalent to

$$\gamma_2 = 0, \qquad \beta = -\frac{1}{6}\left(2\gamma_1 + v_4\frac{\partial\gamma_1}{\partial v_4}\right)$$

and (7.11) is equivalent to

$$\left(\mp\frac{2}{3}v_3\gamma_1 + v_4^3\frac{\partial\gamma_1}{\partial v_2} + v_4\frac{\partial\gamma_1}{\partial v_3} + v_4^2\frac{\partial\gamma_1}{\partial v_3} \mp \frac{1}{3}v_3v_4\frac{\partial\gamma_1}{\partial v_4}\right) + \left(v_4\frac{\partial\gamma_3}{\partial v_2} + \frac{\partial\gamma_3}{\partial v_3}\right) = 0.$$

Thus we have

Proposition 7.3.

$$\mathcal{H}_{\bar{F}_\pm} = \langle\Delta_\pm^2\rangle_{\mathcal{E}_v} + \left\{\Delta_\pm\beta + \gamma \ \mid \beta, \gamma \in \mathcal{E}_{v_2,v_3,v_4} \text{ and satisfy the following conditions}\right\}$$

γ being expressed in the form (7.16),

$$\gamma_2 = 0, \qquad \beta = -\frac{1}{6}\left(2\gamma_1 + v_4\frac{\partial\gamma_1}{\partial v_4}\right), \tag{7.14}$$

$$\left(\mp\frac{2}{3}v_3\gamma_1 + v_4^3\frac{\partial\gamma_1}{\partial v_2} + v_4\frac{\partial\gamma_1}{\partial v_3} + v_4^2\frac{\partial\gamma_1}{\partial v_3} \mp \frac{1}{3}v_3v_4\frac{\partial\gamma_1}{\partial v_4}\right) + \left(v_4\frac{\partial\gamma_3}{\partial v_2} + \frac{\partial\gamma_3}{\partial v_3}\right) = 0. \tag{7.15}$$

Proof. Next we express γ in the form

$$\gamma(v_2, v_3, v_4) = v_4^2\gamma_1(v_2, v_3, v_4) + v_4\gamma_2(v_2, v_3) + \gamma_3(v_2, v_3). \tag{7.16}$$

Then (7.10) is equivalent to

$$6v_4\beta(v_2, v_3, v_4) + 2v_4\gamma_1(v_2, v_3, v_4) + v_4^2\frac{\partial\gamma_1}{\partial v_4}(v_2, v_3, v_4) + \gamma_2(v_2, v_3) = 0,$$

which is equivalent to

$$\gamma_2(v_2, v_3) = 0, \quad 6v_4\beta(v_2, v_3, v_4) + 2v_4\gamma_1(v_2, v_3, v_4) + v_4^2\frac{\partial\gamma_1}{\partial v_4}(v_2, v_3, v_4) = 0.$$

since $\gamma_2(v_2, v_3)$ are function of v_2, v_3 while the other terms have v_4 as a factor. Thus (7.3) is equivalent to

$$\gamma_2 = 0, \qquad 6\beta + 2\gamma_1 + v_4\frac{\partial\gamma_1}{\partial v_4} = 0,$$

and hence to

$$\gamma_2 = 0, \qquad \beta = -\frac{1}{6}\left(2\gamma_1 + v_4\frac{\partial\gamma_1}{\partial v_4}\right).$$

Now we consider the equality

$$v_4\frac{\partial\gamma}{\partial v_2} \pm 2v_3\beta + \frac{\partial\gamma}{\partial v_3} = 0.$$

From (7.16) and (7.10), (7.12) is equivalent to

$$v_4\left(v_4^2\frac{\partial\gamma_1}{\partial v_2}+\frac{\partial\gamma_3}{\partial v_2}\right)\mp\frac{1}{3}v_3\left(2\gamma_1+v_4\frac{\partial\gamma_1}{\partial v_4}\right)+\left(v_4^2\frac{\partial\gamma_1}{\partial v_3}+\frac{\partial\gamma_3}{\partial v_3}\right)=0,$$

which is equivalent to

$$\left(\mp\frac{2}{3}v_3\gamma_1+v_4^3\frac{\partial\gamma_1}{\partial v_2}+v_4\frac{\partial\gamma_1}{\partial v_3}+v_4^2\frac{\partial\gamma_1}{\partial v_3}\mp\frac{1}{3}v_3v_4\frac{\partial\gamma_1}{\partial v_4}\right)+\left(v_4\frac{\partial\gamma_3}{\partial v_2}+\frac{\partial\gamma_3}{\partial v_3}\right)=0.$$

Thus we see that $h\in\mathcal{H}_{\bar{F}_{5\pm}}$ if and only if (7.10) and (7.15) hold. This completes the proof of Proposition 7.3. Q.E.D.

From this proposition we see that β is determined by γ. In this sense Proposition 7.3 is more advanced than Proposition 7.2. However the equation (7.15) is complicated.

To have More detailed representation, we express β in the form

$$\beta(v_2,v_3,v_4)=v_4\beta_1(v_2,v_3,v_4)+\beta_2(v_2,v_3). \tag{7.17}$$

Recall that from (7.16) and from (7.14), γ has the form

$$\gamma(v_2,v_3,v_4)=v_4^2\gamma_1(v_2,v_3,v_4)+\gamma_3(v_2,v_3). \tag{7.18}$$

From (7.17) and (7.18),

$$v_4\frac{\partial\gamma}{\partial v_2}\pm 2v_3\beta+\frac{\partial\gamma}{\partial v_3}=0. \tag{7.19}$$

is equivalent to

$$v_4\left(v_4^2\frac{\partial\gamma_1}{\partial v_2}+\frac{\partial\gamma_3}{\partial v_2}\right)\pm 2v_3\Big(v_4\beta_1(v_2,v_3,v_4)+\beta_2(v_2,v_3)\Big)+v_4^2\frac{\partial\gamma_1}{\partial v_3}+\frac{\partial\gamma_3}{\partial v_3}=0.$$

Dividing (7.19) into those terms containing v_4 as a factor and those terms not containing v_4, (7.19) is equivalent to the pair of

$$\frac{\partial\gamma_3}{\partial v_3}(v_2,v_3)\pm 2v_3\beta_2(v_2,v_3)=0, \tag{7.20}$$

and

$$v_4^2\frac{\partial\gamma_1}{\partial v_2}+\frac{\partial\gamma_3}{\partial v_2}+v_4\frac{\partial\gamma_1}{\partial v_3}\pm 2v_3\beta_1=0. \tag{7.21}$$

Expressing γ_3 in the form

$$\gamma_3(v_2,v_3)=v_3\gamma_{31}(v_2,v_3)+\gamma_{32}(v_2), \tag{7.22}$$

(7.20) is equivalent to

$$\gamma_{31}(v_2,v_3)+v_3\frac{\partial\gamma_{31}}{\partial v_3}(v_2,v_3)\pm 2v_3\beta_2(v_2,v_3)=0$$

which is equivalent to the pair of

$$\gamma_{31}(v_2,v_3)=v_3\bar{\gamma}_{31}(v_2,v_3),\quad \exists\bar{\gamma}_{31}\in\mathcal{E}_{v_2,v_3},$$

$$v_3\bar{\gamma}_{31}(v_2,v_3)+v_3\bar{\gamma}_{31}(v_2,v_3)+v_3^2\frac{\partial\bar{\gamma}_{31}}{\partial v_3}(v_2,v_3)\pm 2v_3\beta_2(v_2,v_3)=0.$$

Thus we see that (7.20) is equivalent to the pair of conditions

(7.23) γ_3 has the form $\quad \gamma_3(v_2,v_3)=v_3^2\bar{\gamma}_{31}(v_2,v_3)+\gamma_{32}(v_2),$

$$\text{(7.24)} \qquad \text{and} \quad \beta_2(v_2,v_3) = \mp\left(\bar{\gamma}_{31}(v_2,v_3) + \frac{1}{2}v_3\frac{\partial\bar{\gamma}_{31}}{\partial v_3}(v_2,v_3)\right).$$

Now we observe (7.21):

$$v_4^2\frac{\partial\gamma_1}{\partial v_2} + \frac{\partial\gamma_3}{\partial v_2} + v_4\frac{\partial\gamma_1}{\partial v_3} \pm 2v_3\beta_1 = 0.$$

We express β_1 in the form

$$\beta_1(v_2,v_3,v_4) = v_4\beta_{11}(v_2,v_3,v_4) + \beta_{12}(v_2,v_3). \tag{7.25}$$

Then (7.21) becomes of the form

$$v_4^2\frac{\partial\gamma_1}{\partial v_2} + \frac{\partial\gamma_3}{\partial v_2}(v_2,v_3) \pm 2v_3(v_4\beta_{11}(v_2,v_3,v_4) + \beta_{12}(v_2,v_3)) + v_4\frac{\partial\gamma_1}{\partial v_3} = 0,$$

dividing into those terms containing v_4 as a factor and those not containing v_4, which is equivalent to the pair of

$$\frac{\partial\gamma_3}{\partial v_2}(v_2,v_3) \pm 2v_3\beta_{12}(v_2,v_3) = 0 \tag{7.26}$$

and

$$v_4\frac{\partial\gamma_1}{\partial v_2} \pm 2v_3\beta_{11}(v_2,v_3,v_4) + \frac{\partial\gamma_1}{\partial v_3} = 0.$$

From (7.23), (7.26) is equivalent to

$$v_3^2\frac{\partial\bar{\gamma}_{31}}{\partial v_2}(v_2,v_3) + \frac{\partial\gamma_{32}}{\partial v_2}(v_2) \pm 2v_3\beta_{12}(v_2,v_3) = 0$$

which is equivalent to the pair of

$$\frac{\partial\gamma_{32}}{\partial v_2}(v_2) = 0,$$

and

$$v_3^2\frac{\partial\bar{\gamma}_{31}}{\partial v_2}(v_2,v_3) \pm 2v_3\beta_{12}(v_2,v_3) = 0,$$

which are subsequently equivalent to

$$\gamma_{32}(v_2) = \text{const},$$

$$\beta_{12}(v_2,v_3) = \mp\frac{1}{2}v_3^2\frac{\partial\bar{\gamma}_{31}}{\partial v_2}(v_2,v_3).$$

From the above equation, (7.23) is equivalent to

$$\gamma_3 \text{ has the form} \qquad \gamma_3(v_2,v_3) = v_3^2\bar{\gamma}_{31}(v_2,v_3) + \text{constant}.$$

Now we observe

$$v_4\frac{\partial\gamma_1}{\partial v_2} \pm 2v_3\beta_{11}(v_2,v_3,v_4) + \frac{\partial\gamma_1}{\partial v_3} = 0. \tag{7.27}$$

We express $\gamma_1(v_2,v_3,v_4)$ in the form

$$\gamma_1(v_2,v_3,v_4) = v_3^2\gamma_{11}(v_2,v_3,v_4) + v_3\gamma_{12}(v_2,v_4) + \gamma_{13}(v_2,v_4). \tag{7.28}$$

Then (7.27) is equivalent to

$$v_4\left(v_3^2\frac{\partial\gamma_{11}}{\partial v_2}+v_3\frac{\partial\gamma_{12}}{\partial v_2}+\frac{\partial\gamma_{13}}{\partial v_2}\right)\pm 2v_3\beta_{11}(v_2,v_3,v_4)$$
$$+\left(2v_3\gamma_{11}+v_3^2\frac{\partial\gamma_{11}}{\partial v_3}+\gamma_{12}\right)=0,$$

which is equivalent to the pair of

$$v_4\frac{\partial\gamma_{13}}{\partial v_2}(v_2,v_4)+\gamma_{12}(v_2,v_4)=0,$$

$$v_4\left(v_3\frac{\partial\gamma_{11}}{\partial v_2}+\frac{\partial\gamma_{12}}{\partial v_2}\right)\pm 2\beta_{11}(v_2,v_3,v_4)+\left(2\gamma_{11}+v_3\frac{\partial\gamma_{11}}{\partial v_3}\right)=0.$$

Thus (7.27) is equivalent to the pair of

$$\gamma_{12}(v_2,v_4)=-v_4\frac{\partial\gamma_{13}}{\partial v_2}(v_2,v_4),$$

$$\beta_{11}(v_2,v_3,v_4)=\mp\frac{1}{2}\left(2\gamma_{11}+v_3v_4\frac{\partial\gamma_{11}}{\partial v_2}+v_3\frac{\partial\gamma_{11}}{\partial v_3}+v_4\frac{\partial\gamma_{12}}{\partial v_2}\right).$$

We list here all the equalities concerning β obtained so far:

$$\beta=-\frac{1}{6}\left(2\gamma_1+v_4\frac{\partial\gamma_1}{\partial v_4}\right).$$

Expressing β in the form

$$\beta(v_2,v_3,v_4)=v_4\beta_1(v_2,v_3,v_4)+\beta_2(v_2,v_3),$$

we have

$$\beta_2(v_2,v_3)=\mp\left(\bar\gamma_{31}(v_2,v_3)+\frac{1}{2}v_3\frac{\partial\bar\gamma_{31}}{\partial v_3}(v_2,v_3)\right).$$

where

$$\gamma_3 \text{ has the form}\qquad \gamma_3(v_2,v_3)=v_3^2\bar\gamma_{31}(v_2,v_3)+\gamma_{32}(v_2).$$

Expressing β_1 in the form

$$\beta_1(v_2,v_3,v_4)=v_4\beta_{11}(v_2,v_3,v_4)+\beta_{12}(v_2,v_3)$$

$$\beta_{12}(v_2,v_3)=\mp\frac{1}{2}v_3^2\frac{\partial\bar\gamma_{31}}{\partial v_2}(v_2,v_3).$$

$$\beta_{11}(v_2,v_3,v_4)=\mp\frac{1}{2}\left(2\gamma_{11}+v_3v_4\frac{\partial\gamma_{11}}{\partial v_2}+v_3\frac{\partial\gamma_{11}}{\partial v_3}+v_4\frac{\partial\gamma_{12}}{\partial v_2}\right).$$

Combining all together, we obtain

$$\begin{aligned}
\beta(v_2,v_3,v_4) &= v_4\beta_1(v_2,v_3,v_4)+\beta_2(v_2,v_3)\\
&= v_4\big(v_4\beta_{11}(v_2,v_3,v_4)+\beta_{12}(v_2,v_3)\big)\mp\left(\bar\gamma_{31}(v_2,v_3)+\frac{1}{2}v_3\frac{\partial\bar\gamma_{31}}{\partial v_3}(v_2,v_3)\right)\\
&= \mp\frac{1}{2}v_4^2\left(2\gamma_{11}+v_3v_4\frac{\partial\gamma_{11}}{\partial v_2}+v_3\frac{\partial\gamma_{11}}{\partial v_3}+v_4\frac{\partial\gamma_{12}}{\partial v_2}\right)\mp\frac{1}{2}v_4v_3^2\frac{\partial\bar\gamma_{31}}{\partial v_2}(v_2,v_3)\\
&\quad\mp\left(\bar\gamma_{31}(v_2,v_3)+\frac{1}{2}v_3\frac{\partial\bar\gamma_{31}}{\partial v_3}(v_2,v_3)\right)\\
&= \mp\frac{1}{2}v_4^2\left(2\gamma_{11}(v_2,v_3,v_4)+v_3v_4\frac{\partial\gamma_{11}}{\partial v_2}(v_2,v_3,v_4)+v_3\frac{\partial\gamma_{11}}{\partial v_3}(v_2,v_3,v_4)\right.\\
&\qquad\left.+v_4\frac{\partial\gamma_{12}}{\partial v_2}(v_2,v_4)\right)\\
&\quad\mp\left(\bar\gamma_{31}(v_2,v_3)+\frac{1}{2}v_4v_3^2\frac{\partial\bar\gamma_{31}}{\partial v_2}(v_2,v_3)+\frac{1}{2}v_3\frac{\partial\bar\gamma_{31}}{\partial v_3}(v_2,v_3)\right) \qquad (7.29)
\end{aligned}$$

Now we list equalities concerning $\gamma(v_2,v_3,v_4)$. We expressed $h(v_1,v_2,v_3,v_4)$ in the form

$$h=\Delta_\pm^2\alpha(v)+\Delta_\pm\beta(v_2,v_3,v_4)+\gamma(v_2,v_3,v_4),$$

and $\gamma(v_2,v_3,v_4)$ in the form

$$\gamma(v_2,v_3,v_4)=v_4^2\gamma_1(v_2,v_3,v_4)+v_4\gamma_2(v_2,v_3)+\gamma_3(v_2,v_3).$$

Then the condition

$$\frac{\partial h}{\partial v_4}\in\langle\Delta_\pm\rangle_{\mathcal{E}_v}$$

is equivalent to

$$\gamma_2=0,\qquad \beta=-\frac{1}{6}\left(2\gamma_1+v_4\frac{\partial\gamma_1}{\partial v_4}\right).$$

$$\gamma_3 \text{ has the form}\qquad \gamma_3(v_2,v_3)=v_3^2\bar\gamma_{31}(v_2,v_3)+\gamma_{32}(v_2),$$
$$\gamma_{32}(v_2)=\text{const},$$

and

$$\gamma_3 \text{ has the form}\qquad \gamma_3(v_2,v_3)=v_3^2\bar\gamma_{31}(v_2,v_3)+\text{constant}.$$

We expressed $\gamma_1(v_2,v_3,v_4)$ in the form

$$\gamma_1(v_2,v_3,v_4)=v_3^2\gamma_{11}(v_2,v_3,v_4)+v_3\gamma_{12}(v_2,v_4)+\gamma_{13}(v_2,v_4).$$

Then

$$\gamma_{12}(v_2,v_4)=-v_4\frac{\partial\gamma_{13}}{\partial v_2}(v_2,v_4),$$

from which

$$\gamma_1(v_2,v_3,v_4)=v_3^2\gamma_{11}(v_2,v_3,v_4)-v_3v_4\frac{\partial\gamma_{13}}{\partial v_2}(v_2,v_4)+\gamma_{13}(v_2,v_4).$$

Then the second equality in (7.14)

$$\beta=-\frac{1}{6}\left(2\gamma_1+v_4\frac{\partial\gamma_1}{\partial v_4}\right)$$

is equivalent to

$$\begin{aligned}
\beta &= -\frac{1}{6}\left\{2\left(v_3^2\gamma_{11}(v_2,v_3,v_4) - v_3v_4\frac{\partial\gamma_{13}}{\partial v_2}(v_2,v_4) + \gamma_{13}(v_2,v_4)\right)\right. \\
&\quad \left. +v_4\left(v_3^2\frac{\partial\gamma_{11}}{\partial v_4} - v_3\frac{\partial\gamma_{13}}{\partial v_2} - v_3v_4\frac{\partial^2\gamma_{13}}{\partial v_2\partial v_4} + \frac{\partial\gamma_{13}}{\partial v_4}\right)\right\} \\
&= \left(-\frac{1}{3}v_3^2\gamma_{11} + v_3^2v_4\frac{\partial\gamma_{11}}{\partial v_4}\right) + \left(-\frac{1}{3}\gamma_{13} - \frac{2}{3}v_3v_4\frac{\partial\gamma_{13}}{\partial v_2} + v_4\frac{\partial\gamma_{13}}{\partial v_4} - v_3v_4\frac{\partial^2\gamma_{13}}{\partial v_2\partial v_4}\right)
\end{aligned} \tag{7.30}$$

On the other hand, $\gamma_{12}(v_2,v_4) = -v_4\frac{\partial\gamma_{13}}{\partial v_2}(v_2,v_4)$,

$$\begin{aligned}
\beta(v_2,v_3,v_4) &= \mp\frac{1}{2}v_4^2\left(2\gamma_{11}(v_2,v_3,v_4) + v_3v_4\frac{\partial\gamma_{11}}{\partial v_2}(v_2,v_3,v_4) + v_3\frac{\partial\gamma_{11}}{\partial v_3}(v_2,v_3,v_4)\right. \\
&\qquad \left. +v_4\frac{\partial\gamma_{12}}{\partial v_2}(v_2,v_4)\right) \\
&\quad \mp\left(\bar{\gamma}_{31}(v_2,v_3) + \frac{1}{2}v_4v_3^2\frac{\partial\bar{\gamma}_{31}}{\partial v_2}(v_2,v_3) + \frac{1}{2}v_3\frac{\partial\bar{\gamma}_{31}}{\partial v_3}(v_2,v_3)\right) \\
&= \mp\frac{1}{2}v_4^2\left(2\gamma_{11}(v_2,v_3,v_4) + v_3v_4\frac{\partial\gamma_{11}}{\partial v_2}(v_2,v_3,v_4) + v_3\frac{\partial\gamma_{11}}{\partial v_3}(v_2,v_3,v_4)\right. \\
&\qquad \left. -v_4^2\frac{\partial^2\gamma_{13}}{\partial v_2^2}(v_2,v_4)\right) \\
&\quad \mp\left(\bar{\gamma}_{31}(v_2,v_3) + \frac{1}{2}v_4v_3^2\frac{\partial\bar{\gamma}_{31}}{\partial v_2}(v_2,v_3) + \frac{1}{2}v_3\frac{\partial\bar{\gamma}_{31}}{\partial v_3}(v_2,v_3)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(-\frac{1}{3}v_3^2\gamma_{11} + v_3^2v_4\frac{\partial\gamma_{11}}{\partial v_4}\right) + \left(-\frac{1}{3}\gamma_{13}(v_2,v_4) - \frac{2}{3}v_3v_4\frac{\partial\gamma_{13}}{\partial v_2} + v_4\frac{\partial\gamma_{13}}{\partial v_4} - v_3v_4\frac{\partial^2\gamma_{13}}{\partial v_2\partial v_4}\right) \\
&\quad = \mp\frac{1}{2}v_4^2\left(2\gamma_{11} + v_3v_4\frac{\partial\gamma_{11}}{\partial v_2} + v_3\frac{\partial\gamma_{11}}{\partial v_3} - v_4^2\frac{\partial^2\gamma_{13}}{\partial v_2^2}\right) \\
&\qquad \mp\left(\bar{\gamma}_{31}(v_2,v_3) + \frac{1}{2}v_4v_3^2\frac{\partial\bar{\gamma}_{31}}{\partial v_2} + \frac{1}{2}v_3\frac{\partial\bar{\gamma}_{31}}{\partial v_3}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(\left(-\frac{1}{3}v_3^2 \pm v_4^2\right)\gamma_{11}(v_2,v_3,v_4) \pm \frac{1}{2}v_3v_4^3\frac{\partial\gamma_{11}}{\partial v_2} \pm \frac{1}{2}v_3v_4^2\frac{\partial\gamma_{11}}{\partial v_3} + v_3^2v_4\frac{\partial\gamma_{11}}{\partial v_4}\right) \\
&\quad + \left(-\frac{1}{3}\gamma_{13}(v_2,v_4) - \frac{2}{3}v_3v_4\frac{\partial\gamma_{13}}{\partial v_2} + v_4\frac{\partial\gamma_{13}}{\partial v_4} \mp \frac{1}{2}v_4^4\frac{\partial^2\gamma_{13}}{\partial v_2^2} - v_3v_4\frac{\partial^2\gamma_{13}}{\partial v_2\partial v_4}\right) \\
&= \mp\left(\bar{\gamma}_{31}(v_2,v_3) + \frac{1}{2}v_4v_3^2\frac{\partial\bar{\gamma}_{31}}{\partial v_2} + \frac{1}{2}v_3\frac{\partial\bar{\gamma}_{31}}{\partial v_3}\right).
\end{aligned} \tag{7.31}$$

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