Almost sure and moment stability of stochastic partial differential equations.

ANNA A. KWIECIŃSKA

Institute of Mathematics, Polish Academy of Sciences,
ul. Śniadeckich 8, 00-950 Warszawa, Poland.
email: akwiecin@impan.gov.pl

Abstract

We study the almost sure and moment stability of a class of stochastic partial differential equations and we present an infinite-dimensional version of a theorem proved for stochastic ordinary differential equations by Arnold, Oeljeklaus and Pardoux. We also investigate how adding a term with white noise influences the stability of a deterministic system. The outcome is quite surprising. It turns out that regardless whether the deterministic system was stable or unstable, after adding a term with sufficiently large noise, it becomes pathwise exponentially stable and unstable in the p-th mean, for $p > 1$.

Keywords: stochastic partial differential equation, almost sure stability, moment stability, deterministic partial differential equation, stabilization by noise, destabilization by noise.

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1. Introduction.

The purpose of this paper is to study the almost sure and moment stability of a class of stochastic partial differential equations and how the white noise influences the stability of a deterministic system. To do so, we need some preparation in the deterministic set-up.

Accordingly, we study first the deterministic Dirichlet problem for the following equation (for a formal setting see Section 2):

$$\frac{\partial u}{\partial t} = Au,$$

with the initial condition f, where the operator A is as in Section 2. The Lyapunov exponent of the deterministic system is defined as

$$\lambda^{det}(f) = \limsup_{t\to\infty} \frac{1}{t} \log \|u(t)\|,$$

we prove that it exists as a limit.

Then, in Section 3, we consider the Dirichlet problem for the following stochastic perturbation of the above equation:

$$dv(t) = Av(t)dt + \sigma v(t)d\beta(t),$$

with the initial condition f, where $\beta(\cdot)$ is a one-dimensional real-valued Wiener processes and $\sigma \neq 0$ is a constant. The almost sure Lyapunov exponent of the stochastic system is defined as

$$\lambda_\sigma^{st}(f,\omega) = \limsup_{t\to\infty} \frac{1}{t} \log \|v(t,\omega)\|,$$

we prove that it exists as a limit and is non-random. The following formula holds (see Theorem 2):

$$\lambda_\sigma^{st}(f) = \lambda_\sigma^{st}(f,\omega) = \lambda^{det}(f) - \frac{1}{2}\sigma^2 \quad a.s.$$

Next we consider the Lyapunov exponent of the p-th moment of the solution to the stochastic problem. For $p > 0$ it is defined as:

$$g_\sigma(p,f) = \limsup_{t\to\infty} \frac{1}{t} \log E\|v(t,f)\|^p.$$

We prove that it exists as a limit and the following equality holds (see Theorem 3):

$$g_\sigma(p,f) = p\lambda^{det}(f) + \frac{p}{2}(p-1)\sigma^2.$$

In Section 3 we also derive a modification of Theorem 2.1 from [2] - see Theorem 4.

In Section 4, we conclude that:

$$\lambda_\sigma^{st}(f) \to -\infty \quad (\sigma \to \infty),$$

it follows that for σ big enough the stochastic system is pathwise exponentially stable – this generalizes our result from [6];
for $p < 1$

$$g_\sigma(p, f) \to -\infty \quad (\sigma \to \infty),$$

it follows that for σ big enough the stochastic system is stable in the p-th mean;
for $p = 1$

$$g_\sigma(1, f) = \lambda^{det}(f), \quad \sigma > 0,$$

it follows that adding a term with white noise does not influence the p-th mean stability;
for $p > 1$

$$g_\sigma(p, f) \to \infty \quad (\sigma \to \infty),$$

it follows that for σ big enough the system is unstable in the p-th mean.
All that holds regardless whether the deterministic system was stable or unstable and it is quite surprising.

In Section 5 we provide an example.

In Section 6 we consider the stochastic problem with the Itô differential replaced by the Stratonovich differential. In that case the almost sure Lyapunov exponent of the stochastic system is equal to the deterministic one, i.e. adding the stochastic term does not influence the stability. On the other hand we have destabilization in the p-th mean for any $p > 0$. For $p > 1$ the destabilization is even faster than in the case of the Itô differential.

2. The deterministic problem.

Let $\mathcal{O}$ be a bounded domain in $\mathbf{R}^d$: a ball or a set that can be mapped into a ball by a regular mapping of class $C^2(\bar{\mathcal{O}})$.

We will study the following parabolic equation:

$$\frac{\partial u}{\partial t} = Au, \tag{1}$$

where $u = u(t, x)$, $t \in \mathbf{R}^+$ (where $\mathbf{R}^+$ denotes the interval $[0, \infty)$), $x \in \mathcal{O}$. We set the initial condition

$$u(0, x) = f(x) \tag{2}$$

and the Dirichlet boundary condition

$$u(t, x) = 0, \quad x \in \partial\mathcal{O}. \tag{3}$$

The function f in (2) takes real values. We consider only deterministic real-valued initial conditions $f \in \mathcal{D}(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$.

The operator A is given by the following formula

$$Au = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a(x)u,$$

where the coefficients satisfy the following assumptions:
(C1) a_{ij} are differentiable
(C2) $a_{ij} = a_{ji}$
(C3) $\mu_1\xi^2 \leq \sum_{i,j} a_{ij}\xi_i\xi_j \leq \mu_2\xi^2$
(C4) $\left|\frac{\partial a_{ij}}{\partial x_k}\right| \leq \mu_3$
(C5) $|a| \leq \mu_4$,
where μ_l are constants and $\mu_l > 0$, $l = 1, \ldots, 4$.

It is easy to verify that the operator A is symmetric on the space $H_0^1(\mathcal{O})$.

Let $W_{2,0}^2(\mathcal{O})$ be a subspace of $H^2(\mathcal{O})$, where the functions belonging to $C^2(\bar{\mathcal{O}})$ and vanishing on $\partial\mathcal{O}$ are dense. Later on we will prove that in our case $W_{2,0}^2(\mathcal{O}) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$.

There exists an orthonormal basis of $L^2(\mathcal{O})$, $\{e_j\}$, $j = 1, 2, \ldots$, consisting of the eigenvectors of the operator A (see [7], p.181) such that

$$Ae_j = \lambda_j e_j, \quad \text{where} \quad \lambda_j \searrow -\infty \ (j \to \infty)$$

and

$$< e_k, e_l >= (\lambda_0 - \lambda_k)^2 \delta_k^l,$$

where $< u, u >^{\frac{1}{2}}$ is equivalent to the norm in $W_{2,0}^2(\mathcal{O})$. The constant λ_0 is chosen such that $\lambda_k < \lambda_0$, $k = 1, 2, \ldots$.

Lemma 1 *Under our assumptions we have the following characterization of the space* $W^2_{2,0}(\mathcal{O})$

$$W^2_{2,0}(\mathcal{O}) = \{u = \sum_{j=1}^{\infty} u_j e_j(x) \in L^2(\mathcal{O}) : \sum_{j=1}^{\infty} u_j^2(\lambda_0 - \lambda_j)^2 < \infty\}.$$

Proof. Obviously $\left\{\frac{1}{\lambda_0-\lambda_j}e_j\right\}$ is an orthonormal basis of $W^2_{2,0}(\mathcal{O})$ with the scalar product $< \cdot, \cdot >$.

If $u \in L^2(\mathcal{O})$ we can write

$$u = \sum_{j=1}^{\infty} u_j e_j = \sum_{j=1}^{\infty} (\lambda_0 - \lambda_j) u_j \left(\frac{1}{\lambda_0 - \lambda_j} e_j\right)$$

and the equality follows. □

The following Proposition may be known, but as we are unable to provide a reference we present our own proof.

Proposition 1 *Under our assumptions the following equality holds:*

$$W^2_{2,0}(\mathcal{O}) = H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}).$$

Proof. Obviously we have

$$W^2_{2,0}(\mathcal{O}) \subset H^2(\mathcal{O}) \quad (*)$$

The inclusion

$$W^2_{2,0}(\mathcal{O}) \subset H^1_0(\mathcal{O}) \quad (**)$$

follows from the trace theorem (see [8], p.48):

Theorem (trace theorem): Let $\mathcal{O}$ be a bounded set of class C^1. There exists a linear continuous operator $\gamma_0 \in \mathcal{L}(H^1(\mathcal{O}), L^2(\partial\mathcal{O}))$ such that:

$$\gamma_0 u = u|_{\partial\mathcal{O}}, \ \forall u \in C^1(\bar{\mathcal{O}}).$$

We have

$$u \in W^2_{2,0}(\mathcal{O}) \Longleftrightarrow \exists\{u_m\} \in C^2(\bar{\mathcal{O}}), \ u_m|_{\partial\mathcal{O}} = 0, \ u_m \to u \quad \text{in} \quad H^2(\mathcal{O}).$$

Since $\gamma_0 u_m = 0$, γ_0 is continuous in $H^1(\mathcal{O})$ and $u_m \to u$ in $H^1(\mathcal{O})$, we deduce that

$$\gamma_0 u = u|_{\partial\mathcal{O}} = 0.$$

In the case of bounded regions with smooth boundary $u \in H_0^1(\mathcal{O})$ is equivalent to the conditions $u \in H^1(\mathcal{O})$ and $u|_{\partial\mathcal{O}} = 0$, see [4], p.122. Therefore we conclude that $u \in H_0^1(\mathcal{O})$, which proves $(**)$.

The inclusions $(*)$ and $(**)$ give

$$W_{2,0}^2(\mathcal{O}) \subset H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}).$$

Let us now take $u \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$. The operator A is symmetric on the space $H_0^1(\mathcal{O})$ and therefore is closable, see [5], p. 269. We have

$$Au = \bar{A}u = \bar{A}\sum_{j=1}^{\infty} u_j e_j(x) = \sum_{j=1}^{\infty} u_j \bar{A} e_j(x) = \sum_{j=1}^{\infty} u_j A e_j(x)$$

$$= \sum_{j=1}^{\infty} u_j \lambda_j e_j(x) \in L^2(\mathcal{O}).$$

Note that $\sum_{j=1}^{\infty} u_j \lambda_j e_j(x) \in L^2(\mathcal{O})$ if and only if $\sum_{j=1}^{\infty} u_j^2 \lambda_j^2 < \infty$, which implies $\sum_{j=1}^{\infty} u_j^2 (\lambda_0 - \lambda_j)^2 < \infty$, and by Lemma 1 it gives $u \in W_{2,0}^2(\mathcal{O})$. Hence the inclusion

$$H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}) \subset W_{2,0}^2(\mathcal{O})$$

follows. □

If $f \in L^2(\mathcal{O})$, we can write

$$f = \sum_{j=1}^{\infty} f_j e_j, \quad \text{where} \quad f_j =< f, e_j > . \tag{4}$$

It is easy to check that the unique strong solution to the problem (1)-(3) is given by the following formula

$$u(t) = S(t)f = \sum_{j=1}^{\infty} \exp(t\lambda_j) f_j e_j. \tag{5}$$

The Lyapunov exponent of the above system is defined as

$$\lambda^{det}(f) = \limsup_{t\to\infty} \frac{1}{t} \log \|u(t)\|.$$

The following theorem holds.

Theorem 1 *Let $f \neq 0$ and let j_0 be the smallest integer $j \geq 1$ in the expansion (4) of f such that $f_{j_0} \neq 0$.*

Then the Lyapunov exponent of (1)-(3) exists as a limit and is given by

$$\lambda^{\text{det}}(f) = \lambda_{j_0}.$$

Therefore the top Lyapunov exponent is equal to λ_1.

Proof. On the one hand,

$$\frac{1}{t}\log \|\sum_{j=1}^{\infty} \exp(t\lambda_j) f_j e_j\| \leq \frac{1}{t}\log(\sum_{j=j_0}^{\infty} |\exp(t\lambda_{j_0}) f_j|^2)^{\frac{1}{2}} = \lambda_{j_0} + \frac{1}{t}\log\|f\|$$

while on the other hand,

$$\frac{1}{t}\log \|\sum_{j=1}^{\infty} \exp(t\lambda_j) f_j e_j\| \geq \frac{1}{t}\log|\exp(t\lambda_{j_0}) f_{j_0}| = \lambda_{j_0} + \frac{1}{t}\log|f_{j_0}|.$$

The existence of the limit and the equality follow. □

3. The stochastic problem

Let us now consider the following stochastic perturbation of the deterministic problem studied in Section 2.

$$dv(t) = Av(t)dt + \sigma v(t) d\beta(t), \tag{6}$$

where $\beta(\cdot)$ is a one-dimensional, real valued Wiener processes and $\sigma \neq 0$ is a constant, with the initial condition

$$v(0, x, \omega) = f(x), \quad x \in \mathcal{O} \tag{7}$$

and the Dirichlet boundary condtion

$$v(t, x, \omega) = 0, \quad x \in \partial\mathcal{O}. \tag{8}$$

Conditions (7) and (8) hold for a.a. $\omega \in \Omega$.

It is easy to check that the unique strong solution to the stochastic problem (6)-(8) is given by the following formula

$$v(t) = \exp(\sigma\beta(t))\exp(-\frac{1}{2}\sigma^2 t)u(t), \tag{9}$$

where $u(t)$ is the solution to the deterministic problem.

The almost sure Lyapunov exponent of the system (6)-(8) is defined pathwise as

$$\lambda_\sigma^{st}(f,\omega) = \limsup_{t\to\infty} \frac{1}{t}\log\|v(t,\omega)\|.$$

In the stochastic we can prove that:

Theorem 2 *The Lyapunov exponent of the system (6)-(8) almost surely exists as a limit, is non-random and the following formula holds:*

$$\lambda_\sigma^{\rm st}(f,\omega) = \lambda^{\rm det}(f) - \frac{1}{2}\sigma^2 = \lambda_{j_0} - \frac{1}{2}\sigma^2 \quad a.s.$$

Therefore the top Lyapunov exponent is equal to $\lambda_1 - \frac{1}{2}\sigma^2$.

Proof. It follows from Theorem 1, the formula (9) and the fact that $\lim_{t\to\infty}\beta(t)/t = 0$ a.s. for $k = 1, 2, \ldots, N$ (see [1], p.46). □

For stochastic systems different kinds of stability are investigated. Another one is the stability in the p-th mean ($p = 2 =$ mean square). The problem (6)-(8) is stable in the p-th mean, for $p > 0$, if there exist constants $M > 0$ and $\delta > 0$ such that

$$E\|v(t,f)\|^p \leq M\exp(-\delta t)\|f\|^p,$$

for each $f \in \mathcal{D}(A)$ and $t \geq 0$.

The stability in the p-th mean can also be expressed in terms of the Lyapunov exponents of the p-th moment of the solution. The Laypunov exponent of the p-th moment of the solution to (6)-(8) is defined, for $p > 0$, as

$$g_\sigma(p,f) = \limsup_{t\to\infty} \frac{1}{t}\log E\|v(t,f)\|^p.$$

Lemma 2 *The system (6)-(8) is stable in the p-th mean if and only if the top Lyapunov exponent of the p-th moment of the solution is smaller than zero.*

Proof. We omit the proof as it is straightforward.

Remark 1 *For deterministic systems the stability in the p-th mean is equivalent to the top classical Lyapunov exponent being smaller than zero.*

More precisely, for deterministic systems, the following equality holds

$$g(p,f) = p\lambda^{det}(f).$$

Theorem 3 *The Lyapunov exponent of the p-th moment of the solution to the problem (6)-(8) exists as a limit as is equal to*

$$\begin{aligned} g_\sigma(p,f) &= p\lambda^{det}(f) + \frac{p}{2}(p-1)\sigma^2 \\ &= p\lambda_{j_0} + \frac{p}{2}(p-1)\sigma^2. \end{aligned}$$

Proof. Formula (9) gives

$$v(t) = \exp(-\frac{1}{2}\sigma^2 t)\exp(\sigma\beta(t))u(t),$$

and hence

$$E|v(t)|^p = |u(t)|^p \exp(-\frac{p}{2}\sigma^2 t) E\exp(p\sigma\beta(t)). \tag{10}$$

We compute

$$E\exp(p\sigma\beta(t)) = \exp(\frac{1}{2}p^2\sigma^2 E\beta^2(t)) = \exp(\frac{1}{2}p^2\sigma^2 t).$$

After substituting the above quantity to (10) we obtain

$$\begin{aligned} E|v(t)|^p &= \exp[(\frac{p^2}{2} - \frac{p}{2})\sigma^2 t]|u(t)|^p \\ &= \exp[\frac{p}{2}(p-1)\sigma^2 t]|u(t)|^p. \end{aligned}$$

We deduce that the Lyapunov exponent of the p-th moment of the solution to the problem (6)-(8) exists as a limit as is equal to

$$\begin{aligned} g(p,f) &= p\lambda^{det}(f) + \frac{p}{2}(p-1)\sigma^2 \\ &= p\lambda_{j_0} + \frac{p}{2}(p-1)\sigma^2. \end{aligned}$$

□

Let us denote by
V_i – the subspace of $L^2(\mathcal{O})$ spanned by the vectors $e_1, e_2, \ldots, e_i$:

$$V_i = \mathrm{span}[e_1, \ldots, e_i], \quad i = 1, 2, \ldots.$$

We put

$$H_1 = L^2(\mathcal{O}) \backslash (L^2(\mathcal{O}) \backslash V_1)$$

and

$$H_{i+1} = (L^2(\mathcal{O}) \backslash V_i) \backslash (L^2(\mathcal{O}) \backslash V_{i+1}).$$

From Theorems 2 and 3 it follows that both $g_\sigma(p,f)$ and $\lambda_\sigma^{st}(f)$ are constant on the subspaces H_i. Let us denote by

$$g_{\sigma,i}(p) = g_\sigma(p,f), \quad f \in H_i$$

and by

$$\lambda_{\sigma,i}^{st} = \lambda_\sigma^{st}(f), \quad f \in H_i.$$

From Theorems 2 and 3 we can also deduce that the following modification of Theorem 2.1 from [2] holds.

Theorem 4 *Under the assumptions from Section 2 and Section 3 we have:*
(i) $g_\sigma(p,f) = \lim_{t\to\infty} \frac{1}{t} \log E\|v(t,f)\|^p$,
(ii) $g_\sigma(p,f) = g_{\sigma,i}(p)$ *for* $f \in H_i$, $i = 1, 2, \ldots$
(iii) $g_{\sigma,i}(p) \in \mathbf{R}$ *for all* $p \in \mathbf{R}$ *and* $g_{\sigma,i} : \mathbf{R} \to \mathbf{R}$ *is convex and analytic,* $i = 1, 2, \ldots$.
(iv) $g_{\sigma,i}(p)/p$ *is increasing,* $g_{\sigma,i}(0) = 0$ *and* $g'_{\sigma,i}(O) = \lambda_{\sigma,i}^{st}$.

4. Stabilization and destabilization by noise.

It follows from Theorem 2 that the stochastic system (6)-(8) is stable, in terms of the almost sure Lyapunov exponents, for σ such that

$$\lambda_1 - \frac{1}{2}\sigma^2 < 0.$$

We even have that for each initial value f

$$\lambda_\sigma^{st}(f) \to -\infty \ (\sigma \to \infty).$$

It means that every deterministic system given by the parabolic problem (1)-(3) can be stabilized pathwise exponentially by noise.

As for the p-th mean stability it follows from Theorem 3 that it depends on the parameter p. Precisely, we can conclude from Theorem 3 that

Corollary 1 *For $p < 1$*

$$g_\sigma(p, f) \to -\infty \ (\sigma \to \infty),$$

it follows that for σ big enough the system is stable in the p-th mean.

For $p = 1$

$$g_\sigma(1, f) = \lambda^{det}(f),$$

it follows that adding a term with white noise does not influence the p-th mean stability.

For $p > 1$

$$g_\sigma(p, f) \to +\infty \ (\sigma \to \infty),$$

it follows that for σ big enough the system is unstable in the p-th mean.

These conclusions are quite surprising. In fact, regardless whether the deterministic system was stable or unstable, after adding a stochastic term with σ big enough, it becomes pathwise exponentially stable and unstable in the p-th mean for $p > 1$.

Even more

$$\lambda_\sigma^{st}(f) \to -\infty \ (\sigma \to \infty),$$

while

$$g_\sigma(p, f) \to +\infty \ (\sigma \to \infty), \ p > 1.$$

On the other hand, from Theorem 3 we can also conclude that

Corollary 2 *Let us fix an arbitrary $\sigma > 0$, then*

$$g_\sigma(p, f) \to +\infty \ (p \to \infty),$$

which means that for each σ there exists a p_0 such that for $p \geq p_0$, the stochastic system is unstable in the p-th mean.

5. An example.

Let us take

$$\mathcal{O} = (a, b), \quad \text{where} \quad a < b$$

and

$$A = \triangle + \alpha \mathrm{Id}, \quad \text{where } \alpha \text{ is a constant.}$$

Then the eigenvalues and the eigenfunctions of the operator A can be computed explicitely and are equal to, respectively:

$$\alpha_j = -\frac{j^2\pi^2}{(b-a)^2} + \alpha$$

and

$$e_j(x) = (\frac{2}{b-a})^{\frac{1}{2}} \sin(\frac{j\pi(x-a)}{(b-a)}),$$

$j = 1, 2, \ldots$, see [3] Example 1.2.3, p.10.

We deduce that the Lyapunov exponents of the deterministic system and the stochastic system are equal to, respectively:

$$\lambda^{\mathrm{det}}(f) = -\frac{j_0^2\pi^2}{(b-a)^2} + \alpha,$$

$$\lambda_\sigma^{\mathrm{st}}(f) = -\frac{j_0^2\pi^2}{(b-a)^2} + \alpha - \frac{1}{2}\sigma^2$$

and

$$g_\sigma(p, f) = \frac{-pj_0^2\pi^2}{(b-a)^2} + p\alpha + \frac{p}{2}(p-1)\sigma^2.$$

We conclude that the stochastic system is pathwise exponentially stable if and only if

$$-\frac{\pi^2}{(b-a)^2} + \alpha - \frac{1}{2}\sigma^2 < 0,$$

while it is stable in the p-th mean for p and σ satisfying

$$\frac{-p\pi^2}{(b-a)^2} + p\alpha + \frac{p}{2}(p-1)\sigma^2 < 0.$$

6. Remarks on the Stratonovich differential.

Let us now consider equation (6) with the Itô differential replaced by the Stratonovich differential:

$$d\tilde{v}(t) = A\tilde{v}(t)dt + \sigma\tilde{v}(t) \circ d\beta(t), \tag{11}$$

with the initial condition (7) and the Dirichlet boundary condition (8). The unique strong solution to the problem (11)(7)(8) is given by the following formula

$$\tilde{v}(t) = \exp(\sigma\beta(t))u(t),$$

where $u(t)$ is the solution to the deterministic problem (1)-(3).

Then the almost sure Lyapunov exponent of the stochastic system is equal to the deterministic one, i.e. for all σ:

$$\tilde{\lambda}^{st}_{\sigma}(f,\omega) = \lambda^{det}(f) \quad a.s,$$

i.e. adding the stochastic term does not influence the stability of the deterministic system.

The Lyapunov exponent of the p-th moment of the solution is equal to

$$\tilde{g}_{\sigma}(p,f) = p\lambda^{det}(f) + \frac{p^2}{2}\sigma^2, \tag{12}$$

which implies that in the Stratonovich case

$$\tilde{g}_{\sigma}(p,f) \to \infty \ (\sigma \to \infty), \quad p > 0. \tag{13}$$

It means that we have destabilization in the p-th mean for any $p > 0$. Also formula (12) implies that for $p > 1$ the destabilization is faster than in the case of the Itô differential.

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References

[1] L. Arnold: Stochastic Differential Equations: Theory and Applications. A Wiley - Interscience Publication, Wiley, New York, 1974.

[2] L. Arnold, E. Oeljeklaus and E. Pardoux: Almost sure and moment stability for linear Itô Equations, in: Lyapunov Exponents, Lecture Notes in Mathematics 1186, Ed.: L. Arnold and V. Wihstutz, Springer-Verlag, 1984, 129-159.

[3] E. B. Davies: Spectral Theory and Differential Operators. Cambridge studies in advanced mathematics vol.42, Cambridge University Press, Cambridge, 1996.

[4] Yu. V. Egorov and M. A. Shubin: Partial differential equations I. Foundations of the classical theory. Encyclopaedia of Mathematical Sciences vol.30, Springer-Verlag, Berlin, 1992.

[5] T. Kato: Perturbation Theory for Linear Operators. A Series of Comprehensive Studies in Mathematics vol. 132, Springer-Verlag, Berlin Heilderberg New York, 1980, corrected printing of the second edition.

[6] A. A. Kwiecińska: Stabilization of partial differential equations by noise. Stochastic Process. Appl. 79, 1999, 179-184.

[7] O. A. Ladyzhenskaya: The boundary value problems of mathematical physics (in Russian). Nauka, Moscow, 1973. (English translation available – Springer-Verlag, 1985).

[8] R. Temam: Infinite-dimensional dynamical systems in mechanics and physics. Applied Mathematical Sciences vol.68, Springer-Verlag, New York, 1997, second edition.